A DYNAMICAL PROOF OF VAN DER WAERDEN'S THEOREM, SECOND APPROACH

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Abstract. In this lecture, the second approach of Van Der Waerden's theorem will be presented. It is based on Birkhoff's theorem which will also be proved. The essence of Van Der Waerden's theorem is that if the set of natural numbers $\mathbb{N}$ is partitioned in some way into finitely many classes in any way whatever, then one of these classes necessarily contains arbitrarily long arithmetic progressions.

1. Introduction

This result was originally conjectured by Baudet and proved by Van der Waerden in 1927 [1, 2]. The theorem gained a wider audience when it was included in Khintchine's famous book *Three pearls in number theory* [4]. The dynamical proof is due to Furstenberg and Weiss [3] (from 1978).

2. Preliminaries

We'll start with introducing several definitions and theorems, which will be used later.

Definition 2.1. A homeomorphism $T : X \to X$ is minimal if for every $x \in X$ the orbit $\{T^n x : n \in Z\}$ is dense in X.

Now we introduce the theorem which gives equivalent definitions.

Theorem 2.2. *(without proof) Let $T : X \to X$ be a homeomorphism of a compact metric space. The following properties are equivalent:*
(i) T is minimal.
(ii) If $TE = E$ is a closed T-invariant set, then either $E = \emptyset$ or $E = X$.
(iii) If $U \neq \emptyset$ is an open set then $X = \bigcup_{n\in Z} T^n U$.

Using property (ii) we get the following result that every compact dynamical system contains a minimal subsystem.

Theorem 2.3. *Let $T : X \to X$ be a homeomorphism of a compact metric space X. There exists a non-empty closed set $Y \subset X$ with $TY = Y$ and $T : Y \to Y$ is minimal.*

Proof. This follows from an application of Zorn's Lemma. Let $\mathcal{E}$ denote the family of all closed T-invariant subsets of X with the partial ordering by inclusion, i.t $Z_1 \leq Z_2$ iff $Z_1 \subset Z_2$.

Every totally ordered subset $\{Z_\alpha\}$ has a least element $Z = \bigcap_\alpha Z_\alpha$ (which is non-empty by compactness of X). Thus by Zorn's Lemma there exists a minimal element

$Y \subset X$ (i.e. $Y \in \mathcal{E}$ and $Y' \in \mathcal{E}$ with $Y' \leq Y$ implies that $Y' = Y$). By property (ii) of Theorem 2.2 this can be reinterpreted as saying that $T : Y \to Y$ is minimal. □

As a corollary we get the following simple but elegant result.

Corollary 2.4. *(Birkhoff recurrence theorem). Let $T : X \to X$ be a homeomorphism of a compact metric space X. We can find $x \in X$ such that $T^{n_i}x \to x$ for a subsequence of the integers $n_i \to +\infty$.*

Proof. By Theorem 2.3 we can choose a T-invariant subset $Y \subset X$ such that $T : Y \to Y$ is minimal. For any $x \in Y \subset X$ we have the required property. □

Now we can present the following interesting example:

Example 2.5. Consider the case $X = R/Z$ and $T : X \to X$ defined by $Tx = x + \alpha$ (mod 1), where α is an irrational number.

Let $\epsilon > 0$; then we can find $n > 0$ (by Birkhoff's theorem) such that $|\alpha n$ (mod 1)$| \leq \epsilon$ (since we know that we can find $x \in X$, such that $T^n x = x + \alpha n$ (mod 1) $\to x$), i.e. there exists $p \in N$ such that $-\epsilon \leq \alpha n - p \leq \epsilon$. Rewriting this, we have that for any irrational $\alpha, \exists p, n \in N$ such that $|\alpha - \frac{p}{n}| \leq \frac{\epsilon}{n}$. This means that we can find a rational number which differs from irrational α for no more then ϵ/n. This is an important result in Numbers Theory, and what's important, that there is no fixed starting point, we can start from any point we choose.

Let $T_1, ..., T_N : X \to X$ be commuting homeomorphisms on a compact metric space X, i.e. $T_i T_j = T_j T_i$ for $1 \leq i, j \leq N$. We can consider all closed *simultaneously invariant* sets $A \subset X$, i.e. $T_i A = A$, $i = 1, ..., N$. By a similar argument to that before, we can consider the partial order by inclusion on all such closed sets and by applying Zorn's Lemma (just as in the proof of Theorem 2.3) we can deduce that there exists a closed set $X_0 \subset X$ such that

(i) $T_i X_0 = X_0$, $i = 1, ..., N$.

(ii) whenever $A \subset X_0$ with A closed and $T_i A = A$ for $i = 1, ..., N$ then necessarily $A = X_0$.

Now we can prove the following useful lemma:

Lemma 2.6. *For each open set $U \subset X_0$ we can choose a finite number M and $n_{ij} \in Z$ with $1 \leq i \leq N$, $1 \leq j \leq M$ with $X_0 = \bigcup_{j=1}^{M} (T_1^{n_{1j}} \circ ... \circ T_N^{n_{Nj}})U$.*

Proof. Clearly $X_0 = \bigcup_{n_1 \in Z} ... \bigcup_{n_N \in Z} (T_1^{n_1} \circ ... \circ T_N^{n_N})U$ (since otherwise the difference $X_0 - (\bigcup_{n_1 \in Z} ... \bigcup_{n_N \in Z} (T_1^{n_1} \circ ... \circ T_N^{n_N})U)$ is a closed non-empty set invariant under $T_1, ..., T_N$, contradicting property (ii) above). Now by compactness we can choose a *finite* subcover. This completes the proof. □

3. Van der Waerden Theorem

Now, we'll formulate Van Der Waerden theorem.

Theorem 3.1. *Consider a finite partition $Z = B_1 \bigcup ... \bigcup B_k$. At least one element B_r in the partition will contain arithmetic progressions of arbitrary length (i.e $\exists 1 \leq r \leq k, \forall N > 0, \exists a, b \in Z (b \neq 0)$ such that $a + jb \in B_r$ for j = 0,...,N - 1).*

Since an arithmetic progression of length N contains arithmetic progressions of all shorter lengths, this is equivalent to: $\exists N_i \to +\infty, \exists a_i, b_i \in Z$ such that $a_i + jb_i \in B_r$ for $j = 0, ..., N_i - 1$.

We give below some simple examples.

Example 3.2. If the sets $B_2, ..., B_k$ in the partition are finite then it is easy to see that B_1 is the element with arithmetic progressions of arbitrary length.

Example 3.3. If $Z = B_1 \bigcup B_2$ where $B_1 = \{\text{odd numbers}\}$ and $B_2 = \{\text{even numbers}\}$ then both contain arithmetic progressions of arbitrary length.

Example 3.4. If $B_1 =$ prime numbers and $B_2 = \{\text{non-prime numbers}\}$ then B_2 contains arithmetic progressions of arbitrary length. However, the problem as to whether B_1 contains arithmetic progressions of arbitrary length wasn't resolved until recently, and, indeed, such progression can be found.

The key to proving Van der Waerden's theorem is the following generalization of Birkhoff's theorem.

Theorem 3.5. *Let $T_1, ..., T_N : X \to X$ be homeomorphisms of a compact metric space such that $T_iT_j = T_jT_i$ for $1 \leq i, j \leq N$. There exists $x \in X$ and $n_j \to +\infty$ such that $d(T_i^{n_j}x, x) \to 0$ for each $i = 1, ..., N$.*

We shall first prove Theorem 3.1 assuming Theorem 3.5 and then return to the proof of Theorem 3.5.

Proof. PROOF OF THEOREM 3.1. We want to begin by associating to the partition $Z = B_1 \bigcup ... \bigcup B_k$ a suitable homeomorphism $T : X \to X$ (and then we set $T_j = T^j, j = 1, ..., N$).

Let $\Omega = \prod_{n \in Z}\{1, ..., k\}$ and then we can associate to the partition $Z = B_1 \bigcup ... \bigcup B_k$ a sequence $z = (z_n)_{n \in Z} \in \Omega$ by $z_n = i$ if and only if $n \in B_i$.

Let $\sigma : \Omega \to \Omega$ be the shift operator (i.e. $(\sigma x)_n = x_{n+1}, n \in Z$). Consider the orbit $\{\sigma^n z : n \in Z\}$ and its closure $X = cl(\bigcup_{n \in Z} \sigma^n z)$ in Ω. Finally, we define $T_i := T^i = \sigma \circ ... \circ \sigma$ (T composed with itself i times).

By Theorem 3.5 (with $\epsilon = \frac{1}{4}$) we can find $x \in X$ and $b \geq 1$ with

$$d(T_1^b x, x) < \frac{1}{4}, d(T_2^b x, x) < \frac{1}{4}, ..., d(T_N^b x, x) < \frac{1}{4}.$$

Since $X = cl(\bigcup_{n \in Z} \sigma^n z)$ we can choose $a \in Z$ such that

$$d(x, T^a z) < \frac{1}{4}, d(T_1^b x, T^a T_1^b z) < \frac{1}{4}, ..., d(T_N^b x, T^a T_N^b z) < \frac{1}{4}.$$

Thus, for each $i = 1, ..., N$ we have that

$$d(T^a T_i^b x, T^a z) \leq d(T^a T_i^b x, T_i^b x) + d(T_i^b x, x) + d(x, T^a z) < \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

Since $d(x, y) = (\frac{1}{2})^{N(x,y)}$ (where $N(x, y) = min\{|N| \geq 0 : x_N \neq y_N$, or $x_{-N} \neq y_{-N}\}$) we see that $(T^a T_i^b x)_0 = x_{a+ib} = z_a \in \{1, ..., k\}$ for $i = 1, ..., N$ (we started from point 0 to get to z_a). This means that $a + ib \in B_{z_a}$ for $i = 1, ..., N$, and completes the proof of Theorem 3.1.

$\square$

4. Birkhoff Theorem

All that remains is to prove Theorem 3.5. This is a fairly detailed proof and to help clarify matters we shall divide it into sublemmas.

Proof of Theorem 3.5 We shall use a proof by induction.

Case $N = 1$. For $N = 1$ the multiple Birkhoff recurrence theorem reduces to the usual Birkhoff recurrence theorem (Corollary 2.4).

Inductive step. Assume that the result is known for $N - 1$ commuting homeomorphisms. We need to show that it holds for N commuting homeomorphisms.

Simplifying fact. We can assume that X is the *minimal* closed set invariant under each of $T_1, ..., T_N$. If this is not the case we can restrict to such a set (using Zorn's lemma).

In order to establish the Birkhoff multiple recurrence theorem for these N commuting homeomorphisms, the following simple alternative formulation of this result is useful.

Alternative formulation. Let $\mathcal{X}_N = X \times ... \times X$ be the N-fold cartesian product of X and let $\mathcal{D}_N = \{(x, ..., x) \in \mathcal{X}_N\}$ be the diagonal of the space. Let $\mathcal{S} : \mathcal{X}_N \to \mathcal{X}_N$ be given by $\mathcal{S}(x_1, ..., x_N) = (T_1x_1, ..., T_Nx_N)$. Then the following are equivalent:

$(i)_N$ the Birkhoff multiple recurrence holds for $T_1, ..., T_N$;

$(ii)_N$ $\exists \underline{z} = (z, ..., z) \in \mathcal{D}_N$ such that $d_{\mathcal{X}_N}(S^{n_i}\underline{z}, \underline{z}) \to 0$ as $n_i \to +\infty$ (where $d_{\mathcal{X}_N}(\underline{z}, \underline{w}) = sup_{1\leq i\leq N} d(z_i, x_i)$).

We can apply the inductive hypothesis to the $(N-1)$ commuting homeomorphisms $T_1T_N^{-1}, ..., T_{N-1}T_N^{-1}$ and using the equivalence of $(i)_{N-1}$ and $(ii)_{N-1}$ above we have that for the map $R := T_1T_N^{-1} \times ... \times T_{N-1}T_N^{-1} : \mathcal{X}_{N-1} \to \mathcal{X}_{N-1}$ defined by

$$R : (x_1, ..., x_{N-1}) \mapsto (T_1T_N^{-1}x_1, ..., T_{N-1}T_N^{-1}x_{N-1})$$

there exists $\underline{z} = (z, ..., z) \in \mathcal{D}_{N-1} \subset \mathcal{X}_{N-1}$ with $d_{\mathcal{X}_{N-1}}(R^{n_i}\underline{z}, \underline{z}) \to 0$ as $n_i \to +\infty$. In particular, $d_{\mathcal{X}_N}(S^{n_i}\underline{z}', \underline{z}) \to 0$ as $n_i \to +\infty$ where $\underline{z} = (z, ..., z), \underline{z}' = (T_N^{-n_i}z, ..., T_N^{-n_i}z) \in \mathcal{D}_N$.

Thus we have proved the following result.

Lemma 4.1. $\forall \epsilon > 0, \exists z, z' \in \mathcal{D}_N, \exists n \geq 1$ *such that* $d_{\mathcal{X}_N}(S^n z, z') < \epsilon$.

Unfortunately, this is not quite in the form of $(ii)_N$ we need for the inductive step. (For example, we would like to take $z = z'$). To get a stronger result, we break the argument up into steps represented by the following lemmas.

Lemma 4.2. *$\forall \epsilon > 0, \forall x \in \mathcal{D}_N, \exists y \in \mathcal{D}_N$ and $\exists n \geq 1$ such that $d(S^n y, x) < \epsilon$.*

(This changes one of the quantifiers $\exists$ to $\forall$).

Lemma 4.3. *$\forall \epsilon > 0, \exists z \in \mathcal{D}_N$ and $\exists n \geq 1$ such that $d(S^n z, z) < \epsilon$.*

(This is almost the Theorem 3.5, except that z might still depend on the choice of $\epsilon > 0$).

We will now complete the proof of the Theorem 3.5 assuming Lemma 4.3 (we shall then return to the proofs "Lemma 4.1 $\Rightarrow$ Lemma 4.2" and "Lemma 4.2 $\Rightarrow$ Lemma 4.3" in the next section).

Consider the function $F : \mathcal{D}_N \to R^+ = [0, +\infty)$ defined by $F(x) = inf_{n \geq 1} d(S^n x, x)$. It is easy to see that to complete the proof of Theorem 3.5 we need only to show there exists a point $x_0 \in \mathcal{D}_N$ with $F(x_0) = 0$. To show this fact, the following properties of F are needed.

Lemma 4.4. *The following properties of F should be proved:*
(i) $F : \mathcal{D}_N \to R^+$ is upper semi-continuous (i.e. $\forall x \in \mathcal{D}_N, \forall \epsilon > 0, \exists \delta > 0$ such that $d(x, y) < \delta \Rightarrow F(y) \leq F(x) + \epsilon$).
(ii) $\exists x_0 \in \mathcal{D}_N$ such that $F : \mathcal{D}_N \to R^+$ is continuous at x_0.

Proof. (i) This result can be easily obtained from the definition of F.
(ii) For $\epsilon > 0$ we can define $A_\epsilon = \{x \in \mathcal{D}_N : \forall \eta > 0, \exists y$ such that $d(y, x) < \eta$ and $F(y) \leq F(x) - \epsilon\}$ (i.e $\exists$ point y arbitrarily close to x with $F(y) \leq F(x) - \epsilon$). Notice that
(a) A_ϵ is closed,
(b) A_ϵ has empty interior.

(to see part (b) observe that if $int(A_\epsilon) \neq \emptyset$ we could choose a sequence of pairs $x, x_1 \in int(A_\epsilon)$ with $F(x_1) \leq F(x) - \epsilon, x_1, x_2 \in int(A_\epsilon)$ with $F(x_2) \leq F(x_1) - \epsilon$, etc. Together these inequalities give $F(x_n) \leq F(x) - n\epsilon < 0$ for n arbitrarily large. But this contradicts $F \geq 0$).

The set of points at which F is continuous is

$$\{x \in \mathcal{D}_N : x \notin A_\epsilon, \epsilon > 0\} = \bigcap_{n=1}^{\infty} (\mathcal{D}_N - A_{\frac{1}{n}}).$$

Since this is a countable intersection of open dense sets, it is still dense (by Baire's theorem). Thus there exists at least one point of continuity for $F : \mathcal{D}_N \to R^+$ (in fact, infinitely many). This completes the proof of Lemma 4.4.

$\square$

Let x_0 be such a point of continuity.

Assume for a contradiction that $F(x_0) > 0$. We can the choose $\delta > 0$ and an open neighborhood $U \ni x_0$ such that $F(x) > \delta > 0$ for $x \in U$. However, we also know that for the diagonal actions $T_i : (x_1, ..., x_N) \mapsto (T_i x_1, ..., T_i x_N)$

$$\mathcal{D}_N \subset \bigcup_{j=1}^{M} (T_N^{n_{1j}} \circ ... \circ T_N^{n_{Nj}})^{-1} U$$

(since by the simplifying assumption X is the minimal closed set invariant under $T_1, ..., T_N$ and so we may apply Lemma 2.6).

By (uniform) continuity of the family $\{T_1^{n_{1j}} \circ ... \circ T_N^{n_{Nj}}\}_{j=1}^{M}$ there exists $\eta > 0$ such that

$$d(x, y) < \eta \Rightarrow d(T_1^{n_{1j}} \circ ... \circ T_N^{n_{Nj}} x, T_1^{n_{1j}} \circ ... \circ T_N^{n_{Nj}} y) < \delta \qquad (4.1)$$

(for $1 \leq j \leq M$). Observe that for $y \in (T_1^{n_{1j}} \circ ... \circ T_N^{n_{Nj}})^{-1} U$ $(j = 1, ..., M)$ we have that $F(y) \geq \eta$. If this were not the case then there would exist $n \geq 1$ with $d(y, S^n y) < \eta$, from the definition of F. This item implies that $d(T_1^{n_{1j}} \circ ... \circ T_N^{n_{Nj}} y, T_1^{n_{1j}} \circ ... \circ T_N^{n_{Nj}} S^n y) < \delta$ by (4.1). Choosing $x := T_1^{n_{1j}} \circ ... \circ T_N^{n_{Nj}} y \in U$ gives $F(x) = inf_{n \geq 1} d(x, S^n x) < \delta$ which contradicts our hypothesis.

Finally we see that by (4.1) we have $F(y) \geq \eta$ for all $y \in \mathcal{D}_N$. However, this contradicts Lemma 4.3 and we conclude that $F(x_0) = 0$.

The proof of Theorem 3.5 is finished (given the proofs of Lemma 4.2 and Lemma 4.3).

5. The proofs of Lemma 4.2 and Lemma 4.3

We now supply the missing proofs of Lemma 4.2 and Lemma 4.3.

Proof. Proof of Lemma 4.2 (assuming Lemma 4.1). Consider the N commuting maps $\hat{T}_1, \hat{T}_2, ..., \hat{T}_N : \mathcal{D}_N \to \mathcal{D}_N$ defined by

$$\begin{cases} \hat{T}_1 = T_1 \times ... \times T_1 : \mathcal{D}_N \to \mathcal{D}_N, \\ \hat{T}_2 = T_2 \times ... \times T_2 : \mathcal{D}_N \to \mathcal{D}_N, \\ \qquad ... \\ \hat{T}_N = T_N \times ... \times T_N : \mathcal{D}_N \to \mathcal{D}_N. \end{cases}$$

We want to apply Lemma 2.6 to these commuting maps with the choice of open set $U = \{w \in \mathcal{D}_N : d_{\mathcal{D}_N}(x, w) < \frac{\epsilon}{2}\}$. This allows us to conclude that there exist $n_{1j}, ..., n_{Nj}$ $(j = 1, ..., M)$ such that

$$\mathcal{D}_N = \bigcup_{j=1}^{M} \hat{T}^{-n_{1j}} ... \hat{T}^{-n_{Nj}} U.$$

Thus for any $z \in \mathcal{D}_N$ we have some $1 \le j \le M$ such that

$$d_{\mathcal{D}_N}(\hat{T}^{n_{1j}} ... \hat{T}^{n_{Nj}} z, x) < \epsilon/2. \qquad (5.1)$$

Next we can use (uniform) continuity of $\hat{T}^{n_{1j}} \circ ... \circ \hat{T}^{n_{Nj}}$ to say that there exists $\delta > 0$ such that whenever $d_{\mathcal{D}_N}(z, z') < \delta$ for $z, z' \in \mathcal{D}_N$ then we have that

$$d_{\mathcal{D}_N}(\hat{T}^{n_{1j}} \circ ... \circ \hat{T}^{n_{Nj}} z, \hat{T}^{n_{1j}} \circ ... \circ \hat{T}^{n_{Nj}} z') < \frac{\epsilon}{4}. \qquad (5.2)$$

By Lemma 4.1 $\exists z, z' \in \mathcal{D}_N$ and $\exists n \ge 1$ such that $d_{\mathcal{D}_N}(S^n z, z') < \delta$. Therefore by inequality (5.2) we have that

$$d_{\mathcal{D}_N}(S^n(\hat{T}^{n_{1j}} \circ ... \circ \hat{T}^{n_{Nj}} z), \hat{T}^{n_{1j}} \circ ... \circ \hat{T}^{n_{Nj}} z') < \frac{\epsilon}{4}. \qquad (5.3)$$

Writing $y = \hat{T}^{n_{1j}} \circ ... \circ \hat{T}^{n_{Nj}} z$ and comparing (5.1), (5.2) and (5.3) gives that

$$d_{\mathcal{D}_N}(S^n y, x) \le d_{\mathcal{D}_N}(S^n y, \hat{T}^{n_{1j}} \circ ... \circ \hat{T}^{n_{Nj}} z') + d_{\mathcal{D}_N}(\hat{T}^{n_{1j}} ... \hat{T}^{n_{Nj}} z', x) \le$$

$$\le d_{\mathcal{D}_N}(S^n y, \hat{T}^{n_{1j}} \circ ... \circ \hat{T}^{n_{Nj}} z') +$$

$$+ d_{\mathcal{D}_N}(\hat{T}^{n_{1j}} \circ ... \circ \hat{T}^{n_{Nj}} z, \hat{T}^{n_{1j}} \circ ... \circ \hat{T}^{n_{Nj}} z') + d_{\mathcal{D}_N}(\hat{T}^{n_{1j}} ... \hat{T}^{n_{Nj}} z, x) <$$

$$< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.$$

This completes the proof of Lemma 4.2.

□

Proof. Proof of Lemma 4.3 (assuming Lemma 4.2). Fix $z_0 \in \mathcal{D}_N$ and let $\epsilon_1 = \frac{\epsilon}{2}$. By Lemma 4.2 we can choose $n_1 \ge 1$ and $z_1 \in \mathcal{D}_N$ with $d(T^{n_1} z_1, z_0) < \epsilon_1$.

By continuity of T^{n_1} we can find $\epsilon_1 > \epsilon_2 > 0$ such that $d(z, z_1) < \epsilon_2$ implies that $d(T^{n_1} z, z_0) < \epsilon_1$.

We can now continue inductively (for $k \ge 2$):

(a) By Lemma 4.3 we can choose $n_k \ge 1$ and $z_k \in \mathcal{D}_N$ with $d(T^{n_k} z_k, z_{k-1}) < \epsilon_k$.

(b) By continuity of T^{n_k} we can find $\epsilon_k > \epsilon_{k+1} > 0$ such that $d(z, z_k) < \epsilon_{k+1}$ implies that $d(T^{n_k} z, z_{k-1}) < \epsilon_k$.

This results in sequences $z_0, z_1, z_2, ... \in \mathcal{D}_N, n_0, n_1, n_2, ... \in N, \epsilon_0 > \epsilon_1 > \epsilon_2 > ...$ such that $d(T^{n_k} z_k, z_{k-1}) < \epsilon_k, k \ge 1$ and $d(z, z_i) < \epsilon_{k+1} \Rightarrow d(T^{n_k} z, z_{k-1}) < \epsilon_k$.

In particular we get that whenever $j < i$ then

$$d(T^{n_i+n_{i-1}+\ldots+n_{j+2}+n_{j+1}} z_i, z_j) < \epsilon_{i+1} \leq \frac{\epsilon}{2}.$$

By compactness of $\mathcal{D}_N$ we can find $d(z_i, z_j) < \frac{\epsilon}{2}$ for some $j < i$.
By the triangle inequality we have that for $N = n_i + n_{i-1} + \ldots + n_{j+1}$

$$d(T^N z_i, z_i) \leq d(T^N z_i, z_j) + d(z_i, z_j) < \epsilon.$$

Thus the choice $z = z_i$ completes the proof of Lemma 4.3.

□

6. Extensions of Van der Waerden Theorem

In this section, we'll introduce several theorems which extend Van der Waerden Theorem (without proof).

Theorem 6.1. *(I. Schur, A. Brauer) For any finite partition $N = C_1 \cup C_2 \cup \ldots \cup C_r$ there is a C_j such that for any $l = 1, 2, 3, \ldots$ there is a number $d \in C_j$ and a number e such that the arithmetic progression $e + id$, $0 \leq i \leq l$, is contained in C_j.*

The definitive result in this direction was obtained by R. Rado who characterized the systems of equations which could be solved in one of the classes of an arbitrary finite partition. Rado defines a *regular system of equations*

$$\sum_{j=1}^{L} a_{ij} x_j = 0, \qquad 1 \leq i \leq I, \quad a_{ij} \in Q = rationals \quad (6.1)$$

as a system which has a "monochromatic" solution for any finite "coloring" of the integers.

Theorem 6.2. *(R. Rado) The system (6.1) is regular if and only if there is a partition $\{1, 2, \ldots, L\} = J_0 \cup J_1 \cup \ldots \cup J_k$ such that*

$$\sum_{j \in J_0} a_{ij} = 0, \qquad 1 \leq i \leq I;$$

$$\sum_{j \in J_1} a_{ij} = \sum_{j \in J_0} c_j^1 a_{ij}, \qquad 1 \leq i \leq I, \quad c_j^1 \in Q;$$

$$\sum_{j \in J_l} a_{ij} = \sum_{m=0}^{l-1} \sum_{j \in J_m} c_j^l a_{ij}, \qquad 1 \leq i \leq I, \quad c_j^l \in Q.$$

Also, Van der Waerden's theorem was extended to higher dimensional configurations by S. Grünwald.

Theorem 6.3. *(Grünwald) For any finite partition $N^m = C_1 \cup C_2 \cup ... \cup C_r$ and any $k = 1, 2, 3, ...,$ there is some C_j, some $d \in N$ and some $b \in N^m$ so that*

$$b + d(x_1, x_2, ..., x_m) \in C_j \qquad 1 \leq x_i \leq k, \quad 1 \leq i \leq m.$$

This result implies that for any finite configuration $S \subset N^m$, some C_j contains a configuration similar to S.

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