TOPOLOGICAL TRANSFORMATION GROUPS: SELECTED TOPICS

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1. Introduction

In this paper all topological spaces are Tychonoff. A topological transformation group, or a G-space, as usual, is a triple (G, X, π), where $\pi\colon G \times X \to X$, $\pi(g,x) := gx$ is a continuous action of a topological group G on a topological space X. Let G act on X_1 and on X_2. A continuous map $f\colon X_1 \to X_2$ is a *G-map* (or, an *equivariant map*) if $f(gx) = gf(x)$ for every $(g,x) \in G \times X_1$.

The Banach algebra of all continuous real valued bounded functions on a topological space X will be denoted by $C(X)$. Let (G, X, π) be a G-space. It induces the action $G \times C(X) \to C(X)$, with $(gf)(x) = f(g^{-1}x)$. A function $f \in C(X)$ is said to be *right uniformly continuous*, or also *π-uniform*, if the map $G \to C(X)$, $g \mapsto gf$ is norm continuous. The latter means that for every $\varepsilon > 0$ there exists a neighborhood V of the identity element $e \in G$ such that $\sup_{x\in X} |f(vx) - f(x)| < \varepsilon$ for every $v \in V$. The set $\mathrm{RUC}_G(X) := \mathrm{RUC}(X)$ of all right uniformly continuous functions on X is a uniformly closed G-invariant subalgebra of $C(X)$.

A *transitive* action in this paper means an action with a single orbit. Let H be a closed subgroup of G and G/H be the (left) coset space endowed with the quotient topology. In the sequel we will refer to G/H as a *homogeneous* G-space. In this particular case $f \in \mathrm{RUC}_G(X)$ iff f is a uniformly continuous bounded function with respect to the natural *right uniform structure* on G/H (this explains Fact 2.1.1 below). A G-space X will be called:

1. *G-compactifiable*, or *G-Tychonoff*, if X is a G-subspace of a compact G-space.
2. *G-homogenizable*, if there exists an equivariant embedding of (G, X) into a homogeneous space $(G', G'/H)$ (i.e., there exists a topological group embedding $h\colon G \hookrightarrow G'$ and a topological embedding $\alpha\colon X \hookrightarrow G'/H$ such that $h(g)\alpha(x) = gx$).
3. *G-automorphic*, if X is a topological group and each $\tilde{g} = \pi(g, \cdot)\colon X \to X$ is a group automorphism. We say also that X is a *G-group*.
4. *G-automorphizable*, if X is a G-subspace of an automorphic G-space. In particular, if Y is a locally convex G-space with a continuous linear action of G on Y then we say that X is *G-linearizable*.

2. Equivariant compactifications

A *G-compactification* of a G-space X is a G-map $\nu\colon X \to Y$ with a dense range into a compact G-space Y. A compactification is *proper* when ν is a topological embedding. The study of equivariant compactifications goes back to J. de Groot, R. Palais, R. Brook, J. de Vries, Yu. Smirnov and others.

The Gelfand–Raikov–Shilov classical functional description of compactifications admits a natural generalization for G-spaces in terms of G-subalgebras of $\mathrm{RUC}(X)$ (see for example [76, 11, 4]). The G-algebra $V := \mathrm{RUC}(X)$ defines the corresponding Gelfand (maximal ideal) space $\beta_G X \subset V^*$ and the, possibly improper, *maximal G-compactification* $i_{\beta_G}\colon X \to \beta_G X$. Consider the natural homomorphism $h\colon G \to \mathrm{Is}(\mathrm{RUC}(X))$, where $\mathrm{Is}(\mathrm{RUC}(X))$ is the group of all linear isometries of $\mathrm{RUC}(X)$ and $h(g)(f) := gf$. The pair (h, i_{β_G}) defines a *representation* (in the sense of Definition 7.1) of the G-space X on the Banach space $\mathrm{RUC}(X)$.

A G-space is G-Tychonoff iff it can be equivariantly embedded into a compact Hausdorff G-space iff i_{β_G} is proper iff $\mathrm{RUC}_G(X)$ separates points and closed subsets iff (G, X) is Banach representable (cf. Definition 7.1 and Fact 7.2).

Unless G is discrete, the usual maximal compactification $X \to \beta X$ (which always is a G_d-compactification for every G-space X, where G_d is the group G endowed with the discrete topology) fails to be a G-compactification, in general. However several standard compactifications are compatible with actions. For instance it is true for the one-point compactifications [75]. The Samuel compactification of an *equiuniform* G-spaces (X, μ) is a G-compactification (see [18, 75, 37]). Here 'μ is an equiuniformity on a G-space X' means that every translation $\tilde{g}\colon X \to X$ is μ-uniform and for every entourage $\varepsilon \in \mu$ there exists a neighborhood U of the identity e such that $(gx, x) \in \varepsilon$ for every $(g, x) \in U \times X$. Equiuniform precompact uniformities correspond to G-compactifications. For *G-proximities* see Smirnov [11]. It is easy to see that *Gromov's compactification*[1] of a bounded metric space (X, d) with a continuous G-invariant action is a proper G-compactification. The reason is that the function $f_z\colon X \to \mathbb{R}$ defined by $f_z(x) := d(z, x)$ is π-uniform for every $z \in X$.

By J. de Vries' well known result [77] if G is locally compact then every Tychonoff G-space is G-Tychonoff. See Palais [55] for the case of a compact Lie group G, and Antonyan [11] for compact G.

We call a group G, a *V-group*, if every Tychonoff G-space is G-Tychonoff. In [75], de Vries posed the 'compactification problem' which in our terms becomes: is every topological group G a V-group? Thus every locally compact group is a V-group. An example of [39] answers de Vries' question negatively: there exists a topological transformation group (G, X) such that both G and X are Polish and X is not G-Tychonoff.

Fact 2.1. *Recall some useful situations when G-spaces are G-Tychonoff:*

(1) *every coset G-space G/H* (de Vries [75]; see also Pestov [61])*;*
(2) *every automorphic G-space X (and, hence, every linear G-space X),* [40]*;*

[1]The corresponding algebra is generated by the set of functions $\{f_z\colon X \to \mathbb{R}\}_{z \in X}$, where $f_z(x) := d(z, x)$ (see for example [4, p. 112]).

(3) *every metric G-space (X, d), where G is second category and $\tilde{g}\colon X \to X$ is d-uniformly continuous for every $g \in G$,* [40]*;*
(4) *every G-space X, where X is Baire, G is uniformly Lindelöf and acts transitively on X* (Uspenskij [70])*.*

For some results related to Fact 2.1(4) see Chatyrko and Kozlov [20].

A topological group G is *uniformly Lindelöf* (alternative names: $\aleph_0$-bounded, ω-bounded, ω-narrow, etc.) if for every nonempty open subset $O \subset G$ countably many translates $g_n O$ cover G. By a G-factorization theorem [40] every G-Tychonoff space X with uniformly Lindelöf G admits a proper G-compactification $X \hookrightarrow Y$ with the same weight and dimension $\dim Y \leq \dim \beta_G X$.

The following two results are proved in [52].

(1) If G is Polish then it is a V-group iff G is locally compact.
(2) If G is uniformly Lindelöf and *not* locally precompact, then G is not a V-group. Furthermore there exists a Tychonoff G-space X such that $i_{\beta_G} : X \to \beta_G X$ is not injective.

The following longstanding question remains open.

Question 2.2 (Yu.M. Smirnov, 1980)**.** *Find a nontrivial Tychonoff G-space X such that every G-compactification of X is trivial.* **1001 ?**

The compactification problem is still open for many natural groups.

Question 2.3 ([52])**.** **1002–1003 ?**

(1) *Is there a locally precompact group G which is not a V-group?*
(2) *What if G is the group $\mathbb{Q}$ of rational numbers? What if G is the precompact cyclic group $(\mathbb{Z}, \tau_p)$ endowed with the p-adic topology?*

Question 2.4 (Antonyan and Sanchis [10])**.** *Is every locally pseudocompact group a V-group?* **1004 ?**

Stoyanov gave (see [21, 67]) a geometric description of G-compactifications for the following natural action: $X := \mathbb{S}_H$ is the unit sphere of a Hilbert space H and $G := U(H)$ is the unitary group endowed with the strong operator topology. Then the maximal G-compactification is equivalent to the natural inclusion of X into the weak compact unit ball $\mathbb{B}_H$ of H.

Question 2.5. *Let V be a separable reflexive Banach space. Consider the natural action of the group* $\mathrm{Is}(V)$ *on the sphere S_V. Is it true that the maximal G-compactification is equivalent to the natural inclusion of X into the weak compact unit ball $\mathbb{B}_V$ of V?* **1005 ?**

For more information about the question: 'whether simple geometric objects can be maximal equivariant compactifications?' we refer to Smirnov [66].

Question 2.6 (H. Furstenberg and T. Scarr)**.** *Let X be a Tychonoff G-space with the transitive action. Is it true that X is G-Tychonoff?* **1006 ?**

Uspenskij's result (see Fact 2.1(4)) implies that the answer is 'yes' if X is Baire and G is uniformly Lindelöf.

Very little is known about the dimension of $\beta_G X$. Even in the case of the left regular action of G on $X := G$ the dimension of $\beta_G G$ (the so-called *greatest ambit* for G) may be greater than $\dim G$ (take a cyclic dense subgroup G of the circle group $\mathbb{T}$; then $\dim G = 0$ and $\dim \beta_G G = \dim \mathbb{T} = 1$). It is an old folklore result that $\dim \beta_G G = 0$ iff G is non-Archimedean[2] (see for example, [57, 53]). It follows by [35, Thm 5.12] that in the case of the Euclidean group $G = \mathbb{R}^n$, we have $\dim \beta_G G = \dim G$.

? 1007 **Question 2.7.** *Does the functor β_G preserve the covering dimension in case of compact Lie acting group G?*

If G is a compact Lie group then for every G-space X the inequality $\dim X/G \leq \dim X$ holds. For second countable X this is a classical result of Palais [55]. For general Tychonoff X this was done in [40] using a G-factorization theorem. This inequality does not remain true for compact (even 0-dimensional) groups. This led us [39] to an example of a locally compact Polish G-space X such that $\dim X = 1$ and $\dim \beta_G X \geq 2$, where G is a 0-dimensional compact metrizable group.

Fact 2.8 ([11, 9])**.** *$(\beta_G X)/G = \beta(X/G)$ for every G-space X and compact G.*

? 1008 **Question 2.9** (Zambakhidze)**.** *Let G be a compact group, X a G-space, and $B(X/G)$ a proper compactification of the orbit space X/G. Does there exist a proper G-compactification $B_G(X)$ of X such that $B_G(X)/G = B(X/G)$?*

For some partial results see Antonyan [6] and Ageev [1].

? 1009 **Question 2.10.** *Let G be a Polish group and X be a second countable G-Tychonoff G-space. Does there exist a metric G-completion of X with the same dimension?*

If G is not Polish then it is not true. The answer is affirmative if G is locally compact [41].

3. Equivariant normality

Definition 3.1 ([36, 38, 52])**.** Let (G, X, π) be a topological transformation group.

(1) Two subsets A and B in X are *π-disjoint* if $UA \cap UB = \emptyset$ for some neighborhood U of the identity $e \in G$.

(2) X is *G-normal* (or, *equinormal*) if for every pair of π-disjoint closed subsets A and B there exists a pair of π-disjoint neighborhoods $O_1(A)$ and $O_2(B)$. It is equivalent to say that every pair of π-disjoint closed subsets can be separated by a function from $\mathrm{RUC}_G(X)$ (*Urysohn lemma for G-spaces*).

(3) X is *weakly G-normal* if every pair of π-disjoint closed *G-invariant* subsets in X can be separated by a function from $\mathrm{RUC}_G(X)$.

Another version of the Urysohn lemma for G-spaces appears in [30, Theorem 3.9].

Every G-normal space is G-Tychonoff. The action of $G := \mathbb{Q}$ on $X := \mathbb{R}$ is not G-normal. One can characterize locally compact groups in terms of G-normality.

Fact 3.2 ([52])**.** *For every topological group G the following are equivalent:*

[2] *Non-Archimedean* means having a local base at the identity consisting of open subgroups,

(1) *Every normal G-space is G-normal.*
(2) *G is locally compact.*

It is unclear if 'G-normal' can be replaced by 'weakly G-normal'.

Question 3.3. *Is every second countable G-space weakly G-normal for the group $G := \mathbb{Q}$ of rational numbers?* **1010 ?**

If not, then by [52, Theorem 3.2] one can construct for $G := \mathbb{Q}$ a Tychonoff G-space X which is not G-Tychonoff. That is, it will follow that $\mathbb{Q}$ is not a V-group (see Question 2.3).

Fact 3.4. *Every coset G-space G/H is G-normal.*

Then the following 'concrete' actions (being coset spaces) are equinormal:

(1) $(U(H), \mathbb{S}_H)$ for every Hilbert space H;
(2) $(\mathrm{Is}(\mathbb{U}), \mathbb{U})$ (where $\mathrm{Is}(\mathbb{U})$ is the isometry group of the Urysohn space $\mathbb{U}$ with the pointwise topology);
(3) $(\mathrm{GL}(V), V \setminus \{0\})$ for every normed space V (see [44]);
(4) $(\mathrm{GL}(V), \mathbb{P}_V)$ for every normed space V and its projective space $\mathbb{P}_V$.

It follows in particular by (4) that $\mathbb{P}_V$ is $\mathrm{GL}(V)$-Tychonoff. This was well known among experts and easy to prove (cf. e.g. Pestov [59]) using equiuniformities.

Question 3.5. *Is it true that the following (G-Tychonoff) actions are G-normal: $(U(\ell_2), \ell_2)$, $(\mathrm{Is}(\ell_p), \mathbb{S}_{\ell_p}))$, $p > 1$, ($p \neq 2$)?* **1011 ?**

4. Universal actions

Let $\mathcal{A}$ be some class of continuous actions (G, X). We say that a pair (G_u, X_u) from $\mathcal{A}$ is *(equivariantly) universal* for the class $\mathcal{A}$ if for every $(G, X) \in \mathcal{A}$ there exists an equivariant pair (h, f) such that $h \colon G \hookrightarrow G_u$ is a topological group embedding and $f \colon X \hookrightarrow X_u$ is a topological embedding. If, in addition we require that $G = G_u$ and $h = \mathrm{id}_G$ then we simply say that X_u is *G-universal*.

For a compact space X denote by $H(X)$ the group of all homeomorphisms of X endowed with the compact open topology.

Fact 4.1.

(1) (Antonyan and de Vries [12]; Tychonoff theorem for G-spaces) *For every locally compact sigma-compact group G and a cardinal τ there exists a universal G-space of weight τ.*
(2) (Megrelishvili [40]; G-space version of Nagata's universal space theorem) *Let G be a locally compact sigma-compact group of weight $w(G) \leq \tau$. For every integer $n \geq 0$ there exists, in the class of metrizable G-spaces of dimension $\leq n$ and weight $\leq \tau$, a universal G-space.*
(3) (Hjorth [33]) *If G is a Polish group, then the class of Polish G-spaces has a G-universal object.*
(4) (Megrelishvili and Scarr [53]; Equivariant universality of the Cantor cube) *Let $K := \{0,1\}^{\aleph_0}$ be the Cantor cube. Then $(H(K), K)$ is equivariantly universal for the class of all 0-dimensional compact metrizable G-spaces, where G is second countable and non-Archimedean.*

See also results of Becker and Kechris [14, Section 2.2.6], Vlasov [74] and questions posed by Iliadis in [34, p. 502].

? 1012 **Question 4.2.** *Let G be a Polish group. Is it true that there exists a universal G-space in the class of all Polish G-spaces with dimension $\leq n$?*

Fact 4.3 ([43])**.**
(1) $(H(I^{\aleph_0}), I^{\aleph_0})$ *is equivariantly universal for the class of all G-compactifiable actions (G, X) with second countable G and X.*
(2) *Let G be a uniformly Lindelöf group. Then every G-Tychonoff space X is equivariantly embedded into $(H(I^{\tau}), I^{\tau})$ where $\tau \leq w(X)w(G)$.*

A direct corollary of Fact 4.3(1) is a well known result of Uspenskij [69] about universality of the group $H(I^{\aleph_0})$. Another proof of Uspenskij's result (see [7, Corollary 4]) follows by the following theorem of Antonyan.

Fact 4.4 ([7])**.** *Let G be a uniformly Lindelöf group. Then for every G-Tychonoff space X there exists a family of convex metrizable G-compacta $\{K_f\}_{f\in F}$ such that $|F| = w(X)$ and X possesses a G-embedding into the product $\prod_{f\in F} K_f$.*

The following natural question of Antonyan remains open (even for $\tau = \aleph_0$).

? 1013 **Question 4.5** (Antonyan [7, 8])**.** *Let G be a uniformly Lindelöf group of weight $\aleph_0 \leq wG \leq \tau$. Does there exist a G-universal compact G-space of weight τ?*

? 1014–1015 **Question 4.6.** *Let τ be an uncountable cardinal.*
(1) *Is it true that there exists an equivariantly universal topological transformation group (G_u, X_u) in the class of all topological transformation groups (G, X) where X is G-Tychonoff and* $\max\{w(G), w(X)\} \leq \tau$*?*
(2) *What if G_u and G are abelian?*

A positive answer on (1) will imply the solution of the following question.

? 1016 **Question 4.7** (Uspenskij [72])**.** *Does there exist a universal topological group of every given infinite weight τ?*

Fact 4.8. *G-Compactifiable $\supset$ G-Homogenizable $\supset$ G-Automorphizable.*

Every G-group X is naturally identified with the coset P-space P/G, where $P := X \leftthreetimes G$ is the corresponding semidirect product. This explains the second inclusion. The first inclusion follows by Fact 3.4.

If G is locally compact then every G-space is *G-linearizable* (see for example, [11, 78]) and all classes from Fact 4.8 coincide.

It is well known that the action of $H(I^{\aleph_0})$ on $I^{\aleph_0}$ is transitive. Using Effros' theorem one can show that Tychonoff cubes I^{λ} are coset $H(I^{\lambda})$-spaces for every infinite power, [43]. Therefore Fact 4.3 leads to the equality G-Compactifiable = G-Homogenizable for every uniformly Lindelöf group G. It is unclear in general.

? 1017 **Question 4.9.** *Is there a G-Tychonoff non-homogenizable G-space? Equivalently, is every compact G-space G-homogenizable?*

5. Free topological G-groups

Let X be a Tychonoff G-space. By $F_G(X)$ we denote the corresponding *free topological G-group* in the sense of [43]. Recall a link with the *epimorphism problem.* Uspenskij has shown in [71] that in the category of Hausdorff topological groups epimorphisms need not have a dense range. This answers a longstanding problem by K. Hofmann. Pestov gave [56, 58] a useful epimorphism criteria in terms of the free topological G-groups.

Fact 5.1 (Pestov [56])**.** *The natural inclusion $H \hookrightarrow G$ of a topological subgroup H into G is an epimorphism (in the category of Hausdorff groups) if and only if the free topological G-group $F_G(X)$ of the coset G-space $X := G/H$ is trivial (here the triviality means, 'as trivial as possible', isomorphic to the cyclic discrete group).*

For instance by results of [43], $F_G(X)$ is trivial in the following situation: the group $G := H(\mathbb{S})$ is the group of all homeomorphisms of the circle $\mathbb{S}$ which can be identified with the compact coset G-space $G/\operatorname{St}(z)$ (where z is a point of $\mathbb{S}$ and $\operatorname{St}(z)$ is the stabilizer of z). It follows that $\operatorname{St}(z) \hookrightarrow G$ is an epimorphism. This example shows also that not every compact G-space is G-automorphizable.

If G is locally compact then $F_G(X)$ canonically can be identified with the usual free topological group $F(X)$. This suggests the following questions.

Question 5.2. *Let X be G-automorphic (i.e., the canonical map $X \to F_G(X)$ is an embedding). Is it true that the natural map $F(X) \to F_G(X)$ is a homeomorphism?* **1018 ?**

Question 5.3. *Let X be a G-automorphic G-space. Is it true that $F_G(X)$ is algebraically free over X?* **1019 ?**

6. Banach representations of groups

A *representation* of a topological group G on a Banach space V is a homomorphism $h\colon G \to \operatorname{Is}(V)$, where $\operatorname{Is}(V)$ is the topological group of all linear surjective isometries $V \to V$ endowed with the *strong operator topology* inherited from V^V. Denote by V_w the space V in its weak topology. The corresponding topology on $\operatorname{Is}(V)$ inherited from V_w^V is the *weak operator topology.* By [46], for a wide class PCP (*Point of Continuity Property*) of Banach spaces, including reflexive spaces, strong and weak operator topologies on $\operatorname{Is}(V)$ coincide.

Let $\mathcal{K}$ be a 'well behaved' subclass of the class *Ban* of all Banach spaces. Important particular cases for such $\mathcal{K}$ are: *Hilb*, *Ref* or *Asp*, the classes of Hilbert, reflexive or Asplund spaces respectively. The investigation of *Asp* and the closely related *Radon–Nikodým property* is among the main themes in Banach space theory. Recall that a Banach space V is an *Asplund space* if the dual of every separable linear subspace is separable, iff every bounded subset A of the dual V^* is (weak*,norm)-*fragmented*, iff V^* has the Radon–Nikodým property. Reflexive spaces and spaces of the type $c_0(\Gamma)$ are Asplund. Namioka's Joint Continuity Theorem implies that every weakly compact set in a Banach space is norm fragmented. This explains why every reflexive space is Asplund. For more details cf. [54, 17, 24]. For some applications of the fragmentability concept for topological transformation groups, see [45, 49, 46, 27].

We say that a topological group G is $\mathcal{K}$*-representable* if there exists a representation $h\colon G \to \mathrm{Is}(V)$ for some $V \in \mathcal{K}$ such that h is a topological embedding; notation: $G \in \mathcal{K}_r$. In the opposite direction, we say that G is $\mathcal{K}$*-trivial* if every continuous $\mathcal{K}$-representation of G is trivial. Of course, $TopGr = Ban_r \supset Asp_r \supset Ref_r \supset Hilb_r$. As to $TopGr = Ban_r$, it is an old observation due to Teleman [68] (see also [58]) that for every topological group G the natural representation $G \to \mathrm{Is}(V)$ on the Banach space $V := \mathrm{RUC}(G)$ is an embedding.

Every locally compact group is Hilbert representable (Gelfand–Raikov). (We say also, *unitarily representable.*) On the other hand, even for Polish groups very little is known about their representabilty in well behaved Banach spaces.

It is also well known that $TopGr \neq Hilb_r$. Moreover, there are examples of unitarily trivial, so-called *exotic* groups (Herer–Christensen [32] and Banasczyk [13]).

Classical results imply that a group is unitarily representable iff the positive definite functions separate the closed subsets and the neutral element. By results of Shoenberg the function $f(v) = e^{-\|x\|^p}$ is positive definite on $L_p(\mu)$ spaces for every $1 \leq p \leq 2$. An arbitrary Banach space V, as a topological group, cannot be exotic because the group V in the weak topology is unitarily representable. However $C[0,1], c_0 \notin Hilb_r$ (see Fact 6.6 below).

Fact 6.1 ([47])**.** *A topological group G is (strongly) reflexively representable (i.e., G is embedded into* $\mathrm{Is}(V)$ *endowed with the strong operator topology for some reflexive V) iff the algebra* $\mathrm{WAP}(G)$ *of all* weakly almost periodic functions *determines the topology of G.*

A weaker result replacing 'strong' by 'weak' appears earlier in Shtern [65]. The group $G := H_+[0,1]$ of orientation preserving homeomorphisms of the closed interval (with the compact open topology) is an important source for counterexamples.

Fact 6.2.

(1) ([47]) $H_+[0,1]$ *is reflexively (and hence also Hilbert) trivial.*
(2) ([28]) *Moreover,* $H_+[0,1]$ *is even Asplund trivial.*

The question if $\mathrm{WAP}(G)$ determines the topology of a topological group G was raised by Ruppert [64]. (1) means that every wap function on $H_+[0,1]$ is constant. The WAP triviality of $G := H_+[0,1]$ was conjectured by Pestov.

? 1020 **Question 6.3** (Glasner and Megrelishvili)**.** *Is there an abelian group which is not reflexively representable?*

Equivalently: is it true that the algebra $\mathrm{WAP}(G)$ on an abelian group G separates the identity from closed subsets?

? 1021 **Question 6.4.** *Is it true that every Banach space X, as a topological group, is reflexively representable?*

A separable Banach space U is *uniformly universal* if every separable Banach space, as a uniform space, can be embedded into U. Clearly, $C[0,1]$ is linearly universal and hence also uniformly universal. In [2] Aharoni proved that c_0 is uniformly universal. P. Enflo [23], in answer to a question by Yu. Smirnov, found in 1969 a countable metrizable uniform space which is not uniformly embedded into a Hilbert space. That is, ℓ_2 is not uniformly universal[3]. However, it is an open question if 'Hilbert' may be replaced by 'reflexive'.

Question 6.5. *Does there exist a uniformly universal* reflexive *Banach space?* **1022 ?**

There is no linearly universal separable reflexive Banach space (Szlenk). Moreover, there is no Lipschitz embedding of c_0 into a reflexive Banach space (Mankiewicz). For more information on uniform classification of Banach spaces we refer to [15].

Fact 6.6 ([42, 46])**.** *Let G be a (separable) metrizable group and let $\mathcal{U}_L$ denote its left uniform structure. If G is reflexively representable, than $(G, \mathcal{U}_L)$ as a uniform space is embedded into a (separable) reflexive space V. Moreover, if G is unitarily representable then G is uniformly embedded into a (separable) Hilbert space.*

As a corollary it follows that $C[0,1]$ and c_0 are not unitarily representable. A positive answer to the following question will imply a positive answer on 6.5.

Question 6.7. *Are the additive groups c_0 and $C[0,1]$ reflexively representable?* **1023 ?**

A natural question arises about coincidence of Ref_r and $Hilb_r$. The positive answer was conjectured by A. Shtern [65]. By [48], $L_4[0,1] \in Ref_r$ and $L_4[0,1] \notin Hilb_r$. Chaatit [19] proved that every separable $L_p(\mu)$ space $(1 \leq p < \infty)$, is reflexively representable.

By [3], if a metrizable abelian[4] group, as a uniform space, is embedded into a Hilbert space then positive definite functions separate the identity and closed subsets. Combining this with Fact 6.6 we have the following[5].

Fact 6.8. *A metric abelian group is unitarily representable if and only if it is uniformly embedded into a Hilbert space.*

The same observation (for second countable abelian groups) is mentioned by J. Galindo in a recent preprint [25]. Facts 6.6 and 6.8 suggest the following question.

Question 6.9 (See also [48])**.** *Let G be a metrizable group and it, as a uniform space $(G, \mathcal{U}_L)$, is uniformly embedded into a reflexive (Hilbert) Banach space. Is it true that G is reflexively (resp., unitarily) representable?* **1024 ?**

Galindo announced [25] that for every compact space X the free abelian topological group $A(X)$ is unitarily representable. Uspenskij found [73] that in fact this is true for every Tychonoff space X. The case of $F(X)$ is open.

Question 6.10. *Let X be a Tychonoff (or, even a compact) space.* **1025 ?**

(1) *Is it true that the free topological group $F(X)$ is reflexively representable?*
(2) (see also Pestov [62]) *Is it true that $F(X)$ is unitarily representable?*

$U(\ell_2)$ clearly is universal for Polish unitarily representable groups.

Question 6.11. *Does there exist a universal reflexively representable Polish group?* **1026 ?**

Question 6.12. *Is it true that if G is reflexively representable then the factor group G/H is also reflexively representable?* **1027 ?**

[3]This result by Enflo has recently led to some exciting developments in geometric group theory, cf. Gromov [31].

[4]In fact, *metrizable amenable*, is enough.

[5]It was presented on Yaki Sternfeld Memorial International Conference (Israel, May 2002).

It is impossible here to replace 'reflexively' by 'Hilbert' because every Abelian Polish group is a factor-group of a Hilbert representable Polish group (Gao and Pestov [26]). A positive answer to Question 6.12 will imply that every second countable Abelian group is reflexively representable. Also then we will get a negative answer to the following problem.

? 1028 **Question 6.13** (A.S. Kechris)**.** *Is every Polish (nonabelian) topological group a topological factor-group of a subgroup of $U(\ell_2)$ with the strong operator topology?*

A natural test case by Fact 6.2 is the group $H_+[0,1]$. Fact 6.2 of course implies that every bigger group $G \supset H_+[0,1]$ is not reflexively representable. Moreover if G in addition is topologically simple then it is reflexively trivial. For instance the Polish group $\mathrm{Is}(\mathbb{U}_1)$[6] is reflexively trivial (as observed by Pestov [60], this fact follows immediately from results by Megrelishvili [47] and Uspenskij [72]). It follows that every Polish group is a subgroup of a reflexively trivial Polish group.

? 1029 **Question 6.14** (Glasner and Megrelishvili [28])**.** *Is it true that there exists a nontrivial Polish group which is reflexively (Asplund) trivial but does not contain a subgroup topologically isomorphic to $H_+[0,1]$?*

By a recent result of Rosendal and Solecki [63] every homomorphism of $H_+[0,1]$ into a separable group is continuous. Hence every representation (of a discrete group) $H_+[0,1]$ on a separable reflexive space is trivial.

? 1030 **Question 6.15** (Glasner and Megrelishvili)**.** *Find a Polish group G which is reflexively (Asplund) trivial but the discrete group G_d admits a nontrivial representation on a separable reflexive (Asplund) space.*

? 1031 **Question 6.16.** *Is it true that the group $H(I^{\aleph_0})$ is reflexively trivial?*

It is enough to show that the group $H(I^{\aleph_0})$ is topologically simple.

? 1032 **Question 6.17** (Glasner and Megrelishvili)**.** *Is it true that there exists a group G such that $G \in Asp_r$ and $G \notin Ref_r$.*

7. Dynamical versions of Eberlein and Radon–Nikodým compacta

Eberlein compacta in the sense of Amir and Lindenstrauss [5] are exactly the weakly compact subsets in Banach (equivalently, reflexive) spaces V. If X is a weak* compact subset in the dual V^* of an Asplund space V then, following Namioka [54], X is called *Radon–Nikodým compact* (in short: RN). Every reflexive Banach space is Asplund. Hence, every Eberlein compact is RN.

Definition 7.1 ([49])**.** A *(proper) representation* of (G,X) on a Banach space V is a pair (h,α) where $h\colon G \to \mathrm{Is}(V)$ is a continuous homomorphism of topological groups and $\alpha\colon X \to V^*$ is a weak star continuous bounded G-mapping (resp., *embedding*) with respect to the *dual action* $G \times V^* \to V^*$, $(g\varphi)(v) := \varphi(h(g^{-1})(v))$.

Note that the dual action is norm continuous whenever V is an Asplund space, [45]. It is well known that the latter does not remain true in general.

[6]$\mathbb{U}_1$ is a sphere of radius 1/2 in $\mathbb{U}$.

Fact 7.2. *A G-space X is properly representable on some Banach space V if and only if X is G-Tychonoff (consider the natural representation on $V := \mathrm{RUC}_G(X)$).*

The following dynamical versions of Eberlein and Radon–Nikodým compact spaces were introduced in [49]. A compact G-space X is *Radon–Nikodým*, RN for short, if there exists a proper representation of (G, X) on an Asplund Banach space V. If V is reflexive (Hilbert) then we get the definitions of reflexively (resp., Hilbert) representable G-spaces. In the first case we say that (G, X) is an *Eberlein* G-space.

Fact 7.3. *Let X be a metric compact G-space.*

(1) ([49]) *X, as a G-space, is Eberlein (i.e., reflexively G-representable) iff X is a weakly almost periodic G-space in the sense of Ellis* [22].
(2) ([27]) *X, as a G-space, is RN iff X is hereditarily nonsensitive.*

Compact spaces which are not Eberlein are necessarily nonmetrizable, while even for $G := \mathbb{Z}$, there are natural *metric* compact G-spaces which are not RN.

There exists a compact metric $\mathbb{Z}$-space which is reflexively but not Hilbert representable [50]. This answers a question of T. Downarowicz.

Question 7.4. *Is it true that Eberlein (that is, reflexively representable) compact* **1033 ?**
G-spaces are closed under factors?

For the trivial group G (i.e., in the purely topological setting) the answer is affirmative and this is just a well known result by Benyamini, Rudin and Wage [16]. The answer is 'yes' for compact *metric* G-spaces.

Question 7.5. *Is it true that RN (that is, Asplund representable) compact G-spaces* **1034 ?**
are closed under factors?

For the trivial group one can recognize a longstanding open question by Namioka [54]. Again if X is metric then the answer is 'yes'. For Hilbert representable actions the situation is unclear even for the metric case.

Question 7.6. *Is it true that Hilbert representable compact metric G-spaces are* **1035 ?**
closed under factors?

For a compact G-space X denote by $E := E(X)$ the corresponding (frequently, 'huge') compact right topological *(Ellis) enveloping semigroup*. It is the pointwise closure of the set of translations $\{\tilde{g} \colon X \to X\}_{g \in G}$ in the product space X^X.

The enveloping semigroup $E(X)$ of a metric compact RN G-space X is a separable Rosenthal compact (hence, $\mathrm{card}(E(X)) \leq 2^{\aleph_0}$), [27].

Question 7.7 (Glasner and Megrelishvili)**.** *Is it true that for every compact metric* **1036 ?**
RN G-space X the enveloping semigroup $E(X)$ is metrizable?[7]

A function $f \in \mathrm{RUC}(G)$ is *Asplund*, notation: $f \in \mathrm{Asp}(G)$, if f is a (generalized) matrix coefficient of an Asplund representation $h \colon G \to \mathrm{Is}(V)$. This means that V is Asplund and there exists a pair of vectors $(v, \psi) \in V \times V^*$ such that $f(g) = \psi(g^{-1}v)$. Similarly, $\mathrm{WAP}(G)$ is the set of all matrix coefficients of reflexive representations. Recall that if $\mathrm{RUC}(G) = \mathrm{WAP}(G)$ then G is precompact [51].

[7] The answer is positive, see [29].

1037 ? **Question 7.8.** *Assume that* $\mathrm{RUC}(G) = \mathrm{Asp}(G)$. *Is it true that* G *is precompact?*

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