Fragmentability and Continuity of Semigroup Actions

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Let a topologized semigroup S act continuously and linearly on a locally convex space X. We find sufficient conditions for continuity of induced actions on the spaces of linear (compact) operators and on the dual space X^*, for instance. The notion of *fragmentability* in the sense of Jayne and Rogers and its natural uniform generalizations play a major role in this paper. Our applications show that problems concerning the continuity of induced actions have satisfactory solutions for *Asplund* Banach spaces X (without additional restrictions, if S is a topological group) and, moreover, for a new locally convex version of Asplund spaces introduced in the paper. The starting point of this concept was the characterization of Asplund spaces due to Namioka and Phelps in terms of fragmentability.

1. Introduction

Let $\pi\colon S \times X \to X$ be a continuous linear action of a topologized semigroup S on a linear space X. For every linear space Y, denote by $L(X,Y)$ and $K(X,Y)$ the linear spaces of all linear and compact operators respectively, endowed with the strong, that is, the topology of bounded convergence. Consider the induced (right) action:

$$\pi^L\colon L(X,Y) \times S \to L(X,Y), \quad (fs)\ (x) = f(sx).$$

(We write sx, fs instead of $\pi(s,x)$ and $\pi^L(f,s)$, respectively). The *dual action*, that is, the case $Y := \mathbb{R}$ (the field of reals), is denoted by $\pi^*\colon X^* \times S \to X^*$. The subset of all functionals $f \in X^*$ for which the orbit map $\tilde{f}\colon S \to X^*, \tilde{f}(s) = fs$, is continuous (at fixed $s \in S$) is denoted by $X^{\odot}$ (resp.: $X^{\odot}(s)$).

Frequently, $X^{\odot}$ may be a proper subset of X^* even for the semigroups $S := \mathbb{R}$ and $S := [0, \infty)$ (see sections 1.3 and 1.5 in [36]). More generally, the space of all absolutely continuous measures on a locally compact group G is just the set of all such functionals $m \in (C_0(G))^* = M(G)$ for which the orbit mappings $\tilde{m}\colon G \to M(G)$ are continuous (cf. [40, 41, 11]).

We study the following general question:

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Question 1.1. For which restrictions on the spaces X, on the operators $f \in L(X,Y)$ and on the elements $s \in S$, is the pair (f,s) a point of joint (or separate) continuity for π^L? In particular, when do we have $X^{\odot} = X^*$?

For $Y := \mathbb{R}$ we refer, among others, to [5, 18, 26, 30] and to the references there. For the important case of one-parameter semigroups see, [36].

Recall some positive results.

1.2. Let S be a locally compact topological group, and let X be semireflexive. Then π^* is continuous [5, 30].

1.3. If X is a Banach space such that the dual X^* has the Radon-Nikodym property (by Stegall's result [45], it is equivalent to saying that X is Asplund), and if S is the one-parameter semigroup $[0,\infty)$, then π^* is continuous at every (f,t), where $f \in X^*$ and $t > 0$. This is proved by Arendt in [3] and, later, by van Neerven [35] in a somewhat stronger form.

One of the main purposes of the present paper is to provide a unified approach to such results. We generalize 1.2 and 1.3 in several ways:

(a) We have found a more general class (NP) of spaces X, called *Namioka-Phelps* spaces, in which the same conclusions hold;
(b) If X is an Asplund Banach space, then π^* is continuous for *any* topological group S. More generally, if S is a topological subsemigroup of a topological group G, then π^* remains continuous at every (f,t), where $f \in X^*$ and $t \in \operatorname{int}(S) \cdot \operatorname{int}(S)$;
(c) Analogous results are valid sometimes if X^* is replaced by $K(X,Y)$ (see 6.13).

In section 3 we investigate a generalized version of fragmentability. This is an attempt to synthesize some known facts and definitions from [34, 22, 31, 20, 47, 13]. Section 4 is devoted to the class (NP) of Namioka-Phelps spaces. We say that a linear topological space X is (NP) if every equicontinuous subset F of X^* is fragmented. More precisely, this means that for every element ε of the natural uniformity on X^* and every non-empty subset $F_1 \subseteq F$, there exists an w^*-open subset O of X^* such that $O \cap F_1$ is non-empty and ε-small. If X is a Banach space, then this definition gives exactly Asplund spaces. This follows directly from the well-known characterization due to Namioka and Phelps [34]. We show that (NP) contains all Asplund linear spaces in the sense of [4, 44], all semireflexive and all nuclear spaces. The class (NP) is closed under taking subspaces, arbitrary products and direct sums. Moreover, using the Diestel-Morris-Saxon result [9], it can be proved that (NP) contains the variety generated by the class of all Asplund Banach spaces.

Section 5 contains some useful "transport arguments." Our main applications are concentrated in section 6. In particular, we generalize the above-mentioned results 1.2 and 1.3. Moreover, some "small-orbit results" are discussed there. It is a well-known observation that the smallness (separability, for instance) of the orbit of a functional $f \in X^*$ frequently implies that $f \in X^{\odot}$. This happens, for example, for measures on a locally compact group G. Such a result was obtained first in [25] for a second countable G and in its full generality in [46, 11]. Another example in this spirit, when $f \in L^{\infty}(G) = (L^1(G))^*$, can be found in [39]. Jan van Neerven [35] proved that for one-parameter semigroups on a Banach space X, the orbit of $f \in X^*$ is (weak) separable iff $f \in X^{\odot}(t)$ for every $t > 0$. We show that some results of this kind can be unified in the framework of the general machinery developed in sections 3 and 5.

2. Preliminary results

Throughout the paper, all spaces are assumed to be Hausdorff and all linear spaces are real. The filter of all neighborhoods ($nbd's$) of a point x is denoted by N_x. The identity of a monoid and the origin of a linear space are denoted by e and 0, respectively.

Let $\pi\colon P \times X \to Y$, $\pi(p,x) = px$ be a function. Then the rules $\tilde{x}(p) = px = \tilde{p}(x)$ define the functions: $\tilde{x}\colon P \to Y, \tilde{p}\colon X \to Y$. We use the following notation:

$$\mathrm{Con}_p^{\ell}(\pi) = \left\{x \in X \;\middle|\; \tilde{x} \text{ is continuous at } p\right\}, \mathrm{Con}^{\ell}(\pi) = \cap \left\{\mathrm{Con}_p^{\ell}(\pi) \;\middle|\; p \in P\right\},$$

$$\mathrm{Con}_x^{r}(\pi) = \left\{p \in P \;\middle|\; \tilde{p} \text{ is continuous at } x\right\}, \mathrm{Con}^{r}(\pi) = \cap \left\{\mathrm{Con}_x^{r}(\pi) \;\middle|\; x \in X\right\},$$

$$\mathrm{Con}_p(\pi) = \left\{x \in X \;\middle|\; \pi \text{ is continuous at } (p,x)\right\},$$

$$\mathrm{Con}_x(\pi) = \left\{p \in P \;\middle|\; \pi \text{ is continuous at } (p,x)\right\},$$

$$\mathrm{Con}(\pi) = \left\{(p,x) \in P \times X \;\middle|\; \pi \text{ is continuous at } (p,x)\right\}.$$

The following useful lemma is well known (see, for example [42, p. 47]).

Lemma 2.1. *Let $\mathrm{Con}^r(\pi) = P, X$ compact and Y a uniform space. Then for every $p_0 \in P$, the family $\tilde{X} := \{\tilde{x}\colon P \to Y \mid x \in X\}$ is equicontinuous at p_0 iff $X = \mathrm{Con}_{p_0}(\pi)$.*

The following definition generalizes the notion of uniform equicontinuity and, as we will see below, in some sense is good enough.

Definition 2.2. Let $\pi\colon P \times X \to Y$ be a function, where P is a topological space, and (X,μ) and (Y,ξ) are uniform spaces. A subset $A \subseteq X$ is called *π-uniform at $p_0 \in P$* if for every entourage $\varepsilon \in \xi$, there exists $(\delta, U) \in \mu \times N_{p_0}$ such that for every $p \in U$ and every $(a,b) \in \delta \cap (A \times A)$, holds $(pa, pb) \in \varepsilon$. If A is π-uniform at every $p \in P$, then A is called π-uniform. Analogously, subsets $A \subseteq X$ which are π-uniform at $p_0 \in P$ can be defined for functions π of the form $X \times P \to Y$.

In 2.3, 2.4 and 2.5 we will keep the notation of 2.2.

Examples 2.3.

(a) Let $U \in N_{p_0}$ and $\tilde{U} = \{\tilde{p}\colon X \to Y \mid p \in U\}$ be uniformly equicontinuous. Then X is π-uniform at p_0.

(b) Let $\tilde{A} = \{\tilde{a}\colon P \to Y \mid a \in A\}$ be equicontinuous at p_0 and the restriction $\tilde{p}_0\big|_A\colon A \to Y$ be uniformly continuous. Then A is π-uniform at p_0.

(c) If X is compact, $\mathrm{Con}^r(\pi) = P$ and $X = \mathrm{Con}_{p_0}(\pi)$, then X is π-uniform at p_0.

(d) If $\pi\colon G \times X \to X$ is an action of a topological group G, and μ is a uniformity on X, then X is π-uniform at e iff π is *quasibounded* in the sense of [27].

(e) Let $\pi\colon G \times X \to X$ be an action of a Baire topological group G on a metric space (X,d). Suppose that each $\tilde{g}\colon X \to X (g \in G)$ is d-uniformly continuous and every $\tilde{y}\colon G \to X (y \in Y)$ is continuous for a certain dense subset $Y \subseteq X$. Then X is π-uniform.

Proof. (a) is trivial and (b) is straightforward, (c) follows from (b) and Lemma 2.1. For (e), use (d) and Theorem 1.10 from [27]. $\square$

Lemma 2.4. *Let $A \subseteq X$ be π-uniform at $p_0 \in P$ and suppose $\mathrm{Con}^r(\pi) = P$. Then*

(i) *The closure $c\ell(A)$ is also π-uniform at p_0.*

(ii) *If $A \subseteq c\ell(A \cap \mathrm{Con}^{\ell}_{p_0}(\pi))$, then the restricted map $P \times c\ell(A) \to Y$ is continuous at (p_0, x) for every $x \in c\ell(A)$.*

(iii) *If $A = X$, then the subset $\mathrm{Con}_{p_0}(\pi)$ is closed and coincides with $\mathrm{Con}^{\ell}_{p_0}(\pi)$.*

Proof. (i) For a fixed $\varepsilon \in \xi$, choose $\varepsilon_1 \in \xi$ such that $\varepsilon_1^3 \subseteq \varepsilon$. According to Definition 2.2 there exists $(\delta, U) \in \mu \times N_{p_0}$ such that:

$$(pa, pb) \in \varepsilon_1 \quad \text{for every} \quad (a, b) \in \delta \cap (A \times A) \quad \text{and every} \quad p \in U.$$

Take $\delta_1 \in \mu$ with $\delta_1^3 \subseteq \delta$. Then (δ_1, U) is the desired pair satisfying Definition 2.2 for $c\ell(A)$. Indeed, for an arbitrary but *fixed* pair (x, y) from $\delta_1 \cap (c\ell(A) \times c\ell(A))$ and a *fixed* element $p \in U$, by the continuity of $\tilde{p}$, we can choose elements $a, b \in A$ such that:

$$(a, x) \in \delta_1, \quad (y, b) \in \delta_1, \quad (px, pa) \in \varepsilon_1, \quad (pb, py) \in \varepsilon_1.$$

Clearly, $(a, b) \in \delta_1^3 \subseteq \delta$. Then $(pa, pb) \in \varepsilon_1$. Finally, observe that $(px, py) \in \varepsilon_1^3 \subseteq \varepsilon$.

(ii) By (i) we may assume that $c\ell(A) = A$. For a fixed $(a_0, \varepsilon) \in A \times \xi$, choose $(\varepsilon_1, \delta) \in \xi \times \mu$ and $U_1 \in N_{p_0}$ such that $\varepsilon_1^3 \subseteq \varepsilon$ and $(px, py) \in \varepsilon_1$ for every $(x, y) \in \delta \cap (A \times A)$ and $p \in U_1$. Choose a symmetric $\delta_1 \in \mu$ such that $\delta_1^2 \subseteq \delta$ and take an element $a \in A \cap \mathrm{Con}^{\ell}_{p_0}(\pi)$ such that $(a_0, a) \in \delta_1$. For a certain $U_2 \in N_{p_0}$, $(p_0 a, pa) \in \varepsilon_1$ for every $p \in U_2$. Then if $(a_0, x) \in \delta_1$, $x \in A$ and $p \in U_1 \cap U_2$, we have $(pa, px) \in \varepsilon_1$ because $(a, x) \in \delta_1^2 \subseteq \delta$. Then $(p_0 a_0, px) \in \varepsilon_1^3 \subseteq \varepsilon$.

(iii) Follows easily from (ii). □

Corollary 2.5. *Let every $\tilde{p}\colon X \to Y (p \in P)$ be uniformly continuous. Denote by $\hat{X}$ and $\hat{Y}$ the corresponding completions, and let $\hat{\pi}\colon P \times \hat{X} \to \hat{Y}$ be the induced function.*

(i) *If X is π-uniform at $p_0 \in P$, then $\hat{X}$ is $\hat{\pi}$-uniform at p_0. If, in addition, $X = \mathrm{Con}_{p_0}(\pi)$, then $\hat{X} = \mathrm{Con}_{p_0}(\hat{\pi})$.*

(ii) *If π is continuous and X is π-uniform , then $\hat{\pi}$ is continuous and $\hat{X}$ is $\hat{\pi}$-uniform.*

Proof. For (i) apply Lemma 2.4 to $\hat{\pi}$. The case (ii) follows from (i). □

Proposition 2.6. *Let X, Y be topological groups, each endowed with its left uniformity, and let $\pi\colon P \times X \to Y$ be a function such that every $\tilde{p}\colon X \to Y (p \in P)$ is a homomorphism.*

(i) *X is π-uniform at $p_0 \in P$ iff $(p_0, e) \in \mathrm{Con}(\pi)$.*

(ii) *If $(p_0, e) \in \mathrm{Con}(\pi)$ and $P = \mathrm{Con}^r(\pi)$, then $\mathrm{Con}_{p_0}(\pi) = \mathrm{Con}^{\ell}_{p_0}(\pi)$.*

(iii) *If $X = \mathrm{Con}_{p_0}(\pi)$ and $P = \mathrm{Con}^r(\pi)$, then $\hat{X} = \mathrm{Con}_{p_0}(\hat{\pi})$.*

(iv) *If π is continuous, then $\hat{\pi}$ is also continuous.*

Proof. (i) is straightforward. For (ii), use (i) and Lemma 2.4 (iii). In order to establish (iii), combine (i) with Corollary 2.5 (i). Clearly, (iii) $\Rightarrow$ (iv). □

In general, $\hat{\pi}$ is not continuous even for group actions (see [28]).

Let X be a linear topological space, and let $\pi\colon P \times X \to X$ be right linear (*i.e.*, each $\tilde{p}$ is linear). We say that π is *locally bounded at* $p_0 \in P$ if for every bounded subset B of X there exists $U \in N_{p_0}$ such that $UB := \{px \mid p \in U, x \in B\}$ is bounded. The function π is called *locally equicontinuous at* p_0 if there exists $U \in N_{p_0}$ such that $\tilde{U}$ is equicontinuous.

Lemma 2.7.

(i) *If $\pi\colon P \times X \to X$ is locally equicontinuous at p_0, then π is locally bounded at p_0.*

(ii) *If π is locally bounded at p_0, then $\pi^*\colon X^* \times P \to X^*$ is continuous at $(0, p_0)$ (and therefore X^* is π^*-uniform at p_0 by Proposition 2.6(i)).*

(iii) *If π is continuous and C is a compact subset of P, then $\tilde{C}$ is equicontinuous.*

Proof. (i) and (ii) are straightforward, and (iii) is a variant of Lemma 2.1. $\square$

Proposition 2.8. *Let $(X, \|\ \|)$ be a normed space, and let $\pi\colon P \times X \to X$ be a right linear function. Then the following conditions are equivalent:*

(i) *π is continuous at $(p_0, 0)$.*

(ii) *π is locally equicontinuous at p_0.*

Proof. (i) $\Rightarrow$ (ii) Since π is continuous at $(p_0, 0)$, there exist $U \in N_{p_0}$ and a number $\delta > 0$ such that the inequality $\|a\| \le \delta$ implies that $\|pa\| \le 1$ for each $p \in U$. Then the norm of $\tilde{p}$ is not greater than $\frac{1}{\delta}$ for every $p \in U$. Indeed, observe that $\left\|\frac{\delta}{\|x\|}\ x\right\| = \delta$ for every $x \neq 0$. By our choice, $\left\|p\frac{\delta}{\|x\|}\ x\right\| \le 1$. Therefore, $\|px\| \le \frac{1}{\delta}\ \|x\|$.

The implication (ii) $\Rightarrow$ (i) is trivial. $\square$

Let P_1, P_2 be topological spaces, X_1, X_2 normed spaces, and $\pi_k\colon P_k \times X_k \to X_k$, $k \in \{1,2\}$, functions, where $\tilde{p}_k\colon X \to X$ is a linear bounded operator for every $k \in \{1,2\}$ and every $p_k \in P_k$. Consider the right linear map:

$$\pi_1 \otimes \pi_2\colon (P_1 \times P_2) \times (X_1 \otimes X_2) \to X_1 \otimes X_2,$$
$$\left((p_1, p_2), \sum_{j=1}^{n} x_j \otimes y_j \right) \longmapsto \sum_{j=1}^{n} p_1 x_j \otimes p_2 y_j.$$

The projective and injective tensor norms $\|m\|_p$, $\|m\|_i$ of $m \in X_1 \otimes X_2$ are defined by the rules:

$$\|m\|_p = \inf \left\{ \sum_{j=1}^{n} \|x_j\| \cdot \|y_j\| \;\Big|\; m = \sum_{j=1}^{n} x_j \otimes y_j \right\},$$
$$\|m\|_i = \sup \left\{ \sum_{j=1}^{n} f_1(x_j) f_2(y_j) \;\Big|\; m = \sum_{j=1}^{n} x_j \otimes y_j, \|f_1\| \le 1, \|f_2\| \le 1, f_1 \in X_1^*, f_2 \in X_2^* \right\}.$$

The projective and injective tensor products of X_1 and X_2 are the completions of $X_1 \otimes X_2$ with respect to the norms $\|\ \|_p$ and $\|\ \|_i$ respectively, which we denote by $X_1\hat{\otimes}_p X_2$ and $X_1\hat{\otimes}_i X_2$. For every $(p_1, p_2) \in P_1 \times P_2$, the corresponding $\pi_1 \otimes \pi_2-$translation of $X_1 \otimes X_2$ is bounded with respect to both norms. Therefore, $\pi_1 \otimes \pi_2$ can be extended to right linear functions:

$$\pi_1\hat{\otimes}_p\pi_2\colon (P_1 \times P_2) \times \bigl(X_1\hat{\otimes}_p X_2\bigr) \longrightarrow X_1\hat{\otimes}_p X_2,$$
$$\pi_1\hat{\otimes}_i\pi_2\colon (P_1 \times P_2) \times \bigl(X_1\hat{\otimes}_i X_2\bigr) \longrightarrow X_1\hat{\otimes}_i X_2.$$

Moreover, we have the following useful result.

Proposition 2.9. *With the above notation, let π_k be continuous at $\bigl(p_k^0, x_k\bigr)$ for every $x_k \in X_k$ and fixed $p_k^0 \in P_k$ $(k \in \{1,2\})$. Then $\pi_1\hat{\otimes}_p\pi_2$ and $\pi_1\hat{\otimes}_i\pi_2$ are continuous at $\bigl(p_1^0, p_2^0, m\bigr)$ for every $m \in X_1\hat{\otimes}_p X_2$ or $m \in X_1\hat{\otimes}_i X_2$, respectively.*

Proof. By Proposition 2.8, each p_k^0 has an *nbd* U_k such that $\tilde{U}_k$ is norm-bounded with respect to the norm of X_k $(k \in \{1,2\})$. The explicit description of the norms $\|\ \|_p, \|\ \|_i$ shows that for the *nbd* $U_1 \times U_2$ of $\bigl(p_1^0, p_2^0\bigr)$, the family of operators $\widetilde{U_1 \times U_2}$ will be norm-bounded with respect to both norms. Therefore, $\pi_1\hat{\otimes}_p\pi_2$, $\pi_1\hat{\otimes}_i\pi_2$ are continuous at $\bigl(p_1^0, p_2^0, 0\bigr)$. By assertions (ii) and (iii) of Proposition 2.6, it suffices to prove that the orbit mappings $\tilde{m}\colon P_1 \times P_2 \to X_1\hat{\otimes}_p X_2$, $\tilde{m}\colon P_1 \times P_2 \to X_1\hat{\otimes}_i X_2$ are continuous at (p_1^0, p_2^0) for every m from the dense subset $X_1 \otimes X_2$. Since $\|m\|_i \le \|m\|_p$ (see Proposition 7.2.1 in [36]), we have only to examine the "projective case". This can be done easily for elementary tensors $x_1 \otimes x_2$ using the well known identity $\|x_1 \otimes x_2\|_p = \|x_1\|\,\|x_2\|$. Since such tensors span every $m \in X_1 \otimes X_2$, the proof is completed. $\square$

3. Fragmentable functions and sets

Let (X, τ) be a topological space, and let ρ be a metric on the set X. Following Jayne and Rogers [22], we say that X is *fragmented* by ρ if for each non-empty subset A of X and for each $\varepsilon > 0$, there exists a τ-open subset O of X such that $O \cap A \ne \phi$ and ρ-diam $(O \cap A) \le \varepsilon$.

We need the following generalization.

Definition 3.1. Let (X, τ) be a topological space, Γ a system of subsets in X, and $f\colon X \to Y$ a fixed (not necessarily continuous) function from X into a uniform space (Y, μ). We say that Γ is *fragmented by f* (with respect to the pair (τ, μ)) if for every non-empty $A \in \Gamma$ and element $\varepsilon \in \mu$, there exists a τ-open subset O of X such that $O \cap A \ne \phi$ and $f(O \cap A)$ is ε-small (*i.e.*, $(x, y) \in \varepsilon$ for $x, y \in f(O \cap A)$). In the special cases $\Gamma = N_x$ or $\Gamma = \tau$, we say that X is *locally fragmented at x* or, respectively, *locally fragmented.* If $2^A \subseteq \Gamma$, then we simply say that A is *fragmented.*

Besides the usual notion of fragmentability, Definition 3.1 generalizes one more concept. Namely, f is *cliquish* (at x) in the sense of Thielman [47] iff X is locally fragmented (at x) and μ is a metric. *Huskable sets* [13] are also a particular case of local fragmentability. It is also a remarkable fact that if a function $f\colon X \to Y$ is *quasi-continuous* in the sense of Kempisty [23, 17] (see also *modified continuity* in [15]), then X is locally fragmented.

Lemma 3.2. *Let Γ be a system of subsets in X, and let $f\colon (X,\tau) \to (Y,\mu)$ be a fixed function from a topological space (X,τ) into a uniform space (Y,μ).*

(a) *If the system $c\ell(\Gamma) := \{c\ell(A) \mid A \in \Gamma\}$ is fragmented by f, then Γ is fragmented by f.*
(b) *If $h\colon (X_1,\tau_1) \to (X,\tau)$ is a continuous function, and for a certain $A \subseteq X_1$ the subset $h(A)$ of X is fragmented by f, then A is fragmented by $f \circ h$ w.r.t. (τ_1,μ).*
(c) *If every non-empty closed subspace $X_1 \subseteq X$ is locally fragmented at some point by the restricted function $f\,|_{X_1}$ w.r.t. $\left(\tau\,|_{X_1},\mu\right)$, then X is fragmented by f w.r.t. (τ,μ).*
(d) *If f is locally fragmented, (X,τ) is a Baire space and $\mu = \rho$ is a metric, then f is continuous at the points of a dense G_δ subset D of X.*

Proof. We omit the easy proof of (a), (b), (c). In order to check (d), for a fixed $\varepsilon > 0$ consider the open set O_ε — the union of all τ-open subsets O of X such that ρ-diam $(f(O)) \le \varepsilon$. Then local fragmentability guarantees that O_ε is dense. It is easy to see that $D := \cap\left\{O_{\frac{1}{n}} \mid n \in \mathbb{N}\right\}$ is the desired dense G_δ subset. $\square$

Note that the assertion (d) was actually known. Its variant has been formulated without proof in [47, Theorem IV]. The present proof, which is very close to the proof of part (i) $\Rightarrow$ (ii) in Lemma 1.1 of [33] (see also [15, Theorem 1.1]), is given here for the sake of completeness.

Let a system Φ of subsets in a topological space X be directed (upwards under the inclusion), and let (Y,μ) be a uniform space. By $C_\Phi(X,Y)$ we will denote the set $C(X,Y)$ of all continuous maps from X into Y endowed with the uniformity μ_Φ of uniform convergence on elements of Φ. Recall that a standard base of this uniformity is the system $\{[A,\varepsilon] \mid A \in \Phi,\ \varepsilon \in \mu\}$, where $[A,\varepsilon] := \{(\varphi_1,\varphi_2) \mid (\varphi_1(a),\varphi_2(a)) \in \varepsilon \ \ \forall a \in A\}$. The set $C(X,Y)$ with the pointwise topology will be denoted by $C_p(X,Y)$.

Lemma 3.3. *Let Φ be a directed system of subsets in a topological space Y such that Φ contains a fundamental subsystem Φ_1 consisting of compact subsets. Let (Z,μ) be a uniform space. Suppose that (X,τ) is a Čech-complete space and that the function*

$$f\colon (X,\tau) \to C_p(Y,Z), \quad x \mapsto \varphi_x$$

is continuous. Then X is fragmented by f w.r.t. (τ,μ_Φ).

Proof. Let $\phi \ne X_1 \subseteq X, K \in \Phi$ and $\varepsilon \in \mu$. We have to show that there exists a τ-open subset O of X such that $O \cap X_1 \ne \phi$ and $f(O \cap X_1)$ is $[K,\varepsilon]$-small. We may assume that K is compact (by our assumption on Φ) and X_1 is τ-closed (Lemma 3.2 (a)). There exist a metric space (M,ρ), a uniformly continuous map h of Z onto M and a positive number δ such that the inequality $\rho(h(z_1),h(z_2)) < \delta$ implies that $(z_1,z_2) \in \varepsilon$. Consider the (separately continuous) evaluation map:

$$\pi_{X_1,K}\colon X_1 \times K \to M \quad , \quad (x,y) \mapsto h(\varphi_x(y)).$$

By Namioka's theorem [32, Theorem 1.2], there exists a dense subset D of X_1 such that $\pi_{X_1,K}$ is jointly continuous at every (x,y), where $x \in D$ and $y \in K$. Lemma 1.1 now yields that the family $\{h \circ \tilde{y}\colon X_1 \to M \mid y \in K\}$ is equicontinuous at every

$x \in D$. Choose arbitrarily $x_0 \in D$. Then there exists a τ-open *nbd* O of x_0 such that

$$\rho(h(\tilde{y}(x_0)), h(\tilde{y}(x))) = \rho(h(\varphi_{x_0}(y)), h(\varphi_x(y))) < \frac{\delta}{2} \quad \forall x \in O \cap X_1 \quad \forall y \in K.$$

This implies that $f(O \cap X_1)$ is $[K, \varepsilon]$-small. $\square$

Next, we make some notational conventions. Let X be a linear topological space ($\ell.t.s.$), and let μ be its natural uniformity. By μ^* we denote the uniformity of the strong dual X^*. The weak and weak* topologies on X and X^* will be denoted by w and w^* respectively. If A is a subset of X (X^*), then we say that A is *fragmented* if A is fragmented by the inclusion map $f = 1_A \colon A \hookrightarrow X$ $(A \hookrightarrow X^*)$ with respect to (w, μ) (resp.: (w^*, μ^*)). The systems of all bounded subsets of X and all equicontinuous subsets of X^* are denoted by Φ_b and Φ_{eq} respectively.

The first assertion of the following lemma easily follows from the definitions.

Lemma 3.4. *(a) Let Y be an $\ell.t.s.$ Then $L(X,Y)$ is a uniform subspace of $C_{\Phi_b}(X,Y)$.*

(b) [43, Ch. IV, 1.5, Corollary 4] *Every locally convex space ($\ell.c.s.$) X is a uniform subspace of $C_{\Phi_{eq}}(X^*, \mathbb{R})$.*

The following result is well known for Banach spaces [33, Theorem 1.2].

Proposition 3.5. *Every relatively weakly compact subset A of an $\ell.c.s.$ X is fragmented.*

Proof. By [43, Ch. III, 4.3], the system Φ_{eq} has a fundamental subsystem consisting of weak*-compact subsets. Therefore we can apply Lemma 3.3 to the w-continuous inclusion $c\ell_w(A) \hookrightarrow C_p(X^*, \mathbb{R})$. Then $c\ell_w(A)$ (and, hence, its subset A) is fragmented. $\square$

Proposition 3.6. *If X is semireflexive, then every relatively weak*-compact (and, hence, every equicontinuous) subset A of X^* is fragmented.*

Proof. The semireflexivity of X means that each bounded subset of X is relatively weakly compact. Taking into consideration that every weakly compact subset of X is weakly bounded and, hence, bounded, we can apply Lemma 3.3 to the weak*-continuous inclusion map $c\ell_{w^*}(A) \hookrightarrow C_p(X, \mathbb{R})$ and the system Φ_b. $\square$

Definition 3.7. Let (X, μ) be a uniform space, and let $\varepsilon \in \mu$. We say that X is *ε-Lindelöf* if the uniform cover $\{\varepsilon(x) \mid x \in X\}$, where $\varepsilon(x) = \{y \in X \mid (x,y) \in \varepsilon\}$, has a countable subcover. If X is ε-Lindelöf for each $\varepsilon \in \mu$, then it will be called *uniformly Lindelöf.*

We mention that (X, μ) is uniformly Lindelöf iff it is $\aleph_0$-*precompact* in the sense of Isbell [20] iff X is $\aleph_0$-*bounded* in the sense of Guran (cf. Definition 2.4 in [48]). We prefer the name “uniformly Lindelöf” in order to avoid possible misunderstandings in linear spaces. If X , as a topological space, is either separable, Lindelöf or *ccc* (see [20, p.24]), then (X, μ) is uniformly Lindelöf. For a metrizable uniformity $\mu, (X, \mu)$ is uniformly Lindelöf iff X is separable. Uniformly continuous maps move uniformly Lindelöf subspaces onto uniformly Lindelöf subspaces. Guran showed [16] that a topological group G endowed with its left or right uniformity is uniformly Lindelöf iff G is a topological subgroup in a product of second countable groups. As is well known, every ℓ.c.s. X is a linear topological subspace in a product of normed spaces. Combining these facts we obtain

Proposition 3.8. *An ℓ.c.s. X is uniformly Lindelöf iff X is a linear topological subspace in a product of separable normed spaces.*

Definition 3.9. Let $f\colon X \to Y$ be a map from a topological space X into a uniform space (Y, μ). We say that f is *locally uniformly Lindelöf at a point $x \in X$*, if for every $\varepsilon \in \mu$ there exists $U \in N_x$ such that $f(U)$ is ε-Lindelöf (this holds, for example, if $f(U_0)$ is uniformly Lindelöf for a certain $U_0 \in N_x$).

This definition is closely related to a concept from [31] called the *index of non-separability*. If μ is a metric, then the condition in Definition 3.9 can be reformulated by saying that for every $\varepsilon > 0$ there exists $U \in N_x$ such that the index of non-separability $\beta(U)$ is less than ε. Hence, this gives a single-valued variant of the "β upper semi-continuity" [31, p. 70].

Proposition 3.10. *Let Φ be a directed system of subsets in a topological space Y, and let (Z, μ) be a uniform space. Suppose that (X, τ) is Baire and the function $f\colon X \to C_p(Y, Z), x \mapsto \varphi_x$ is continuous.*

- (i) *If $f\colon X \to C_\Phi(Y, Z)$ is locally uniformly Lindelöf at $x_0 \in X$, then X is locally fragmented by f at x_0 w.r.t. (τ, μ_Φ).*
- (ii) *If X is hereditarily Baire (i.e., each closed subset is Baire) and $f\colon X \to C_\Phi(Y, Z)$ is locally uniformly Lindelöf at each point, then X is fragmented by f w.r.t. (τ, μ_Φ).*

Proof. (i). Let W be an open nbd of $x_0, A \in \Phi$ and $\varepsilon \in \mu$. Our aim is to find an open subset O of W such that $f(O)$ is $[A, \varepsilon]$-small. Choose symmetric $\delta \in \mu$ such that $\delta^2 \subseteq \varepsilon$. We can suppose that δ is a closed subset of $Z \times Z$. Since f is locally uniformly Lindelöf at x_0, there exists an open nbd U of x_0 such that $U \subseteq W$ and $f(U)$ is $[A, \delta]$-uniformly Lindelöf. Thus, there exists a sequence (x_n) in U such that

$$f(U) \subseteq \cup \left\{ [A, \delta] \ \ (f_{x_n}) \;\middle|\; n \in \mathbb{N} \right\},$$

where

$$[A, \delta] \, (f_{x_n}) = \left\{ \varphi \in C(Y, Z) \;\middle|\; (f_{x_n}(y), \varphi\ (y)) \in \delta \quad \forall y \in A \right\}.$$

Then

$$U = \cup \{ M_n \mid n \in \mathbb{N} \} \text{ for } M_n := \{ x \in U \mid (f_{x_n}(y), f_x(y)) \in \delta \quad \forall y \in A \}.$$

Since δ is closed and f is (τ, p)-continuous, one can easily see that each M_n is closed in U. By the Baire category theorem, a certain M_{n_0} contains a non-empty τ-open subset O. Then, $\left(f_{x_{n_0}}, f_x\right) \in [A, \delta]$ for every $x \in O$. This implies that $f(O)$ is $[A, \varepsilon]$-small, because $\delta^2 \subseteq \varepsilon$.

For (ii), use (i) and Lemma 3.2 (c). $\square$

Corollary 3.11. *Let Y be an ℓ.t.s. (or an ℓ.c.s.), and let X be a uniformly Lindelöf subset of Y^* (resp.: Y). If X is hereditarily Baire in the weak* (resp.: weak) topology, then X is fragmented.*

Proof. Apply Proposition 3.10(ii) to the inclusion $(X, w^*) \hookrightarrow C_p(Y, \mathbb{R})$ and the system $\Phi = \Phi_b$ (resp.: $(X, w) \hookrightarrow C_p((Y^*, \mathbb{R})$ and $\Phi = \Phi_{eq}$). $\square$

Corollary 3.12. *Let X be a Baire topological space.*

(i) *Suppose that (Y,τ) is an ℓ.c.s and that $f\colon X\to Y$ is weakly continuous. If $f(X)$ is second countable w.r.t. τ, then f is τ-continuous at each point of a dense G_δ subset of X.*

(ii) *Suppose that (Y,τ) is an ℓ.t.s., (Y^*,τ^*) denotes its dual, and $f\colon X\to Y^*$ is weak*-continuous. If $f(X)$ is second countable w.r.t. τ^*, then f is τ^*-continuous at each point of a dense G_δ subset of X.*

Proof. Use Proposition 3.10 (i) and Lemmas 3.4, 3.2. □

Remark 3.13. If, in the assertion (i) of Corollary 3.12, the space Y is assumed to be normed, then we obtain a result of Alexiewicz and Orlicz [1].

Recall that a real-valued function f defined on an open convex subset U of a linear topological space X is called *Fréchet differentiable* at $x_0\in U$ whenever there exists $u\in X^*$ such that for every bounded $B\subseteq X$ and every $\varepsilon>0$, there exists $\delta>0$ such that for all $x\in B$ and for all t with the property $0<|t|<\delta$, the following inequality is satisfied:

$$\left|\frac{f(x_0+tx)-f(x_0)}{t}-u(x)\right|<\varepsilon.$$

The function u is denoted by $f'(x_0)$. In the definition, δ can be chosen so small that for every $t\in(0,\delta)$ and every $x\in B$, holds

$$\text{(D)}\qquad f(x_0+tx)+f(x_0-tx)-2f(x_0)<t\frac{\varepsilon}{3}.$$

A *weak* slice* of a nonempty subset $A\subseteq X^*$ is a subset of A of the form:

$$S(x_0,A,\alpha)=\left\{f\in A \;\middle|\; f(x_0)>\sigma_A(x_0)-\alpha\right\},$$

where $x_0\in X,\alpha>0$ and $\sigma_A(x):=\sup\left\{f(x)\;\middle|\; f\in A\right\}$.

Proposition 3.14. *Let X be a linear topological space, and let F be an equicontinuous subset of X^*. Suppose that for every non-empty relatively w^*-closed subspace A of F, the sublinear functional $\sigma_A\colon X\to\mathbb{R}$, defined by the rule $\sigma_A(x)=\sup\left\{f(x)\;\middle|\; f\in A\right\}$, is Fréchet differentiable at some point (depending on A) x of X. Then F is fragmented.*

Proof. Fix $\varepsilon>0$ and a bounded set $B\subseteq X$. We have to show that for every non-empty subset A of F, there exists a relatively w^*-open non-empty subset O which is $[B,\varepsilon]$-small. It suffices to find a $[B,\varepsilon]$-small weak* slice of A. By Lemma 3.2 (a), we may suppose that A is relatively w^*-closed in F. Consider the function σ_A. It is continuous because A is equicontinuous. Suppose that every weak* slice of A is not $[B,\varepsilon]$-small. As in the proof of Lemma 2.18 in [38], we will show that σ_A is nowhere Fréchet differentiable. Let $x\in X$. By our assumption, for each $n\geq 1$ the weak* slice $S\left(x,A,\frac{\varepsilon}{3n}\right)$ is not $[B,\varepsilon]$-small. Therefore, there exist $f_n,h_n\in S\left(x,A,\frac{\varepsilon}{3n}\right)$ and $x_n\in B$ which satisfy

$$|f_n(x_n)-h_n\;(x_n)|>\varepsilon.$$

On the other hand, by the definition of slice we have:

$$f_n(x) > \sigma_A(x) - \frac{\varepsilon}{3n} \quad , \quad h_n(x) > \sigma_A(x) - \frac{\varepsilon}{3n}.$$

Then

$$\begin{aligned} &\sigma_A\left(x + \frac{x_n}{n}\right) + \sigma_A\left(x - \frac{x_n}{n}\right) - 2\sigma_A(x) \geq \\ &\geq f_n\left(x + \frac{x_n}{n}\right) + h_n\left(x - \frac{x_n}{n}\right) - (f_n + h_n)\,(x) - \frac{2\varepsilon}{3n} = \\ &= \frac{1}{n}(f_n - h_n)\,(x_n) - \frac{2\varepsilon}{3n} > \frac{\varepsilon}{n} - \frac{2\varepsilon}{3n} = \frac{1}{n}\cdot\frac{\varepsilon}{3}. \end{aligned}$$

This contradicts the inequality (D). □

4. Namioka-Phelps spaces

Recall [4, 44] that a linear topological space X is said to be *Asplund* if every continuous convex real function defined on an open convex subset of X is Fréchet differentiable on a dense G_δ subset of its domain. If, in the definition, "dense G_δ" is replaced by "dense", X is called a *Fréchet differentiable* space.

For Banach spaces, these definitions give the same classes. General Asplund spaces are studied systematically in [44, 14]. For information on Asplund Banach spaces, see for example [34, 45, 6, 38]. Among various characterizations of Asplund Banach spaces, we mention here only two. The first one states that a Banach space X is Asplund iff X^* has the Radon-Nikodym property [45]. In 1975 Namioka and Phelps [34] proved that a Banach space X is Asplund iff every bounded subset of X^* is fragmented. The last criteria justifies the following main definition.

Definition 4.1. We say that an ℓ.t.s. X is *Namioka-Phelps* (abbr.: (NP)) if every equicontinuous subset of X^* is fragmented.

Recall that if X is a normed space then a subset F of X^* is equicontinuous if and only if F is bounded.

Proposition 4.2. *Every Fréchet differentiable (and, hence, every Asplund) space is (NP).*

Proof. Directly follows from Proposition 3.14. □

Proposition 4.3. *Every semireflexive space is (NP).*

Proof. Directly follows from Proposition 3.6. □

An ℓ.c.s. X is called *semi-Montel* [37] if every bounded subset of X is relatively compact. We say that an ℓ.c.s. X is *quasi-Montel* if every bounded subset of X is precompact. By [43, ch. III, §7, Corollary 2], every nuclear space is quasi-Montel.

Proposition 4.4. *Let X be a quasi-Montel space, and let Y be a linear topological space. Then for every equicontinuous subset F of $L(X,Y)$, the topology of pointwise convergence and the strong topology coincide on F.*

Proof. The strong topology is the topology of bounded convergence. By our assumption, every bounded subset of X is precompact. Now our assertion follows from the fact that the topologies of precompact and simple convergence coincide on F (see [43, III, 4.5]). □

Corollary 4.5. *Every quasi-Montel space is (NP).*

Proposition 4.6. *Let X be a linear topological space. If X^* (or, at least, each equicontinuous subset of X^*) is uniformly Lindelöf, then X is (NP).*

Proof. By [43, III, 4.3], every equicontinuous subset F of X^* is contained in a weak*-compact equicontinuous subset $c\ell_{w^*}(F)$, which is fragmented by Corollary 3.11. $\square$

Remark 4.7. (a) The last result generalizes the well-known fact in the theory of Asplund spaces, which states that each Banach space X with the separable dual X^* is Asplund.

(b) Corollary 4.5 can also be derived from Proposition 4.6 because, as follows easily from [43, III, 4.3], each equicontinuous subset of X^* is relatively compact in the strong topology whenever X is quasi-Montel.

The following result is a minor modification of Lemma 2.1 from [33].

Lemma 4.8. *Let (X, τ_X) and (Y, τ_Y) be compact (Hausdorff) spaces, and let μ_X and μ_Y be uniformities on the sets X and Y respectively. Suppose that there is a continuous surjection $f\colon X \to Y$ which is also uniformly continuous w.r.t. μ_X and μ_Y. If X is fragmented by the identity map $1_X\colon X \to X$ w.r.t. (τ_X, μ_X), then Y is fragmented by the identity map $1_Y\colon Y \to Y$ w.r.t. (τ_Y, μ_Y).*

Proof. Let A be a non-empty τ_Y-compact subset of Y, and let $\varepsilon \in \mu_Y$. Choose $\delta \in \mu_X$ such that $(f \times f)\ (\delta) \subseteq \varepsilon$. By Zorn's Lemma, there exists a minimal τ_X-compact subset M of X such that $f(M) = A$. Since X is fragmented, there exists $V \in \tau_X$ such that $V \cap M \neq \phi$ and $V \cap M$ is δ-small. Then the set $f(V \cap M)$ is ε-small. Consider the set $W = A \setminus f(M \setminus (V \cap M))$. Then

(a) W is ε-small, being a subset of $f(V \cap M)$;
(b) W is relatively τ_Y-open in A;
(c) W is non-empty (otherwise $M \setminus (V \cap M)$ is a *proper* τ_X-compact subset of M such that $f(M \setminus (V \cap M)) = A$).

Therefore, by Lemma 3.2 (a), the proof is complete. $\square$

Proposition 4.9. *If E is (NP) and M is a linear subspace of E, then M is (NP).*

Proof. Let Y be an equicontinuous subset of M^*. By [37, 9.11.4 (a)] there exists an equicontinuous (and, hence, fragmented) subset X of E^* such that for the canonical mapping $q^*\colon E^* \to M^*$ (where $q\colon M \hookrightarrow E$ is the inclusion), holds $q^*(X) = Y$. By the Alaoglu-Bourbaki theorem and Lemma 3.2 (a), we may assume that X and Y are weak*-compact. The fragmentability of Y follows from Lemma 4.8, applied to the map $q^*\big|_X\colon X \to Y$, which is weak*-weak* continuous and also uniformly continuous when X and Y carry the uniformities μ_X and μ_Y inherited from the strong uniformities. $\square$

A linear map $q\colon X \to Y$ is said to be *bound covering* if for every bounded subset A of Y, there exists a bounded subset B of X such that $q(B) = A$. A linear open map of a normed space onto a normed space is a bound covering. For more information see [8].

Proposition 4.10. *Let $q \colon X \to Y$ be a continuous bound covering linear map. If X is (NP), then Y is (NP).*

Proof. Let $F \subseteq Y^*$ be equicontinuous. In order to establish the fragmentability of F, fix: a non-empty subset F_1 of F, a bounded subset A of Y, and a number $\varepsilon > 0$. Since q is bound covering, there exists a bounded subset B of X such that $q(B) = A$. The set

$$F_1 q = \{ f \circ q \in X^* \mid f \in F_1 \}$$

is an equicontinuous subset of X^* and, hence, it is fragmented because X is (NP). Therefore, for $[B, \varepsilon]$ there exist a finite sequence $\{x_1, x_2, \ldots, x_n\}$ in X, a number $\delta > 0$ and a functional $h_0 \in F_1 q$ such that for every $h \in F_1 q$ satisfying the condition

$$|h_0(x_i) - h(x_i)| < \delta \qquad \forall\, i \in \{1, 2, \ldots, n\},$$

we have

$$|h_0(x) - h(x)| < \varepsilon \qquad \text{for every} \quad x \in B.$$

Clearly, $h_0 = f_0 \circ q$ for a certain $f_0 \in F_1$. Then

$$|f_0(y) - f(y)| < \varepsilon \qquad \text{for every} \quad y \in A,$$

whenever

$$|f_0(y_i) - f(y_i)| < \delta \qquad \forall\, i \in \{1, 2, \ldots, n\},$$

where $f \in F_1$ and y_i denotes $q(x_i)$. This proves our assertion. $\square$

Example 4.11. The class (NP) is not closed under quotients. In particular, Proposition 4.10 may be false if q is not bound covering. Indeed, there is [43, IV, Ex. 20] a Fréchet *Montel* (and, hence, (NP)) space E in which there exists a closed subspace M such that E/M is isomorphic with the Banach space ℓ^1 which is not Asplund (or, equivalently, is not (NP)) by [34, Corollary 10]. This example also shows that the class of all ℓ.c. (NP)-spaces is not a variety in the sense of [9].

However, we have:

Proposition 4.12. *The class (NP) is closed under products and l.c. direct sums.*

Proof. Let $X = \prod_{i \in I} X_i$ be a topological product of (NP) spaces. Suppose F is an equicontinuous subset of X^*. Fix: a nonempty subset $F_1 \subseteq F$, a bounded subset B of X, and a number $\varepsilon > 0$. Since F_1 is equicontinuous, the polar

$$F_1^0 = \{ x \in X \mid |f(x)| \leq 1 \qquad \forall\, f \in F_1 \}$$

is a neighborhood of 0 in X. Therefore, for a certain finite $J \subseteq I$, the projection $pr_i\left(F_1^0\right)$ is X_i for $i \notin J$. This implies that for every functional $f = \sum_{i \in I} f_i$ from F_1, the functional $f_i \in X_i^*$ is trivial for each $i \notin J$. In fact, f can be represented as $f = \sum_{j \in J} f_j$.

Every projection $pr_j(B)$ is a bounded subset of $X_j (j \in J)$. Since $pr_j(F)$ is an equicontinuous subset of X_j^* and X_j is (NP), for every $j \in J$ there exist:

(a) a functional $h^j \in X_j^*$,
(b) A finite set $\{x_1^j, x_2^j, \ldots, x_{n_j}^j\} \subseteq X_j$,
(c) a number $\delta_j > 0$

such that if

$$\left| f_j\left(x_k^j\right) - h^j\left(x_k^j\right)\right| < \delta_j \qquad \forall\, k \in \{1, 2, \ldots, n_j\},$$

then

$$|f_j(y) - h^j(y)| < \frac{\varepsilon}{|J|} \qquad \forall\, y \in pr_j(B).$$

Now, consider:

(1) the finite set $\left\{\tilde{x}_k^j \mid j \in J, \quad k \in \{1, 2, \ldots, n_j\}\right\} \subseteq X$, where $\tilde{x}_k^j$ denotes the element of $\prod\limits_{i\in I} X_i$ having x_k^j in the j-th coordinate and all other coordinates are zero;
(2) the number $\delta := \min\{\delta_j \mid j \in J\}$;
(3) the functional $h_0 := \sum\limits_{j\in J} h^j \in X^*$.

We claim that for every $f = \sum\limits_{j\in J} f_j \in F_1$, the finite system of inequalities:

$$\left| f\left(\tilde{x}_k^j\right) - h_0\left(\tilde{x}_k^j\right)\right| \leq \delta \quad \forall\, j \in J, \;\; \forall\, k \in \{1, 2, \ldots, n_j\}$$

imply that

$$|f(y) - h_0(y)| \leq \varepsilon \quad \forall\, y \in B.$$

Indeed,

$$\left| f\left(\tilde{x}_k^j\right) - h_0\left(\tilde{x}_k^j\right)\right| = \left| f_j\left(x_k^j\right) - h^j\left(x_k^j\right)\right| \leq \delta \leq \delta_j.$$

Then for every $y = (y_i)_{i\in I} \in B$, holds:

$$|f(y) - h_0(y)| = \left|\sum_{j\in J}\left(f_j(y_j) - h^j(y_j)\right)\right| \leq \sum \left|f_j(y_j) - h^j(y_j)\right| \leq |J| \cdot \frac{\varepsilon}{|J|} = \varepsilon.$$

This proves our assertion for products.

For direct sums, the proof is quite similar. The following standard fact from [43, II, 6.3] plays the major role in the proof.

Fact. For every bounded subset B of a locally convex direct sum $\bigoplus\limits_{i\in I} X_i$, there exists a finite set $J \subseteq I$ such that $pr_i(B)$ is zero for every $i \notin J$.

□

Proposition 4.13. *The class (NP) contains the variety generated by the Asplund Banach spaces. In particular, every quotient of a subspace M of $\prod\limits_{i\in I} X_i$, where each X_i is an Asplund Banach space, belongs to (NP).*

Proof. An easy consequence of [9, Theorem 1.4] and the results of this section.
□

Remark 4.14. (a) By Proposition 4.13, the (NP)-space E from Example 4.11 is not contained in a product of Asplund Banach spaces.

(b) The class (ASP) of Asplund spaces is a proper subclass of (NP). Indeed, (ASP) is not closed under locally convex direct sums [43, Example 6.1], in contrast to (NP).

(c) Recall that if the dual E^* of a Banach space is weakly compactly generated (in the sense of [2]), then E is Asplund (cf. [34]). Is it true that an l.c.s. E is (NP) if E^* is weakly compactly generated (cf. [19]) ?

5. A "transport" argument

Definition 5.1. Let S be a topologized semigroup, and let P, Q be subsets of S. We say that an element $t \in S$ is *left (right) P-reachable from Q* if for every non-empty open subset $O \subseteq Q$ there exists $p \in P$ such that $pO \in N_t$ (resp.: $Op \in N_t$). We write $s \lhd^\ell t$ or $s \lhd^r t$ if there exists a non-empty compact subset $C_{s,t}$ of S such that for every *nbd* P of $C_{s,t}$ there exists an *nbd* Q of s such that t is left or, respectively, right P-reachable from Q. One gets the weaker relations writing: $s \lhd^\ell_w t$ or $s \lhd^r_w t$, whenever there exists $Q \in N_s$ such that t is left or, respectively, right S-reachable from Q. Denote by $S^{\lhd^\ell}$ $\left(S^{\lhd^r}, S^{\lhd^\ell_w}, S^{\lhd^r_w}\right)$ the set of all $t \in S$ such that for a certain $s \in S$, holds $s \lhd^\ell t$ (resp.: $s \lhd^r t, s \lhd^\ell_w t, s \lhd^r_w t$).

Lemma 5.2. *Let S be a topological subsemigroup of a topological group G. Then $\mathrm{int}(S) \cdot \mathrm{int}(S) \subseteq S^{\lhd^\ell} \cap S^{\lhd^r}$. More precisely, if $t = s_1 s_2$, where $s_1, s_2 \in \mathrm{int}(S)$, then $s_2 \lhd^\ell t$ with $C_{s_2,t} = \{s_1\}$ and $s_1 \lhd^r t$ with $C_{s_1,t} = \{s_2\}$.*

Proof. For every *nbd* P of s_1 in S, choose an *nbd* U of e in G so small that $s_1 U \subseteq P$ and $U^{-1} s_2 \subseteq \mathrm{int}(S)$. Consider *nbd* $Q := U^{-1} s_2$ of s_2. Then t is left P-reachable from Q because for every non-empty open subset O of Q, we can take $u \in U$ such that $u^{-1} s_2 \in O$. Then for $p := s_1 u$, holds $t = (s_1 u)\,(u^{-1} s_2) \in pO$ and pO is open in S. This proves $s_2 \lhd^\ell t$ with $C_{s_2,t} = \{s_1\}$. Easy modifications prove the second case. $\square$

Corollary 5.3. *(a) If G is a topological group, then $G^{\lhd^\ell} = G^{\lhd^r} = G$.*

(b) $[0,\infty)^{\lhd^\ell} = [0,\infty)^{\lhd^r} = (0,\infty)$.

A topologized semigroup S is called *left (right) topological* if, for the multiplication $\pi \colon S \times S \to S$, we have $\mathrm{Con}^\ell(\pi) = S$ (resp.: $\mathrm{Con}^r(\pi) = S$).

Lemma 5.4. *Let S be a left (right) topological monoid, and let $H(e)$ denote the group of all units in S. If $sH(e)$ (resp.: $H(e)s$) is dense in S, then $s \in S^{\lhd^r_w}$ (resp.: $s \in S^{\lhd^\ell_w}$).*

Proof. It is trivial to show that $e \lhd^r_w s$ (resp.: $e \lhd^\ell_w s$) . $\square$

Lemma 5.5. *(a) Let $\pi \colon S \times X \to X$ be a semigroup action on a uniform space (X, μ). Assume that $x_0 \in X, s, t \in S$ and the following conditions are satisfied:*

1. $s \lhd^\ell t$;
2. *X is π-uniform at every $c \in C_{s,t}$;*
3. *S is locally fragmented at s by the orbit map $\tilde{x}_0 \colon S \to (X, \mu)$.*

Then $\tilde{x}_0$ is continuous at t.

(b) The same is true for a right action $\pi\colon X\times S\to X$ provided that "$s \lhd^r t$" takes the place of (1).

(c) If $\tilde{S}$ is μ-uniformly equicontinuous, then in (a) and (b), condition (2) can be dropped, and (1) can be replaced by the weaker assumptions: $s \lhd^\ell_w t$ and $s \lhd^r_w t$ respectively.

Proof. We prove only (a). Case (b) is similar, and (c) can be obtained by a minor modification of (a).

Let $\varepsilon \in \mu$. Since $C_{s,t}$ is compact, by elementary compactness arguments, making use (2), we can pick *nbd* P of $C_{s,t}$ such that:
(*) There exists $\delta\in\mu$ such that $(px,py)\in\varepsilon$ for every $(x,y)\in\delta$ and $p\in P$.

According to Definition 5.1, choose for P *nbd* Q of s such that t is P-reachable from Q. By (3) there exists an open non-empty subset O of Q such that the set $\tilde{x}_0(O)=Ox_0$ is δ-small. By our choice, for a certain $p\in P$, holds $pO\in N_t$. Then the set $\tilde{x}_0(pO)=pOx_0$ is ε-small by $(*)$. This proves the continuity of $\tilde{x}_0$ at t. $\square$

Several transport arguments and their applications can be found in [26, 42].

6. Applications

Lemma 6.1. *(i) Let X be an ℓ.t.s., P be a topological space and $\pi\colon P\times X\to X$ a right linear map satisfying $\mathrm{Con}^r(\pi)=P$. For a normed space Z, consider the induced map $\pi^L\colon L(X,Z)\times P\to L(X,Z)$. If $(p_0,0)\in\mathrm{Con}(\pi)$, then for every equicontinuous subset F of $L(X,Z)$, there exists $U\in N_{p_0}$ such that FU is equicontinuous in $L(X,Z)$.*

(ii) If we replace the assumption $(p_0,0)\in\mathrm{Con}(\pi)$ with the assumption that $\tilde{Q}$ is equicontinuous for a certain $Q\in N_{p_0}$, then Z may be an arbitrary ℓ.t.s.

Proof. (i) Since Z is normed, the system $\left\{\frac{1}{n}B \;\middle|\; n\in\mathbb{N}\right\}$, where B is the unit ball in Z, is a local base at 0. As in the case $Z:=\mathbb{R}$, it is easy to show that a subset E of $L(X,Z)$ is equicontinuous iff $E\subseteq W^0$, where W is a certain *nbd* of 0 in X, and W^0 denotes the "polar" of W, *i.e.*, the set:

$$W^0=\{f\in L(X,Z)\mid \|f(x)\|\le 1\qquad \forall x\in W\}.$$

Since F is equicontinuous in $L(X,Z)$, there exists an *nbd* V of 0 such that $F\subseteq V^0$. The continuity of π at $(p_0,0)$ implies that $UW\subseteq V$ for certain *nbd*'s $U\in N_{p_0}$, $W\in N_0$. Then, eventually, $FU\subseteq W^0$.

The proof of (ii) is analogous and even easier and, hence, is omitted. $\square$

Now we are ready to prove the following main lemma.

Lemma 6.2. *Let X be (NP), and let $\pi\colon S\times X\to X$ be a continuous linear action. Denote by π^* the dual action $X^*\times S\to X^*$. If $s\lhd^r t$ ($s\lhd^r_w t$) and π is locally bounded at every $q\in C_{s,t}\cup\{t\}$ (resp.: $\tilde{S}$ is equicontinuous), then $X^*=\mathrm{Con}_t(\pi^*)$.*

Proof. We prove only the "locally bounded" case. The second case is an easy modification.

Fix $f\in X^*$. In order to establish that π^* is continuous at (f,t), by Lemma 2.7(ii) and the 'right' version of Proposition 2.6(ii), it suffices to show that $\tilde{f}\colon S\to$

$X^*, s \mapsto fs$ is continuous at t. By our hypothesis, condition (1) in Lemma 5.5(b) is satisfied. Let $c \in C_{s,t}$. Since π is locally bounded at c, π^* is continuous at $(0, c)$ (Lemma 2.7(ii)). By the 'right' version of Proposition 2.6(i), X^* is π^*-uniform at c. Therefore, condition (2) also holds. In order to check the validity of (3), take, due to Lemma 6.1, *nbd* $U \in N_s$ such that fU is an equicontinuous subset of X^*. Since X is (NP), fU is fragmented. On the other hand, the continuity of π guarantees that $\tilde{f}: S \to X^*$ is weak*-continuous. Therefore, Lemma 3.2(b) establishes that U is fragmented by $\tilde{f}: S \to (X^*, \mu^*)$. Hence, S is locally fragmented at s by $\tilde{f}$. Now we can use Lemma 5.5(b) which yields that $\tilde{f}$ is continuous at t. □

The following applications are divided into several subsections.

A. Locally compact (semi) group actions.

Theorem 6.3. *Let X be (NP), let S be a topological subsemigroup of a locally compact topological group G, and let $\pi: S \times X \to X$ be a continuous linear action. Then $\pi^*: X^* \times S \to X^*$ is continuous at every (f, t), where $f \in X^*$ and $t \in \mathrm{int}(S) \cdot \mathrm{int}(S)$. In particular, if $S = G$, then π^* is continuous.*

Proof. Let $f \in X^*$ and $t = s_1 s_2$, where $s_1, s_2 \in \mathrm{int}(S)$. By Lemma 5.2, $s_1 \lhd^r t$ with $C_{s_1,t} = \{s_2\}$. The points s_2 and t, being in $\mathrm{int}(S)$, have locally compact *nbd*'s in S. Therefore, by Lemma 2.7, π is locally equicontinuous (and, hence, locally bounded) at each $q \in \{s_2, t\}$. Now, we can apply Lemma 6.2 . □

Corollary 6.4. *(Moore* [30, Ch. 5, Theorem 5]*, Bourbaki* [5, Ch. 8, §2, Ex. 3(c)]*) For every locally compact topological group G and every continuous linear action on a semireflexive space X, the dual action is continuous.*

Proof. By Proposition 4.3, X is (NP). □

Corollary 6.5. *(Generalized Arendt Theorem.) Let X be (NP). For every continuous linear one-parameter semigroup action $\pi: [0, \infty) \times X \to X$, the dual action $\pi^*: X^* \times [0, \infty) \to X^*$ is continuous at every (f, t), where $f \in X^*$ and $t > 0$.*

Remark 6.6.

(a) Helmer has shown [18, Corollary 4.3], that if S is a locally compact topological monoid, then for every continuous linear action of S on a semireflexive space X, the dual action is continuous at every (f, t), where $f \in X^*$ and t is a unit.

(b) Arendt [3] proved Corollary 6.5 for Banach spaces X whose duals have the Radon-Nikodym Property. By [45], such X are Asplund. Therefore, Corollary 6.5 is stronger. A "non-adjoint" generalization of Arendt's result was obtained by van Neerven [35, Lemma 3.1].

(c) It is actually well known that some restrictions on points $t \in S$ are really needed. For example, if $S = [0, \infty)$ and X is a (non-reflexive) Asplund Banach space, then "$t > 0$" is essential even for $X = c_0$, or for $X = Y^*$ where Y is the (quasi-reflexive) James space (cf. [36, Examples 2.3.5, 1.5.3]).

(d) Generally, the local boundedness in Lemma 6.2, as well as the local compactness in Theorem 6.3 and Corollary 6.4, cannot be dropped even for a Baire topological group G and a reflexive ℓ.c.s. X. Indeed, if $G = \mathbb{R}_+^{\aleph_0}$ is the

$\aleph_0$-power of the multiplicative group of all positive real numbers, $X = \mathbb{R}^{\aleph_0}$ and $\pi\colon G \times X \to X$ is defined coordinatewise, then it is easy to see that π^* is not continuous at any $(0, g)$, where $g \in G$.

Theorem 6.7. *Let X be (NP) and let S be a compact left topological monoid such that the group of all units $H(e)$ is dense in S. For every linear continuous action $\pi\colon S \times X \to X$, the dual action π^* is continuous at every (f, t), where $f \in X^*$ and $t \in H(e)$.*

Proof. Apply Lemma 6.2, taking into account Lemmas 2.7(iii) and 5.4. □

B. Actions on normed spaces.

Theorem 6.8. *Let X be a normed space whose dual X^* has the Radon-Nikodym property, and let S be an arbitrary topologized semigroup. For every continuous linear action $\pi\colon S \times X \to X$, the dual action π^* is continuous at every (f, t), where $f \in X^*$ and $t \in S^{\lhd^r}$ (or even $t \in S^{\lhd^r_w}$, if the action is contracting).*

Proof. By Proposition 2.6(iv), we may assume that X is Banach. Abovementioned characterization from [45] implies that X is Asplund. By Proposition 4.2, X is (NP). By Proposition 2.8, π is locally equicontinuous and, hence, locally bounded by Lemma 2.7(i). Finally, use Lemma 6.2. □

Corollary 6.9. *For an arbitrary topological group G and any continuous linear action of G on an Asplund Banach space, the dual action is also continuous.*

Remark 6.10. (a) For $G = \mathbb{R}$, see van Neerven [36, Corollary 6.2.6].

(b) Christensen and Kenderov proved in [7] that every weak*-continuous mapping F from a Baire space S into X^*, where X is Banach and X^* has the Radon-Nikodym property, is norm-continuous at any point of a certain dense G_δ subset of S. This result, together with Propositions 2.8, 2.6, Lemma 2.7 and easy transport arguments, provide an alternative proof of Corollary 6.9 in the case of a Baire topological group G.

Let $\mathrm{Is}(X)$ be the group of all linear isometries of a normed space X. Denote by p (and p^*) the group topology on $\mathrm{Is}(X)$ generated by the system of all orbit mappings

$$\{\tilde{x}\colon \mathrm{Is}(X) \to X, \;\; \tilde{x}(g) = gx \mid x \in X\}$$
$$\Big(\text{respectively: } \{\tilde{f}\colon \mathrm{Is}(X) \to X^*, \;\; \tilde{f}(g) = fg \mid f \in X^*\}\Big).$$

Theorem 6.8 implies that if X^* has the Radon-Nikodym property, then the dual action of $(\mathrm{Is}(X), p)$ on X^* is continuous. Therefore, $p^* \subseteq p$. This yields:

Corollary 6.11. *If X is a reflexive Banach space, then the topologies p and p^* on $\mathrm{Is}(X)$ coincide.*

C. Construction of minimal topological groups.

Recall that a topological group G is said to be *minimal* [10] if it does not admit a strictly coarser Hausdorff group topology. If $X \times G$ is minimal for every minimal group X, then G is called *perfectly minimal* [10]. The following theorem provides additional information to the results of [29].

Theorem 6.12. *Let X be an Asplund Banach space. Then every topological subgroup of $(\mathrm{Is}(X), p)$ is a topological group retract of a perfectly minimal topological group.*

Proof. By Corollary 6.9, the dual action of $(\mathrm{Is}(X), p)$ on X^* is continuous. Therefore, G is an HBR-group in the sense of [29, Definitions 4.2, 4.7]. Now our assertion follows directly from Theorem 4.8 of [29]. □

D. Actions on compact operators.

Let X and Y be Banach spaces. If X^* or Y has the approximation property, then the injective tensor product $X^* \hat{\otimes}_i Y$ is naturally isomorphic to $K(X, Y)$ [24, p. 268].

Theorem 6.13. *Let X and Y be Banach spaces. Suppose that X is Asplund and that either X^* or Y has the approximation property. Then for every continuous linear action $\pi\colon S \times X \to X$, the induced action $\pi^L\colon L(X, Y) \times S \to L(X, Y)$ is continuous at every (f, t), where f is a compact operator and $t \in S^{\lhd^r}$. In particular, the induced action $\pi^K\colon K(X, Y) \times G \to K(X, Y)$ is jointly continuous for arbitrary topological group $S = G$.*

Proof. By Proposition 2.8, π is locally equicontinuous. Then the same is true for the right action π^L. Since "equicontinuous" implies "uniformly equicontinuous" in linear spaces, Example 2.3(a) shows that $L(X, Y)$ is π^L-uniform. Fix $(f, t) \in K(X, Y) \times S^{\lhd^r}$. By Lemma 2.4(iii), it suffices to show that $\tilde{f}\colon S \to L(X, Y)$ is continuous at t. Since f is compact, $\tilde{f}(S) \subseteq K(X, Y)$. Therefore, we may assume that $\tilde{f}$ has the form $S \to K(X, Y)$. By the above-mentioned fact, we have $K(X, Y) = X^* \hat{\otimes}_i Y$. Moreover, the action of S on $K(X, Y)$ naturally coincides with the action $\pi^* \hat{\otimes}_i \pi_0$, where π_0 is the trivial action of the one-point monoid. By Proposition 4.2 and Lemma 6.2, π^* is continuous at (f, t). Now, Proposition 2.9 completes the proof. □

The compactness of f_0 is essential. For an appropriate counterexample, take $f_0 = 1_X$ and a *non-uniform* continuous semigroup representation.

E. "Small" orbits.

Theorem 6.14. *Let $\pi\colon S \times X \to X$ be a continuous linear action of a Baire topologized semigroup on a normed space X, let Y be an ℓ.t.s., and let $f_0\colon X \to Y$ be a continuous linear operator. Suppose that for every $s \in S$ there exists $U \in N_s$ such that $f_0 U$ is uniformly Lindelöf (e.g., precompact, separable, Lindelöf, ccc). Then the action $\pi^L\colon L(X, Y) \times S \to L(X, Y)$ is continuous at (f_0, t) for every $t \in S^{\lhd^r}$.*

Proof. As in the proof of Theorem 6.13, it suffices to show that $\tilde{f}_0\colon S \to L(X, Y)$ is continuous at every $t \in S^{\lhd^r}$. This can be done by combining Proposition 3.10(i) and Lemma 5.5(b). □

We list here some known results which can easily be obtained by adapting 6.14.

(a) [35, Theorem 3.4] Let $\pi\colon [0, \infty) \times X \to X$ be a continuous linear action of $[0, \infty)$ on a Banach space X, and let $f_0 \in X^*$. Then $\tilde{f}_0\colon [0, \infty) \to X^*$ is continuous at every $t > 0$ iff the orbit of f_0 is (weakly) separable.

(b) (See [46, 11], [25] (for second countable G)) Let G be a locally compact group. If the G-orbit of a measure $m \in M(G) = (C_0(G))^*$ is separable, then the orbit map $\tilde{m}$ is continuous (and, hence, m is absolutely continuous).

(c) [39] If G is a locally compact group and $f_0 \in L^\infty(G) = (L^1(G))^*$ has a separable orbit, then the orbit map $\tilde{f_0}$ is continuous.

Recall that a Banach space X is said to have the *point of continuity property* (abbr.: (PC)) if each bounded closed set C admits a point of continuity of the identity map $(C, weak) \to (C, norm)$. This property can be characterized in terms of fragmentability. By [13, 3.13] X has the property (PC) iff each weakly closed bounded subset of X is *huskable* (in our terminology: locally fragmented). Another way to say this is: every bounded subset of X is fragmented [21, p. 665]. Banach spaces with the Radon-Nikodym property or with (weakly) Čech-complete ball have the property (PC) [13, 3.14].

Sometimes the weak continuity of an orbit mapping implies its continuity. Several results of this kind can be found in [12, 18, 26, 30].

Theorem 6.15. *Let $\pi\colon S \times X \to X$ be a linear action on an ℓ.c.s. (X, τ) such that π is continuous at $(p, 0)$ for every $p \in S$. Suppose that $s \lhd^\ell t$, $x_0 \in X$ and $\tilde{x}_0\colon S \to X$ is weakly continuous. Then $\tilde{x}_0$ is τ-continuous at t under each of the following conditions:*

(i) X is a Banach space with the property (PC);

(ii) There exists a separable Baire nbd U of s.

If the action is equicontinuous, then in both cases we may assume that $s \lhd^\ell_w t$.

Proof. Proposition 2.6(i) shows that X is π-uniform.

(i) By proposition 2.8, Ux_0 is norm-bounded for a certain *nbd* U of s. Abovementioned characterization of (PC) implies that Ux_0 is fragmented. Then by Lemma 3.2(b) the map $\tilde{x}_0$ is locally fragmented at s. For the rest use Lemma 5.5(a).

(ii) Since $\tilde{x}_0$ is weakly continuous, Ux_0 is weakly separable, and, hence, even τ-subseparable (a subset of a separable set) because τ is locally convex. Therefore, Ux_0 is uniformly Lindelöf. By Lemma 3.4(b) and Proposition 3.10(i), S is locally fragmented by $\tilde{x}_0$ at s. Now Lemma 5.5(a) completes the proof.

If the action is equicontinuous then use Lemma 5.5(c). □

Corollary 6.16. *Let S be a Baire locally separable semitopological group. Then $RUC(S) = WRUC(S)$ (for definitions see, for example, [12]).*

F. Actions on quasi-Montel spaces.

Theorem 6.17. *Let X be a quasi-Montel space, P a topological space and $\pi\colon P \times X \to X$ a continuous right linear function.*

(i) *For every normed space Y, the induced map $\pi^L\colon L(X,Y) \times P \to L(X,Y)$ is separately continuous.*

(ii) *If P is locally compact, then π^L is jointly continuous for every ℓ.t.s. Y.*

Proof. (i) For each $f_0 \in L(X,Y)$, use Lemma 6.1(i) (with $F = \{f_0\}$) and Proposition 4.4.

(ii) Instead of Lemma 6.1(i), use Lemmas 6.1(ii), 2.7 and 2.4(iii). □

G. Sublinear functionals defined over orbits.

Theorem 6.18. *Let $\pi\colon S\times X\to X$ be a continuous linear action of a topologized semigroup S on a normed space X. Suppose that $f_0\in X^*$, $s_0\in S$, $U\in N_{s_0}$, and for every non-empty relatively w^*-closed subset A of f_0U, the sublinear functional σ_A, defined by the rule $\sigma_A(x)=\sup\{f_0(sx)\mid f_0s\in A\}$, has a point of Fréchet differentiability. Then π^* is continuous at (f_0,t) for every t satisfying $s_0\lhd^r t$.*

Proof. Let $s_0\lhd^r t$. Our aim is to show that all conditions of Lemma 5.5(b) are satisfied. First, choose, by Lemma 6.1, *nbd* V of s_0 such that $V\subseteq U$ and f_0V is equicontinuous (equivalently, bounded). Proposition 3.14 states that f_0V is fragmented. By Lemma 3.2(b) and the weak*-continuity of $\tilde{f}_0$, we obtain that V is fragmented by $\tilde{f}_0$. In particular, S is locally fragmented at s_0 by $\tilde{f}_0$. By Proposition 2.8 and Lemma 2.7(ii), X^* is π^*-uniform. Therefore, we can use Lemma 5.5(b), which completes the proof. □

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