Every semitopological semigroup compactification of the group $H_+[0,1]$ is trivial

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Abstract. Let $G = H_+[0,1]$ be the topological group of all orientation-preserving selfhomeomorphisms of the closed interval $[0,1]$ endowed with the usual compact open topology. We show that every weakly almost periodic function on G is constant. Consequently, G does not admit nontrivial (weakly) continuous representations by linear isometries in *reflexive* Banach spaces .

§*1. Introduction*

A topologized semigroup S is said to be *semitopological* if the multiplication function $m : S \times S \to S$ is separately continuous. If we only have continuity of the maps

$$\rho_s : S \to S, \quad \rho_s(x) = m(x,s) = xs,$$

then we call S a *right topological semigroup.*

The *Lawson's joint continuity theorem* implies that a subgroup of a compact semitopological semigroup is always topological [La, Corollary 6.3]. It is a widely explored fact that every Hausdorff topological group G is a topological subgroup of a compact right topological semigroup S (see, for example, *dynamical compactifications* in [Ru]). One of the applications of our main result is to establish that "right topological" cannot be replaced by "semitopological". More precisely, let $G := H_+[0,1]$ be the topological group of all orientation preserving selfhomeomorphisms of the closed interval $[0,1]$ endowed with the compact open topology. We show that every semitopological semigroup compactification of G is trivial.

In order to formulate other applications, we need some background.

Let E be a (real) Banach space. Denote by $Is(E)_s$, $Is(E)_w$ the group of all linear isometries of E endowed with the strong and weak operator topologies, respectively. An important (and closely related to the existence of proper right topological dynamical compactifications) fact is that every Hausdorff topological

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group G can be embedded into the group $Is(E)_s$ of all linear isometries of a suitable Banach space E endowed with the strong operator topology. Indeed, following [Te], take for example $E := C_r^b(G)$, the Banach space of all bounded right uniformly continuous functions on G. The natural question is whether E may be taken to be reflexive or even Hilbert. The latter case, that is the case of unitary representations, in contrast to the case of a reflexive E, has been extensively studied. There are in particular, several counterexamples (see [HC, Ba]). Our example shows that every (weakly) continuous homomorphism $H_+[0,1] \to Is(E)_s$ is trivial whenever E is reflexive.

Recall that a continuous bounded function $f \in C^b(G)$ on a topological group G is called *weakly almost periodic (wap)* in the sense of Eberlein [Eb, dLG] if the orbit of f in $C^b(G)$ is relatively weakly compact. The subset $WAP(G)$ of all wap functions in $C^b(G)$ forms a C^*-algebra. The maximal ideal space G^w of $WAP(G)$ is a compact Hausdorff semitopological semigroup and the natural continuous homomorphism $j : G \to G^w$ enjoys the following universal property: every weakly almost periodic compactification of G, i.e., every continuous homomorphism $\phi : G \to P$ onto a dense subgroup of a compact Hausdorff semitopological semigroup P, is lifted in a unique way to a continuous semigroup homomorphism $\widetilde{\phi} : G^w \to P$ such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & G^w \arrow[d] \\ & P \end{tikzcd}$$

For every reflexive Banach space E, the semigroup

$$cont(E) := \{s \in L(E,E) : \|s\| \leq 1\}$$

of all contractive linear operators forms a compact semitopological semigroup in the weak operator topology. Hence, the same is true for its closed subsemigroups. Conversely, arbitrary compact Hausdorff semitopological semigroup can be obtained in this way (see Fact 1.2).

Definition 1.1. ([Me2]) We say that a Hausdorff topological group G is *reflexively representable* if one of the following equivalent conditions holds:

(i) $WAP(G)$ separates points and closed subsets;

(ii) $j : G \to G^w$ is a topological embedding;

(iii) G is a topological subgroup of a Hausdorff compact semitopological semigroup;

(iv) There exists a reflexive Banach space E such that G is embedded as a topological subgroup into $Is(E)_s$ (equivalently, into $Is(E)_w$).

The equivalence of (i), (ii) and (iii) is actually well known [Ru, BJM].

It is a standard fact that for every reflexive Banach space E and a norm-bounded semigroup S of linear operators on E the *generalized matrix coefficients*

$$\{m_{v,f}\}_{f \in E^*, v \in E} \qquad m_{v,f}(s) = f(sv)$$

all are wap. This proves (iv) $\Longrightarrow$ (i).

The part (iii) $\Longrightarrow$ (iv), in the case of $Is(E)_w$, directly follows from the following result of A. Shtern.

Fact 1.2. ([Sh, Me3]) The following conditions are equivalent:
(a) $WAP(S)$ separates points and closed subsets;
(b) S can be embedded into $cont(E)_w$ for a certain reflexive Banach space E.

As to the case of $Is(E)_s$, note that by [Me2] strong and weak operator topologies coincide on $Is(E)$ for a wide class of Banach spaces including the class of all reflexive spaces. Therefore, for reflexive E we can replace $Is(E)_w$ by $Is(E)_s$.

The main result of the present paper provides an example of a Hausdorff topological group which is not reflexively representable. This result answers a question discussed by W. Ruppert in [Ru] (see p. 115 and p. 242).

For more information about wap and semigroup compactifications, we refer to [BJM]. See also a survey by Pestov [Pe] which among many useful information includes a brief exposition of the present paper, as well as of Fact 1.2.

§*2. Some useful G-spaces, Roelcke completion*

All topological spaces in this paper are assumed to be Hausdorff. The filter of all neighborhoods (nbd's) of a point z in a space X is denoted by $N_z(X)$. Denote by $\mathcal{L}$ and $\mathcal{R}$ the *left* and *right* uniformities on a topological group G. If e is the identity of G and V runs over $N_e(G)$, then the covers of the form $\{ xV \mid x \in G \}$, $\{ Vx \mid x \in G \}$ generate $\mathcal{L}$ and $\mathcal{R}$, respectively. Besides the usual *two-sided uniformity* $\mathcal{L} \vee \mathcal{R}$, there exists also one more natural uniformity $\mathcal{L} \wedge \mathcal{R}$, called in [RD] *lower uniformity.* Recently, V. Uspenskij [Us1, Us2, Us3] found several deep applications of $\mathcal{L} \wedge \mathcal{R}$. Following his suggestion, we will call it the *Roelcke uniformity.* The uniformity $\mathcal{L} \wedge \mathcal{R}$ is the greatest lower bound of $\mathcal{L}$ and $\mathcal{R}$ and is generated by the system of coverings $\{ VxV \mid V \in N_e(G) \}$. Denote by $i : (G, \mathcal{L} \wedge \mathcal{R}) \longrightarrow (\widehat{G}, \widehat{\mathcal{L} \wedge \mathcal{R}})$ the *Roelcke-completion*; that is, the completion of the uniform space $(G, \mathcal{L} \wedge \mathcal{R})$. There are *jointly continuous* (left and right) group actions:

$$G \times \widehat{G} \longrightarrow \widehat{G} \qquad (g_1, u) \mapsto g_1 \circ u$$
$$\widehat{G} \times G \longrightarrow \widehat{G} \qquad (u, g_2) \mapsto u \circ g_2$$

extending the usual left and right actions of G on G. The joint continuity of the extended actions follows immediately from [RD, Proposition 10.12] (or, [Me1, Theorem 3.1]). Since $i(G)$ is dense in the Hausdorff space $\widehat{G}$, the "principle of extension of identities" implies that for every $g_1, g_2 \in G$ and every $u \in \widehat{G}$, $(g_1 \circ u) \circ g_2 = g_1 \circ (u \circ g_2)$. Clearly, $g_1 \circ i(g_2) = i(g_1) \circ g_2 = i(g_1 g_2)$.

In the sequel we will identify g and $i(g)$.

Now we turn again to the universal semitopological compactification $j : G \longrightarrow G^w$. Since the corresponding algebra $WAP(G)$ is closed under left and right translations [dLG], there are actions

$$\begin{aligned} G \times G^w &\longrightarrow G^w & \qquad (g,v) &\mapsto g \bullet v \\ G^w \times G &\longrightarrow G^w & \qquad (v,g) &\mapsto v \bullet g \end{aligned}$$

such that $j(g_1 g_2) = g_1 \bullet j(g_2) = j(g_1) \bullet g_2$ for all $g_1, g_2 \in G$. Lawson's *joint continuity theorem* [La, Corollary 5.2] implies that these actions of G on G^w both are jointly continuous.

Every wap function $f \in WAP(G)$ is simultaneously left and right uniformly continuous [Ru, p.113]. Therefore, such f is $\mathcal{L} \wedge \mathcal{R}$ - uniformly continuous too. Thus the map $j : (G, \mathcal{L} \wedge \mathcal{R}) \longrightarrow G^w$ is uniformly continuous. Denote by π the corresponding (unique) uniformly continuous extension $\pi : (\widehat{G}, \widehat{\mathcal{L} \wedge \mathcal{R}}) \longrightarrow G^w$. Then $j(g) = \pi(i(g)) = \pi(g)$ for every $g \in G$.

Lemma 2.1. *For every $g \in G$ and every $u \in \widehat{G}$, the following hold*

$$\begin{aligned} \pi(g \circ u) &= \pi(g) \cdot \pi(u) = g \bullet \pi(u), \\ \pi(u \circ g) &= \pi(u) \cdot \pi(g) = \pi(u) \bullet g. \end{aligned}$$

Proof. Use the "principle of extension of identities". □

§3. *Main Theorem and sketch of the proof*

We show that the group $H_+[0,1]$ is "as non-reflexively representable as possible".

Main Theorem 3.1. Every wap function on $H_+[0,1]$ is constant.

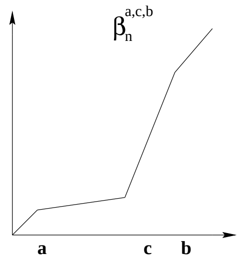
Here we describe the general idea of the proof. Set $G := H_+[0,1]$ and use the notations of §2. In order to prove the triviality of G^w, we show that there exists a zero element θ in G^w and it coincides with the identity $\pi(e_G)$ of the semigroup G^w. Since $\pi : \widehat{G} \longrightarrow G^w$ is continuous, it suffices to construct a double sequence $\phi_{nk} \in G$ $(n, k \in \mathbb{N})$ such that the following two conditions are satisfied:

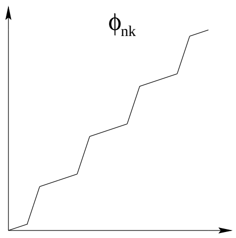
3.2. (i) $\lim_n \lim_k \phi_{nk} = e_G$.
(ii) $\lim_n \lim_k \pi(\phi_{nk}) = \theta$.

Notation 3.3. (i) Let $0 \leq a \leq c \leq b \leq 1$ and $n, k \in \mathbb{N} \setminus \{1\}$. Define $\beta_n^{a,c,b}$ as the homeomorphism of [0,1] the graph of which is piecewise linear and is obtained by connection of the following points: $(0,0)$, (a,a), $(c - \frac{c-a}{n}, a + \frac{c-a}{n})$, $(c + \frac{b-c}{n}, b - \frac{b-c}{n})$, (b,b), $(1,1)$.

(ii) Define

$$\phi_{nk} := \beta_k^{0, \frac{1}{2^n}, \frac{1}{2^n}} \cdot \beta_k^{\frac{1}{2^n}, \frac{1}{2^n}, \frac{2}{2^n}} \cdot \beta_k^{\frac{2}{2^n}, \frac{3}{2^n}, \frac{3}{2^n}} \cdot \beta_k^{\frac{3}{2^n}, \frac{3}{2^n}, \frac{4}{2^n}} \cdots \beta_k^{\frac{2^n-2}{2^n}, \frac{2^n-1}{2^n}, \frac{2^n-1}{2^n}} \cdot \beta_k^{\frac{2^n-1}{2^n}, \frac{2^n-1}{2^n}, \frac{2^n}{2^n}}.$$





§4. *Main Lemmas*

First some further notation. Let (X, d) be a compact metric space, and let $G := H(X)$ be the topological group of all homeomorphisms endowed with the topology of compact convergence. For every $\varepsilon > 0$, set

$$\begin{aligned} \mathcal{U}_\varepsilon : &= \{ \phi \in G \mid d(x, \phi(x)) < \varepsilon \quad \forall x \in X\}, \\ [\varepsilon]_{\mathcal{R}} : &= \{ (f_1, f_2) \in G \times G \mid f_2 = \phi f_1 \text{ for a certain } \phi \in \mathcal{U}_\varepsilon\}, \\ &= \{(f_1, f_2) \in G \times G \mid d(f_1(x), f_2(x)) < \varepsilon \quad \forall x \in X\}, \\ [\varepsilon]_{\mathcal{L}\wedge\mathcal{R}} : &= \{(f_1, f_2) \in G \times G \mid f_2 = \phi f_1 \psi \text{ for some } \phi, \psi \in \mathcal{U}_\varepsilon\}. \end{aligned}$$

Lemma 4.1. *(i) The system of entourages $\{[\varepsilon]_{\mathcal{L}\wedge\mathcal{R}} \mid \varepsilon > 0\}$ generates the uniformity $\mathcal{L} \wedge \mathcal{R}$.*

(ii) $[\varepsilon]_{\mathcal{L}\wedge\mathcal{R}} = \{(f_1, f_2) \in G \times G \mid (f_1, h) \in [\varepsilon]_{\mathcal{R}}, \quad (h^{-1}, f_2^{-1}) \in [\varepsilon]_{\mathcal{R}}$ for a certain $h \in G\}$.

(iii) $(f_1, f_2) \in [\varepsilon]_{\mathcal{L}\wedge\mathcal{R}} \Leftrightarrow (f_1^{-1}, f_2^{-1}) \in [\varepsilon]_{\mathcal{L}\wedge\mathcal{R}}$.

Proof. Straightforward. $\square$

Lemma 4.2. *Let $f_1, f_2, h \in H(X)$ and let A be a subset of X of diameter $\operatorname{diam} A < \varepsilon$ such that*

(i) $d(f_1(x), h(x)) < \varepsilon \quad \forall x \in X$.

(ii) $h(x) = f_2(x) \quad \forall x \notin A$.

Then $(f_1, f_2) \in [\varepsilon]_{\mathcal{L}\wedge\mathcal{R}}$.

Proof. Condition (i) means exactly that $(f_1, h) \in [\varepsilon]_{\mathcal{R}}$. The second condition implies that

$$d(h^{-1}(x), f_2^{-1}(x)) \leq \operatorname{diam} A < \varepsilon \quad \forall x \in X.$$

Therefore $(h^{-1}, f_2^{-1}) \in [\varepsilon]_{\mathcal{R}}$. Now we can use Lemma 4.1 (ii). $\square$

Lemma 4.3. *Let $f_1, f_2 \in H_+[0,1]$, and let there exist $0 \leq a < b \leq 1$ such that*

(i) $|f_1(t) - f_2(t)\ | < \frac{\varepsilon}{2} \quad \forall t \in [0, a] \cup [b, 1]$,

(ii) $|a - b\ | < \varepsilon$.

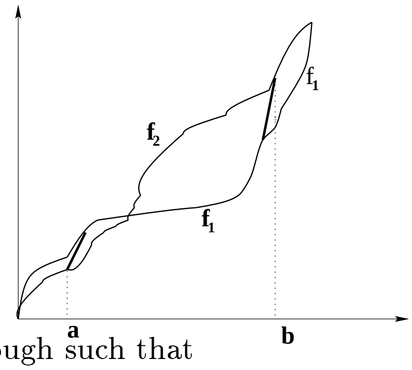
Then $(f_1, f_2) \in [\varepsilon]_{\mathcal{L}\wedge\mathcal{R}}$.

Proof. According to Lemma 4.2, it suffices to construct $h \in H_+[0,1]$ such that the following two conditions are satisfied:

(a) $|f_1(t) - h(t)\ | < \varepsilon \quad \forall t \in [0, 1]$

(b) $h(t) = f_2(t) \quad \forall t \notin (a, b)$.

Restricting the interval (a,b) if necessary, we may suppose that $f_1(a) \neq f_2(a), f_1(b) \neq f_2(b)$. Suppose, $f_1(a) > f_2(a)$ (otherwise, interchange f_1 and f_2). We consider only the case when $f_1(b) < f_2(b)$ (the second case of "$f_1(b) > f_2(b)$" is also easy). The following picture might be useful.



Choose $\delta > 0$ small enough such that

$$a+\delta < b-\delta, \qquad f_1(a+\delta) - f_2(a) < \frac{\varepsilon}{2}, \qquad f_2(b) - f_1(b-\delta) < \frac{\varepsilon}{2}.$$

Define

$$h(t) = \begin{cases} f_2(t), & t \in [0,a] \cup [b,1], \\ \frac{1}{\delta}(f_1(a+\delta) - f_2(a))(t-a) + f_2(a), & t \in [a, a+\delta], \\ f_1(t), & t \in [a+\delta, b-\delta], \\ \frac{1}{\delta}(f_2(b) - f_1(b-\delta))(t-b) + f_2(b), & t \in [b-\delta, b]. \end{cases}$$

Then $h \in H_+[0,1]$ and it has the desired properties (a) and (b). $\square$

For every $0 \leq a \leq b \leq 1$, set

$$\begin{aligned} H_+[a,b] &:= \{g \in H_+[0,1] \mid g(t) = t \quad \forall t \notin (a,b)\}, \\ G^w_{[a,b]} &:= cl_{G^w}(\pi(H_+[a,b])), \end{aligned}$$

where cl_{G^w} denotes the closure operator in the space $G^w = H_+[0,1]^w$.

Lemma 4.4. *Let $g \in H_+[a,b]$. Then*

(i) The sequence $\{\beta_n^{a,c,b} \circ g\}$ is $\mathcal{L} \wedge \mathcal{R}$-Cauchy and equivalent to the sequence $\left\{\beta_n^{a,g^{-1}(c),b}\right\}$ in $(G, \mathcal{L} \wedge \mathcal{R})$.

(ii) The sequence $\{g \circ \beta_n^{a,c,b}\}$ is $\mathcal{L} \wedge \mathcal{R}$-Cauchy and equivalent to the sequence $\{\beta_n^{a,c,b}\}$.

Proof. (i) For a given $g \in H_+[a,b]$ and $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following four conditions are satisfied:

(1) $\frac{c-a}{n} < \frac{\varepsilon}{2}$, $\frac{b-c}{n} < \frac{\varepsilon}{2}$.

(2) $g^{-1}([c - \frac{c-a}{n}, c + \frac{b-c}{n}]) \subset [g^{-1}(c) - \frac{\varepsilon}{2}, g^{-1}(c) + \frac{\varepsilon}{2}]$.

(3) $\beta_n^{a,g^{-1}(c),b}([a, g^{-1}(c) - \frac{\varepsilon}{2}]) \subset [a, a+\frac{\varepsilon}{2})$.

(4) $\beta_n^{a,g^{-1}(c),b}([g^{-1}(c) + \frac{\varepsilon}{2}, b]) \subset (b - \frac{\varepsilon}{2}, b]$.

Every $h \in H_+[0,1]$ is a continuous strictly increasing function. Therefore, $h([p,q]) = [h(p), h(q)]$ for every $p, q \in [0,1]$. Combining this argument with (1) and (2), we get

$$\begin{aligned}
&(\beta_n^{a,c,b} \circ g)([a, g^{-1}(c) - \frac{\varepsilon}{2}]) \\
\subset &(\beta_n^{a,c,b} \circ g)([a, g^{-1}(c - \frac{c-a}{n})] \\
= &[\beta_n^{a,c,b}(g(a)), \beta_n^{a,c,b}(c - \frac{c-a}{n})] = [a, a + \frac{c-a}{n}] \\
\subset &[a, a + \frac{\varepsilon}{2}).
\end{aligned}$$

Analogously,

$$\begin{aligned}
&(\beta_n^{a,c,b} \circ g)([g^{-1}(c) + \frac{\varepsilon}{2}, b]) \\
\subset &(\beta_n^{a,c,b} \circ g)([g^{-1}(c + \frac{b-c}{n}), b]) \\
= &[\beta_n^{a,c,b}(c + \frac{b-c}{n}), \beta_n^{a,c,b}(g(b))] = [b - \frac{b-c}{n}, b] \\
\subset &(b - \frac{\varepsilon}{2}, b].
\end{aligned}$$

Combining the results of these computations with the conditions (3), (4), for every $n, m \geq n_0$, we have

$$|(\beta_n^{a,c,b} \circ g)(t) - \beta_m^{a,g^{-1}(c),b}(t)| < \frac{\varepsilon}{2} \qquad \forall t \notin (g^{-1}(c) - \frac{\varepsilon}{2}, g^{-1}(c) + \frac{\varepsilon}{2}).$$

According to Lemma 4.3, we obtain

$$(\beta_n^{a,c,b} \circ g, \beta_m^{a,g^{-1}(c),b}) \in [\varepsilon]_{\mathcal{L} \wedge \mathcal{R}} \qquad \forall n, m \geq n_0.$$

This proves (i).

The proof of (ii) is similar. $\square$

Notation 4.5.

(i) $\lim \beta_n^{a,c,b} = \beta^{a,c,b} \in \widehat{G}$.
(ii) $\pi(\beta^{a,c,b}) = \alpha^{a,c,b} \in G^w$.

Note that the limit in (i) exists in virtue of Lemma 4.4.

Lemma 4.6. *In the semigroup $G^w_{[a,b]}$ there exists a zero element $\theta_{[a,b]}$ and it coincides with $\alpha^{a,c,b}$ for every $c \in [a,b]$. In particular,*
(i) $\alpha^{a,c,b} = \alpha^{a,a,b} = \alpha^{a,b,b}$.
(ii) $\alpha^{0,1,1} = \alpha^{0,0,1} = \theta_{[0,1]} = \theta$ (a zero element of G^w).

Proof. It suffices to show that $\alpha^{a,c,b}$ is a *right zero* and that $\alpha^{a,a,b}$ is a *left zero*. First we check that $\alpha^{a,c,b}$ is a right zero in $G^w_{[a,b]}$. We have to show that

(1) $$u \cdot \alpha^{a,c,b} = \alpha^{a,c,b} \qquad \forall u \in G^w_{[a,b]}.$$

Since $\pi(H_+[a,b])$ is dense in the *semitopological* semigroup $G^w_{[a,b]}$, we have to check (1) only for the elements $u = \pi(g) \in \pi(G)$. For every $g \in G$, by Lemma 2.1, we get

$$\begin{aligned}\pi(g)\cdot\alpha^{a,c,b} &= g \bullet \alpha^{a,c,b}\\ &= g \bullet \lim \pi(\beta_n^{a,c,b})\\ &= \lim(g \bullet \pi(\beta_n^{a,c,b}))\\ &= \lim \pi(g \circ \beta_n^{a,c,b})\end{aligned}$$

On the other hand, using Lemma 4.4 (ii), we have

$$\lim \pi(g \circ \beta_n^{a,c,b}) = \pi(\lim(g \circ \beta_n^{a,c,b})) = \pi(\lim \beta_n^{a,c,b}) = \pi(\beta^{a,c,b}) = \alpha^{a,c,b}.$$

Analogously, making use Lemmas 4.4 (i) and 2.1 (taking into account that $g^{-1}(a) = a$), we may check that $\alpha^{a,a,b}\cdot u = \alpha^{a,a,b}$ for every $u \in G^w_{[a,b]}$. This establishes that $\alpha^{a,a,b}$ is a left zero in $G^w_{[a,b]}$, as desired. □

Lemma 4.7. *Suppose that $\phi, f_1, f_2 \in H_+[a,b]$ and $g \in H_+[c,d]$. If $(a,b)\cap(c,d) = \emptyset$ then*

(i) $g\phi = \phi g$.

(ii) If $(f_1, f_2) \in [\varepsilon]_{\mathcal{L}\wedge\mathcal{R}}$, then f_1, f_2 are 3ε-close with respect to the Roelcke uniformity of the group $H_+[a,b]$.

(iii) If $(f_1, f_2) \in [\varepsilon]_{\mathcal{L}\wedge\mathcal{R}}$ then $(gf_1, gf_2) \in [3\varepsilon]_{\mathcal{L}\wedge\mathcal{R}}$ and $(f_1g, f_2g) \in [3\varepsilon]_{\mathcal{L}\wedge\mathcal{R}}$.

Proof. (i) is trivial.

(ii) We can suppose that $b - a \geq 3\varepsilon$ (otherwise, use Lemma 4.3). According to Lemma 4.1(b), there exists $h \in H_+[0,1]$ such that for every $t \in [0,1]$ the following hold:

(1) $\mid f_1(t) - h(t) \mid\ < \varepsilon$.

(2) $\mid h^{-1}(t) - f_2^{-1}(t) \mid\ < \varepsilon$.

Denote $s := h^{-1}(a), p := h(b), q := h^{-1}(b)$. We may suppose that $s \geq a$. If not, take h^{-1} instead of h and use Lemma 4.1 (c). By (1) and (2), we get

(3) $\mid s - a \mid\ < \varepsilon,\ \mid p - b \mid\ < \varepsilon,\ \mid q - b \mid\ < \varepsilon$.

Suppose first that $p \leq b$. Choose $\delta > 0$ such that

(4) $\delta < \varepsilon$.

(5) $h([s, s+\delta]) \subset [a, a+\varepsilon)$.

(6) $h([b-\delta, b]) \subset (p-\varepsilon, p]$.

Define

$$\phi(t) = \begin{cases} t, & t \in [0,a]\cup[b,1],\\ \frac{h(s+\delta)-a}{s+\delta-a}(t-a)+a & t \in [a, s+\delta],\\ h(t), & t \in [s+\delta, b-\delta].\\ \frac{b-h(b-\delta)}{\delta}(t-b)+b, & t \in [b-\delta, b].\end{cases}$$

Clearly, $\phi \in H_+[a,b]$. Using the conditions (1)-(6), by elementary computations, we obtain

$$(h,\phi) \in [2\varepsilon]_{\mathcal{R}}, \qquad (\phi^{-1}, h^{-1}) \in [2\varepsilon]_{\mathcal{R}}.$$

Using (1) and (2), by the triangle axiom, eventually we have

$$(f_1, \phi) \in [3\varepsilon]_{\mathcal{R}}, \qquad (\phi^{-1}, f_2^{-1}) \in [3\varepsilon]_{\mathcal{R}}.$$

This proves the first case.

In the second case the assumption is $b < p = h(b)$. Then $q = h^{-1}(b) < h^{-1}(p) = b$. We can choose $\delta > 0$ with the following additional property (7) $[h(q-\delta), h(b)] \subset (b-\varepsilon, p]$.

Define in this case

$$\phi(t) = \begin{cases} t, & t \in [0,a] \cup [b,1], \\ \frac{h(s+\delta)-a}{s+\delta-a}(t-a)+a, & t \in [a, s+\delta], \\ h(t), & t \in [s+\delta, q-\delta], \\ \frac{h(q-\delta)-b}{q-\delta-b}(t-b)+b, & t \in [q-\delta, b]. \end{cases}$$

The other arguments are the same. (iii) By (ii), there exist $\phi, \psi \in U_{3\varepsilon} \cap H_+[a,b]$ such that $f_2 = \phi f_1 \psi$. Using (i), clearly we have

$$gf_2 = g(\phi f_1 \psi) = \phi(gf_1)\psi,$$
$$f_2 g = (\phi f_1 \psi) g = \phi(f_1 g)\psi.$$

□

Lemma 4.8. *Let $\{g_{ni}\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(G, \mathcal{L}\wedge\mathcal{R})$ for every $i \in \{1,2,\cdots s\}$, where $g_{ni} \in H_+[a_i, b_i]$ and $0 \le a_i < b_i \le a_{i+1} < b_{i+1} \le 1$. Then*

(i) $\{g_{n1}g_{n2}\cdots g_{ns}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(G, \mathcal{L}\wedge\mathcal{R})$.

(ii) $\lim \pi(g_{n1}g_{n2}\cdots g_{ns}) = \lim \pi(g_{n1})\cdots \lim \pi(g_{ns})$.

Proof. It suffices to consider the case when $s = 2$.

(i) We have to check that $\{g_{n1}g_{n2}\}$ is $\mathcal{L}\wedge\mathcal{R}$-Cauchy. For a given $\varepsilon > 0$ choose $n_0 \in \mathbb{N}$ such that for every $n, k \ge n_0$, $(g_{n1}, g_{k1}) \in [\frac{\varepsilon}{6}]_{\mathcal{L}\wedge\mathcal{R}}$ and $(g_{n2}, g_{k2}) \in [\frac{\varepsilon}{6}]_{\mathcal{L}\wedge\mathcal{R}}$. For every fixed pair (n,k), according to Lemma 4.7 (ii), there exist $\phi_1, \psi_1 \in U_{\frac{\varepsilon}{2}} \cap H_+[a_1, b_1]$ and $\phi_2, \psi_2 \in U_{\frac{\varepsilon}{2}} \cap H_+[a_2, b_2]$ such that $g_{n1} = \phi_1 g_{k1} \psi_1$ and $g_{n2} = \phi_2 g_{k2} \psi_2$.

By Lemma 4.7 (i), ϕ_2 commutes with g_{k1}, ψ_1, and ψ_1 commutes with g_{k2}. Therefore,

$$g_{n1}g_{n2} = (\phi_1 g_{k1} \psi_1)(\phi_2 g_{k2} \psi_2) = (\phi_1\phi_2)(g_{k1}g_{k2})(\psi_1\psi_2).$$

This proves that $(g_{n1}g_{n2}, g_{k1}g_{k2}) \in [\varepsilon]_{\mathcal{L}\wedge\mathcal{R}}$.

(ii) We use the notations

$$\lim g_{n1} = q_1, \quad \lim g_{n2} = q_2, \quad \lim(g_{n1}g_{n2}) = \gamma.$$

By Lemma 4.7 (iii), it is easy to show that $\{g_{n1} \circ q_2\}$ is a Cauchy sequence in $(\widehat{G}, \widehat{\mathcal{L}\wedge\mathcal{R}})$. The corresponding limit in $\widehat{G}$ will be denoted by μ. We have to prove that $\pi(\gamma) = \pi(q_1)\pi(q_2)$. First we prove that $\mu = \gamma$. Assuming on the contrary that $\mu \neq \gamma$, we can choose disjoint nbd's $U \in N_\mu(\widehat{G}), V \in N_\gamma(\widehat{G})$ and $\varepsilon > 0$ such that the following condition is satisfied:

$$(*) \qquad (h_1, h_2) \notin [\varepsilon]_{\mathcal{L}\wedge\mathcal{R}} \quad \forall h_1 \in U \cap G \quad \forall h_2 \in V \cap G.$$

For sufficiently large $r \in \mathbb{N}$, we have $g_{r1} \circ q_2 \in U$ and $g_{r1}g_{r2} \in V$. Now choose a natural number $k \geq r$ such that $(g_{k2}, g_{r2}) \in [\frac{\varepsilon}{3}]_{\mathcal{L}\wedge\mathcal{R}}$ and $g_{r1}g_{k2} \in U$. The first condition, by Lemma 4.7 (iii), guarantees that

$$(**) \qquad (g_{r1}g_{k2}, g_{r1}g_{r2}) \in [\varepsilon]_{\mathcal{L}\wedge\mathcal{R}}.$$

By our construction, $h_1 := g_{r1}g_{k2} \in U$ and $h_2 := g_{r1}g_{r2} \in V$. Then $(**)$ contradicts $(*)$. This proves the assertion.

Now we can complete the proof of (ii). Since $\gamma = \mu = \lim(g_{n1} \circ q_2)$, we have $\pi(\gamma) = \lim \pi(g_{n1} \circ q_2)$. According to Lemma 2.1, $\pi(g_{n1} \circ q_2) = \pi(g_{n1}) \cdot \pi(q_2)$. Since G^w is semitopological, we get

$$\lim(\pi(g_{n1}) \cdot \pi(q_2)) = (\lim \pi(g_{n1})) \cdot \pi(q_2) = \pi(q_1) \cdot \pi(q_2).$$

□

Lemma 4.9. $\alpha^{a,c,b} = \alpha^{a,c,c} \cdot \alpha^{c,c,b}$.

Proof. According to Lemma 4.8 (ii), we obtain

$$\alpha^{a,c,c} \cdot \alpha^{c,c,b} = \lim \pi(\beta_n^{a,c,c}) \cdot \lim \pi(\beta_n^{c,c,b}) = \lim \pi(\beta_n^{a,c,c} \circ \beta_n^{c,c,b}).$$

On the other hand, by Lemma 4.3, it is clear that the $\mathcal{L} \wedge \mathcal{R}$-Cauchy sequences $\{\beta_n^{a,c,c} \circ \beta_n^{c,c,b}\}$ and $\{\beta_n^{a,c,b}\}$ are equivalent. □

§5. *Proof of the Main Theorem*

By Lemma 4.8, the sequence $\{\phi_{nk}\}_{k\in\mathbb{N}}$ is $\mathcal{L} \wedge \mathcal{R}$-Cauchy for arbitrary $n \in \mathbb{N}$. Denote $u_n := \lim_k \phi_{nk} \in \widehat{G}$. Clearly, $\mid \phi_{nk}(t) - t \mid \; < \frac{1}{2^n}$ for every $t \in [0,1]$. Therefore, $\lim u_n$ is the identity e_G of G. This proves 3.2 (i).

In order to prove the second condition, consider $\lim_k \pi(\phi_{nk})$. By Lemma 4.8 (ii) and Notation 4.5, we have

$$\begin{aligned}
&\lim_k \pi(\phi_{nk}) = \\
&\lim \pi(\beta_k^{0,\frac{1}{2^n},\frac{1}{2^n}}) \cdot \lim \pi(\beta_k^{\frac{1}{2^n},\frac{1}{2^n},\frac{2}{2^n}}) \cdots \lim \pi(\beta_k^{\frac{2^n-1}{2^n},\frac{2^n-1}{2^n},\frac{2^n}{2^n}}) \\
&= \alpha^{0,\frac{1}{2^n},\frac{1}{2^n}} \cdot \alpha^{\frac{1}{2^n},\frac{1}{2^n},\frac{2}{2^n}} \cdots \alpha^{\frac{2^n-1}{2^n},\frac{2^n-1}{2^n},\frac{2^n}{2^n}}.
\end{aligned}$$

On the other hand, by Lemma 4.6 (ii) we know that $\theta = \alpha^{0,1,1}$. By multiple use of Lemmas 4.6 and 4.9, we obtain

$$\begin{aligned}
\theta &= \alpha^{0,1,1} \\
&= \alpha^{0,\frac{1}{2},1} \\
&= \alpha^{0,\frac{1}{2},\frac{1}{2}} \cdot \alpha^{\frac{1}{2},\frac{1}{2},1} \\
&= \alpha^{0,\frac{1}{4},\frac{1}{2}} \cdot \alpha^{\frac{1}{2},\frac{3}{4},1} \\
&= (\alpha^{0,\frac{1}{4},\frac{1}{4}} \cdot \alpha^{\frac{1}{4},\frac{1}{4},\frac{1}{2}}) \cdot (\alpha^{\frac{1}{2},\frac{3}{4},\frac{3}{4}} \cdot \alpha^{\frac{3}{4},\frac{3}{4},1}) \\
&= \cdots \text{(By Lemmas 4.6(i) and 4.9)} \cdots \\
&= (\alpha^{0,\frac{1}{2^n},\frac{1}{2^n}} \cdot \alpha^{\frac{1}{2^n},\frac{1}{2^n},\frac{2}{2^n}}) \cdots (\alpha^{\frac{2^n-2}{2^n},\frac{2^n-1}{2^n},\frac{2^n-1}{2^n}} \cdot \alpha^{\frac{2^n-1}{2^n},\frac{2^n-1}{2^n},\frac{2^n}{2^n}}).
\end{aligned}$$

Therefore, $\lim_k \pi(\phi_{nk}) = \theta$ for every n. This proves 3.2 (ii). $\square$

§*6. Final remarks*

The class of all reflexively representable groups is closed under formation of topological subgroups and arbitrary products. It is unclear if the same is true for quotient groups.

Question 6.1. Let H be a closed normal subgroup of a reflexively representable group G. Is it true that the quotient group G/H is also reflexively representable ?

The non-reflexively representable group $H_+[0,1]$ is very far from being abelian. Therefore, the following question arises.

Question 6.2. Is every *abelian* topological group G reflexively representable ?

If the abelian group G is $\omega-$ *bounded* [Gu] (that is, G is covered with countably many translations of every non-empty open subset) then the latter question can be reduced to the particular case of a *cyclic* second countable group G. Indeed, G is $\omega-$bounded iff G is a topological subgroup of a topological group product $\prod G_i$ of second countable groups G_i (see [Gu]). On the other hand, by a recent result of Morris and Pestov [MP], every second countable abelian group is a topological subgroup of a *monothetic* second countable group. Hence, G_i is a topological subgroup of some monothetic group M_i. Without restriction of generality, we can suppose that M_i is complete, and moreover, that M_i is a completion of some second countable cyclic group C_i. Now, it remains to apply the following lemma.

Lemma 6.3. *Let G be a reflexively representable group. Then its completion $\overline{G}$ with respect to the two-sided uniformity is also reflexively representable.*

Proof. By Definition 1.1, G is a subgroup of a Hausdorff compact semitopological semigroup S. Then every subgroup of S and, in particular, the subgroup $H := H(1)$ of all units in S, is a topological group (by [La, Corollary 6.3]) and is reflexively representable (by our Definition 1.1). On the other hand, by [Ru, Th. 4.6, p. 64], the group H is complete with respect to its two-sided uniformity. Since

G is a dense subgroup of H, we can identify H and $\overline{G}$. Therefore, $\overline{G}$ is reflexively representable. $\square$

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