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COMPACTIFICATIONS OF SEMIGROUP ACTIONS

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Lecture in Seminar "Topology of actions and representations"

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Abstract. In this lecture I'll explain about basic ideas about semigroup action of a topological semigroup S on topological space X and equicompactifications of S-spaces. We give a transparent description of such compactifications (a generalized Smirnov's theorem) in terms of S-proximities, special action compatible proximities on X by analytical way (spacial subalgebras of algebra of all bounded continuous functions $C(X)$). Also we will give some "beautiful" examples of equicompactifications in case of topological groups.

1. Introduction

A *topological transformation group* (*G-space*, or a *G-flow*) is a continuous action of a topological group G on a topological space X. This is a one of the most basic mathematical objects. The study of G-spaces in their own right began systematically in the late 1940s, when W.H. Gottshalk and G.A Hendlund [7] generalized several classical dynamical results from the theory of differential equations and other branches of mathematics. The investigation of the behavior of G-spaces became known as *topological dynamics* or *abstract dynamics*. In the 1960's, this field grew rapidly under the influence of R.Ellis, H. Furstenberg, J. Auslander, and others. Topological dynamics lies at the junction of several other branches of mathematics, including classical dynamics, topology and functional analysis. Thus, results in topological dynamics often have far-reaching consequences.

The next stage in development of topological dynamics was the study of continuous action of a topological semigroup S. This structure is a natural generalization of G-spaces and is known as *topological transformation semigroups* (*S-spaces*, or *S-flows*). They play a major role in several mathematical investigations. For instance in combinatorial number theory (see for example, [20]).

We intend to study some concepts and ideas in topological dynamics which are well known and important in topology and abstract analysis. One of the main objects of our research is the dynamical analogue of the compactification concept. Compactifiability of topological spaces means the existence of topological embeddings into compact spaces. For the compactifiability of flows we require in addition the continuous extendability of the original action. As is known [5, Chapter 8], compactifiable topological spaces are exactly spaces with uniform topology (uniform spaces). Therefore, the dynamical generalization of this problem, must be related with the theory of uniform structures with respect to some action. This area of abstract dynamic is known as the *equivariant topology.* Compactifiable G-spaces are known also as *G-Tychonoff spaces* or spaces with a *proper G-compactification.*

From topological space theory, we know, that the set of all compactifications of a Tychonoff space X can be described in several ways. By Banach subalgebras of algebra $C(X)$ (Gelfand-Naimark 1-1 correspondence, see [21]). By totally bounded (precompact) uniformities on X (see [5, p.563]), or proximities on X (Smirnov's Theorem see in [5, p.563]). The first two correspondences admit dynamical generalizations in the category of G-flows (see for example J. de Vries [22]). The correspondence by proximities for G-spaces was announced without proof by Yu.M. Smirnov in [1].

2. Topological Background

First of all we remind some concepts of topology.

2.1. **Compactifications of topological spaces.**

Now we recall some auxiliary basic concepts from general topology concerning compact extensions of topological spaces. For more detailed description see R. Engelking [5, p.165-280].

Definition 2.1. Let X be a topological space. A pair (Y, c), where Y is compact and $c : X \to Y$ is a homeomorphic embedding of the space X into Y such that $\overline{c(X)} = Y$, is called *proper compactification* (or simply, *compactification*) of the space X.

It is well known that a topological space X has a compactification if and only if X is Tychonoff ($T_{3\frac{1}{2}}$) space. We denote by $\mathscr{C}(X)$ the family of all different compactifications of the given Tychonoff space X. Often we shall identify the space X with the subspace $c(X)$ (homeomorphic to X) of the compactification cX of X.

A partial order can be defined on the family $\mathscr{C}(X)$.

Definition 2.2. We say that c_1X *dominates* c_2X (and write $c_2X \leqslant c_1X$) if and only if there exists a continuous mapping $f : c_1X \to c_2X$ such that $f \circ c_1 = c_2$.

The relation between compactifications c_1 and c_2 of X may be illustrated by the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "c_1"] \arrow[dr, "c_2"'] & c_1X \arrow[d, "f"] \\ & c_2X \end{tikzcd}$$

If two compactifications c_1 and c_2 satisfy $c_1 \leqslant c_2$ and $c_2 \leqslant c_1$, that we say that they are *equivalent* compactifications. Notation: $c_1 \sim c_2$.

Usually Tychonoff spaces have many different compactifications, but there always exists *a maximal compactification* in the set $\mathscr{C}(X)$. This compactification is called the *Stone-Ĉech compactification* of X and is denoted by βX.

2.2. **Proximities and Proximity spaces. Smirnov's Theorem.** *Proximity* on X, intuitively, is a relationship between subsets of X. As a axiomatic theory, it generalizes some geometrical concepts, such as 'near' or 'separation of sets'. In 1908 this structure was introduced by F. Riesz and V.A. Efremovich (see [2, 17, Chapter 1.5]). This construction is a natural generalization of theory of metric spaces and of topological groups. The main interest for us will be the connection between proximities and compactifications (see [5]).

Definition 2.3. Let X be a nonempty set and δ be a relation in the set of all its subsets. We'll write $A\delta B$ if the sets A and B are δ-related and $A\overline{\delta}B$ if they not. The relation δ will be called *proximity* in the set X provided that the following conditions are satisfied:

(P1) $A\delta B$ if and only if $B\delta A$;
(P2) $A\delta(B\cup C)$ if and only if $A\delta B$ or $A\delta C$;
(P3) For every $x, y \in X$ the condition $\{x\}\delta\{y\}$ is equivalent to the condition $x = y$;
(P4) $\varnothing\overline{\delta}X$;
(P5) If $A\overline{\delta}B$, then there exist $C, D \subset X$ such that $C \cup D = X$ and $A\overline{\delta}C$, $B\overline{\delta}D$.

A pair (X, δ) is called a *proximity space*. Two sets $A, B \subset X$ are said to be *close* in the proximity space (X, δ) if $A\delta B$ and *far* if $A\overline{\delta}B$.

We say a set A is *strongly contained* in a set B with respect to δ, if $A\overline{\delta}(X \setminus B)$ and denote by $A \Subset B$. A finite cover $\mathscr{A} = \{A_i\}_{i=1}^n$ is called δ-*cover* if and only if there exist another finite cover $\mathscr{B} = \{B_i\}_{i=1}^n$, such that $B_i \Subset A_i$ for every $i = 1, \dots, n$.

Every proximity space (X, δ) defines a topology $\tau := top(\delta)$, by closure operator:

$$\mathrm{cl}_\delta[A] := \{x : x\delta A\}, \forall A \subset X.$$

Definition 2.4. Let (X, δ_1) and (Y, δ_2) be two proximity space; A mapping $f : X \to Y$ is called *proximity continuous with respect to the proximities δ_1 and δ_2* if for any sets $A, B \subset X$, $A\delta_1 B$, the images: $f(A)\delta_2 f(B)$. Or equivalently, f is *proximity continuous* if and only if :

$$C\overline{\delta}_2 D \Longrightarrow f^{-1}(C)\overline{\delta}_1 f^{-1}(D)$$

or

$$C \Subset_2 D \Longrightarrow f^{-1}(C) \Subset_1 f^{-1}(D).$$

Theorem 2.5. *In a proximity space (X, δ), $A\delta B$ implies that there exist a proximity continuous mapping f, such that $f(A) = 0$ and $f(B) = 1$.*

Theorem 2.6. *Every proximity space (X, δ) is Tychonoff with respect to topology $top(\delta)$.*

First we briefly recall some classical auxiliary facts about compactifications. Due to the Gelfand-Naimark theory there is the 1-1 correspondence (up to the equivalence classes of compactifications) between Banach *unital* (that is, the containing the constants) subalgebras $\mathcal{A} \subset C^*(X)$ and the compactifications $\nu : X \to Y$ of X. Any Banach unital S-subalgebra $\mathcal{A}$ of $C^*(X)$, induces the *canonical $\mathcal{A}$-compactification* $\alpha_{\mathcal{A}} : X \to X_{\mathcal{A}}$, where $X_{\mathcal{A}}$ is the Gelfand space (or, the *spectrum*) of the algebra $\mathcal{A}$. The map $\alpha_{\mathcal{A}} : X \to X_{\mathcal{A}}$ is defined by the *Gelfand transform*, the evaluation at x

multiplicative functional, that is $\alpha(x)(f) := f(x)$. Conversely, every compactification $\nu : X \to Y$ is equivalent to the *canonical* $\mathcal{A}_\nu$*-compactification* $\alpha_{\mathcal{A}_\nu} : X \to X_{\mathcal{A}_\nu}$, where the algebra $\mathcal{A}_\nu$ (corresponding to ν) is defined as the image $j_\nu(C^*(Y))$ of the embedding $j_\nu : C^*(Y) \to C^*(X), \quad \phi \mapsto \phi \circ \nu$.

Proposition 2.7. *Let X be a Tychonoff space and $\mathcal{A}$ be Smirnov's subalgebra of $C^*(X)$. Then following relation:*

$$B\delta_{\mathcal{A}}C \iff \mathrm{cl}[f(B)] \cap \mathrm{cl}[f(C)] \neq \varnothing, \forall f \in \mathcal{A} \tag{2.1}$$

defines proximity on space X. And more over Smirnov's compactifications, which defined by $\delta_{\mathcal{A}}$ is corresponds with algebra $\mathcal{A}$.

Proposition 2.8. *Let (X, δ) be proximity space. Then collection $\mathcal{A}_\delta$ of all δ-proximity continuous real valued bounded functions is unital subalgebra of $C^*(X)$. and $\delta = \delta_{\mathcal{A}_\delta}$. Also satisfy that for every unital subalgebra $\mathcal{A}$ $C^*(X)$ of satisfy following: $\mathcal{A} = \mathcal{A}_{\delta_{\mathcal{A}}}$.*

Theorem 2.9. [5, Thm.8.4.13] *(`The classical Smirnov theorem`) By assigning to any compactification cX of a Tychonoff space X the proximity $\delta(c)$ on the space X there exists a natural one-to-one correspondence between all compactifications of X and all proximities on the space X.*

Remark 2.10. Subalgebra $\mathcal{A}$ is separates closed subsets and points of X iff compactification $\alpha_{\mathcal{A}}$ is proper compactification and iff proximity $\delta_{\mathcal{A}}$ is defines is original topology of space X.

Example 2.11. Let X be a Tychonoff space. For any two nonempty sets $A, B \subset X$ define

$$A\delta_\beta B \Leftrightarrow \nexists f \in C^*(X) \text{ such that } f(x) = 0 \text{ for } x \in A \text{ and } f(x) = 1 \text{ for } x \in B.$$

The relation δ_β defines a proximity on X, which correspondents to the Ĉech-Stone compactification $\beta : X \to \beta X$ of the space X.

3. Dynamical systems

Let $\langle S, \cdot\rangle$ be a semigroup and assume that also S has a topological structure τ. We say, that $\langle S, \cdot\rangle$ is a *topological semigroup*, if the multiplication map: $S \times S \to S$ is continuous. Information about the theory of topological semigroups can be found in [3].

3.1. **Transformations.**

Definition 3.1. *Topological S-flow* (or an *S-space*) is a triple $\langle S, X, \pi\rangle$ where $\pi : S \times X \to X$ is a jointly continuous left action of a topological semigroup S on a topological space X; we write it also as a pair $\langle S, X\rangle$, or simply, X (when π and$S$ are understood). "Action" means that the following conditions are satisfied:

(1) $\forall s_1, s_2 \in S, \forall x \in X : s_1(s_2x) = (s_1s_2)x$;
(2) if S is monoid, e is identity of S, then $\forall x \in X : ex = x$, e.i. e is identity mapping of X.

We define for every $x \in X$ the *corresponding orbit map* $\tilde{x} : S \to X$, by $\tilde{x}(s) := sx$ (or we say $\tilde{x}(S)$ is *S-orbit* of point x) and also for every $s \in S$, *translations* (or $\breve{s}$-transformations) $\breve{s} : X \to X$, by $\breve{s}(x) := sx$.

For every $(s, x) \in S \times X$ we define *inverse set,* $s^{-1}x := \{y \in X : sy = x\}$ and for $A \subset X$, $s^{-1}A := \bigcup\{s^{-1}a : a \in A\}$.

For $U \subset S$ and $A \subset X$ define $UA := \{y = ua; (u, a) \in U \times A\}$, $U^{-1}A := \bigcup\{u^{-1}A : u \in U\}$ and $U \star A := \bigcap\{u^{-1}A : u \in U\}$.

Definition 3.2. Let X be an S-flow.

(1) Let Y be an S-flow, then function $f : X \to Y$ is called *S-map* if $\forall(s, x) \in S \times X \Longrightarrow f(sx) = sf(x)$.
(2) A *topological S-compactification* of X is a S-map $\alpha : X \to Y$ if α is compactification of X.
(3) A flow (S, X) is said to be *compactifiable* if there exists a *proper* topological S-compactification $\alpha : X \hookrightarrow Y$. A topological semigroup S is *compactifiable* if the flow (S, S) (left regular action) is compactifiable .

3.2. **Proximities and functions on S-spaces.**

Definition 3.3. Let X be a S-space. The subsets $A, B \in X$ are *π-disjoint at $s_0 \in S_e$* if there exists $U \in N_{s_o}(S)$ such that $U^{-1}A \cap U^{-1}B = \varnothing$. If this condition holds for every $s_0 \in S$ then we simply say: *π-disjoint sets.*

Definition 3.4. Let X be S-space. Proximity δ on space X is called *S-proximity* if it satisfy the following: *Compatibility with action*:

$$A\overline{\delta}B \Longleftrightarrow \forall s_0 \in S_e, \qquad \exists U \in N_{s_0}(S_e) : \qquad U^{-1}A\overline{\delta}U^{-1}B.$$

We write $A \ll_\pi B$ if A and B^c are π-disjoint (where $B^c := X \setminus B$) that is for every $s_0 \in S$ there exists $U \in N_{s_0}(G)$ such that $s^{-1}A \subset t^{-1}B$ for every $s, t \in U$. It is also equivalent to saying that $U^{-1}A \subset U \star B$.

Remark 3.5. Let B be a S-space and δ be a S-proximity. If $A\overline{\delta}B$, A and B are π-disjoint sets. If $A \Subset B$ with respect to δ, then $A \ll_\pi B$.

Definition 3.6. Let $\pi : S \times X \to X$ be a given action. A bounded function $f \in C^*(X)$ is said to be *π-uniformly continuous* at $s_0 \in S$ if every every $\varepsilon > 0$, there exists a neighborhood $U \in N_{s_0}$ such that $|f(sx) - f(s_0x)| < \varepsilon$ for every $s \in U, x \in X$. Or, equivalently, the orbit map $\tilde{f} : S \to C(X), s \mapsto fs$ is continuous. We denote family of these functions by $C^*_\pi(X)$.

It's equivalently to say, that:

$$\forall \varepsilon > 0, \exists U \in N_{s_0} : |f(s_1x) - f(s_2x)| < \varepsilon, \forall s_1, s_2 \in U.$$

The set $C^*_\pi(X)$ is an S-invariant Banach unital subalgebra of $C(X)$. Easy to see, that if δ a proximity on S-space X is S-proximity, then function $\breve{s} : X \to X$ is δ-proximity continuous.

Lemma 3.7. *For every compact S-space X we have $C_\pi(X) = C(X)$.*

Theorem 3.8. [16] *There exists a natural 1-1 correspondence between S-compactifications of X and closed unital subalgebras of $C_\pi(X)$. In particular, $C_\pi(X)$ determines the maximal S-compactification $\beta_S : X \to \beta_S X$.*

Proof. Sketch. □

Lemma 3.9. *If $F \in C_\pi(X)$ separates A and B in X then $cl[A]$ and $cl[B]$ are π-disjoint.*

Proposition 3.10. *Let δ be S-proximity on space X. Then every function $f \in \mathcal{A}_\delta$ is π-uniform. And moreover subalgebra $\mathcal{A}_\delta$ is unital S-invariant subalgebra of $C^*_\pi(X)$.*

Proof. Let f be proximity continuous function with respect to the proximities δ and δ'(natural proximity on $I = [0,1]$). Let $\varepsilon > 0$, then there exist finite δ'-uniform cover $\mathscr{A} = \{A_i\}_{i=1}^n$ of I with respect to a cover $\mathscr{B} = \{B_i\}_{i=1}^n$ (i.e. $B_i \Subset A_i$) such that for every $i = 1,\dots,n \Longrightarrow diam(A_i) < \varepsilon$. Since function f is proximity continuous, $f^{-1}(\mathscr{A})$ is δ-uniform cover of X with respect to the cover $f^{-1}(\mathscr{B})$ (e.i. $f^{-1}(B_i) \Subset f^{-1}(A_i)$ for every i). Since δ is S-proximity, for every $s_o \in S$ implies, that there exist $U \in N_{s_0}(S)$, such that for every $i = 1,\dots,n$

$$U^{-1}f^{-1}(B_i) \subset U \star f^{-1}(A_i).$$

Now, for every $x \in X$ and $s \in U$ there exist $y \in X$, such that $x \in s^{-1}y$. For y, there exist $j \in \{1,\dots,n\}$, such that: $y \in f^{-1}(B_j)$. We have that for every $s \in U$ following:

$$x \in s^{-1}y \subset U^{-1}f^{-1}(B_j) \subset U \star f^{-1}(A_j) \Longrightarrow sx \in f^{-1}(A_j) \Longrightarrow f(sx) \in A_j.$$

Since $diam(A_j) < \varepsilon$, we have $\forall s', s'' \in U$ satisfy, that $|f(s'x) - f(s''x)| < \varepsilon$.
For every $s \in S$, function $\check{s}$ proximity continues and $\forall f \in \mathcal{A}_\pi$, satisfy, that real valued function $fs = \check{s} \circ f : X \to \mathbb{R}, (fs)(x) := f(sx)$ is composition of two proximity continues functions, therefore $fs \in \mathcal{A}_\delta$. □

Proposition 3.11. *Let $\mathcal{A}$ be a S-invariant subalgebra of $C^*_\pi(X)$ on S-space X. Then $\delta_\mathcal{A}$ is S-proximity on space X.*

Proof. Let us $C\overline{\delta}_\mathcal{A}D$. Now by construction we assume, that there exist $s_0 \in S$, such that:

$$\forall U \in N_{s_0}(S), U^{-1}C\delta_\mathcal{A}U^{-1}D \Longrightarrow \forall f \in \mathcal{A}, \mathrm{cl}[f(U^{-1}C)] \cap \mathrm{cl}[f(U^{-1}D)] \neq \varnothing.$$

Be Remark 3.5, we assume that C and D is not π-disjoint, these mean that:

$$\forall U \in N_{s_0},\;\; U^{-1}C \cap U^{-1}D \neq \varnothing.$$

For every $\varepsilon > 0$ and $g \in \mathcal{A}$ implies $f := s_o g \in \mathcal{A}$ and π-uniformity of g follow, there exist $V \in N_{s_0}$, such that $|g(s_0x) - g(sx)| < \varepsilon/3, \forall (s,x) \in V \times X$.
On the our assumption, for chosen ε, neighborhood V and function $f := s_0 g$ satisfy the following:

$$\mathrm{cl}[f(V^{-1}C)] \cap \mathrm{cl}[f(V^{-1}D)] \neq \varnothing \Longrightarrow \exists(\bar{x},\bar{y}) \in U^{-1}C \times U^{-1}D, |f(\bar{x}) - f(\bar{y})| < \varepsilon/3$$

This mean, that there exist $s', s'' \in V$, such that $(c,d) := (s'\bar{x}, s''\bar{y}) \in C \times D$ and satisfy $|g(s_0\bar{x}) - g(s_0\bar{y})| < \varepsilon/3$. We note, that:

$$\forall \varepsilon > 0, \forall g \in \mathcal{A}; \exists (c,d) \in C \times D,$$

$$|g(c)-g(d)| = |g(s'\bar{x})-g(s''\bar{y})| \leq |g(s'\bar{x})-g(s_0\bar{x})|+|g(s_0\bar{y})-g(s''\bar{y})|+|g(s_0\bar{x})-g(s_0\bar{y})| < \varepsilon.$$

This implies, that:

$$\forall g \in \mathcal{A}, \mathrm{cl}[g(C)] \cap \mathrm{cl}[g(D)] \neq \varnothing \Longrightarrow C\delta_{\mathcal{A}} D.$$

It is in contradiction with our assumption. $\square$

?? Smirnov

4. Some "good" examples of equicompactifications

The following example suggests several interesting questions of an analytical and geometrical nature.

4.1. **Unit ball of Hilbert space.** This example from [4, p.246-258].

Let $\mathscr{H}$ be an infinite-dimensional real or complex Hilbert space. Let $\mathbb{S}$ and $\mathscr{B}$ be the unit sphere and unit ball, both endowed with the weak(Tychonoff) topology, τ_w. A net $\{x_\alpha\}$ converges to x_0 in this topology or we say weak converging, $x_\alpha \to^w x_0$ iff $\langle x_\alpha, y\rangle \to \langle x_o, y\rangle, \forall y \in \mathscr{H}$.

Fix an orthonormal basis basis E in $\mathscr{H}$. It is easy to see that each $x \in \mathscr{B}$ a basic neighborhood of x in $\mathscr{B}$ has form:

$$O_\varepsilon^F = \{z \in \mathscr{B} : |\langle e, x - z\rangle| < e, \forall e \in F\},$$

where $\varepsilon > 0$ and F is finite subset of E. It is well known, that $\mathscr{B}$ is compact and $\mathbb{S}$ is dense in $\mathscr{B}$. In other words, the inclusion $i : \mathbb{S} \hookrightarrow \mathscr{B}$ is proper compactification of $\mathbb{S}$.

Also we define space of action. $G := \mathscr{U}(\mathscr{H})$ be the group of all unitary operators (linear isomorphic function, $T : H \to H$) of $\mathscr{H}$. There are three topologies on space G :

- *the strong topology* or *uniform(norm) topology*,
$$T_\alpha \rightrightarrows T \Longleftrightarrow \|T_\alpha - T\| \to 0;$$
- *the strong* topology*
$$T_\alpha \to^* T \Longleftrightarrow \|T_\alpha x - Tx\| \to 0, \forall x \in \mathscr{H};$$
- *the weak topology*;
$$T_\alpha \to^w T \Longleftrightarrow T_\alpha x \to^w Tx, \forall x \in \mathscr{H} \Longleftrightarrow \langle T_\alpha x, y\rangle \to \langle Tx, y\rangle, \forall x, y \in \mathscr{H};$$

In what follows we will consider G with strong* operator topology. A basic neighborhood of identity operator I in G is given by

$$U_\varepsilon^F = \{T \in G : \|Te - e\| < \varepsilon, \forall e \in F\},$$

where $\varepsilon < 0$ and F is finite subset of E. The natural action $\pi : G \times \mathscr{H} \to \mathscr{H}$ of G on the $\mathscr{H}$ is defined by, $\pi(T, x) = Tx$, for all $T \in G$ and $x \in \mathscr{H}$ is jointly continuous, also $\pi|_{\mathscr{B}}$ and $\pi|_{\mathbb{S}}$ are well defined on $\mathscr{B}$ and $\mathbb{S}$. This means, that $\mathscr{H}$, $\mathscr{B}$, $\mathbb{S}$ are G-spaces. Since $\mathscr{B}$ is compact and $\mathbb{S}$ is dense in it, the inclusion $i : \mathbb{S} \hookrightarrow \mathscr{B}$ is G-compactification and in other words $\mathbb{S}$ is G-Tychonoff space.

Problem 4.1. To describe of all G-compactifications of $\mathbb{S}$, by simple geometrical characterization.

First of all we will describe important step to this problem is to find the greatest G-compactification of $\mathbb{S}$.

Theorem 4.2. *A function $f : \mathbb{S} \to \mathbb{R}(\mathbb{C})$ is π-uniform if and only if there is continuous function $\hat{f} : \mathscr{B} \to \mathbb{R}(\mathbb{C})$. Consequently, the inclusion $i : \mathbb{S} \hookrightarrow \mathscr{B}$ is (up to equivalence) the greatest G-compactification of $\mathbb{S}$.*

To describe all G-compactifications of S we will consider separately the real and complex cases. First suppose that $\mathscr{H}$ is real Hilbert space. For any a, b with $0 \leq b \leq a \leq 1$, consider the following relation $\sim$ on $\mathscr{B}$.

$$x \sim y \text{ iff } \begin{cases} x = y, \text{or} \\ \|x\| \ \leq a \text{ and } y = -x, \text{or} \\ \|x\| \leq b \text{ and } \|y\| \leq b. \end{cases}$$

Easy to see that relation $\sim$ is closed equivalence on $\mathscr{B}$ and invariant under the action. So the quotient space $\mathscr{B}_{a,b} := \mathscr{B}/\sim$ has natural structure of a G-space, such that the natural surjection $p : \mathscr{B} \to \mathscr{B}_{a,b}$ is G-equivariant. By φ_{ab} we denote the restriction of p on sphere $\mathbb{S}$. Clearly, φ_{ab} is a G-compactification of $\mathbb{S}$, and it is equivalent ti φ_{cd} iff $a = c$ and $b = d$.

Theorem 4.3. *Let $\varphi : \mathbb{S} \to Y$ be a G-compactification of $\mathbb{S}$. Then there are unique numbers a and a in interval $[0; 1]$ with $b \leq a$ such that φ is equivalent to φ_{ab}.*

4.2. **Some Maximal G-compactifications of $\mathbb{R}^n$.** This example have given by Yu. Smirnov in [18]. Now we will see some simple geometrical object, which will be compactification of $\mathbb{R}^n$ under natural actions. We will take the sphere $\mathbb{S}^n \approx \mathbb{R}^n \bigcup \{\infty\}$ and ball $\mathbb{B}^n \approx \mathbb{R}^n \bigcup \mathbb{S}^{n-1}$. Easy to see, that $\mathbb{S}^n$ and $\mathbb{B}^n$ are proper compactifications of $\mathbb{R}^n$.

Theorem 4.4. *The sphere $\mathbb{S}^n$, $n \geq 2$ is a maximal G-compactification of $\mathbb{R}^n$ under the action of group G of all homeomorphisms $g : \mathbb{R}^n \to \mathbb{R}^n$, taken with the topology of uniform convergence in the metric of $\mathbb{S}^n$.*

Theorem 4.5. *Then ball $\mathbb{B}^n$, $n \geq 2$ is a maximal G-compactification of $\mathbb{R}^n$ under the group of all uniform homeomorphisms $g : \mathbb{R}^n \to \mathbb{R}^n$, whose inverse g^{-1} are also uniform, taken with the topology of uniform convergence in the metric of $\mathbb{B}^n$.*

Problem 4.6. Find interesting concrete applications and examples in Topology and Functional Analysis for semigroup case.

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