Seminar in Topology and Actions of Groups.

Metrization of topological groups.

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Abstract

It is a well known fact that a metric space satisfies the first axiom of countability. The opposite isn't always true. We show that for topological groups satisfying the first axiom of countability implies metrizability. In addition we give a counter example in the general case.

1 Topology Backgroud

We shall start by mentioning some basic definitions and theorems.

Definition 1.1 (Neighborhood Base). Let X be a topological space, $x \in X$. A collection $\mathscr{B}_x$ of neighborhoods of x is called *neighborhood base* of x, if and only if for every neighborhood N of x there exists a neighborhood $B \in \mathscr{B}_x$ such that $B \subseteq N$.

If every set $B \in \mathscr{B}_x$ is open then $\mathscr{B}_x$ is called *open neighborhood base.*

Definition 1.2 (First Axiom of Countability). A space X fulfill the *First Axiom of Countability* or is called *first countable* when for every $x \in X$ there exist a countable neighborhood base.

It is easy to ensure that:

Theorem 1.1. *Every metrizable space is first countable.*

Indeed, for every $x \in X$, we can take the collection of open spheres $D(x, 1/n)$ to be a countable neighborhood base of x.

Definition 1.3 (Second Axiom of Countability). A topological space X fulfill the *second Axiom of Countability* or is called *second countable* if and only if X's topology has a countable base.

2 Metrization of topological groups

Recall that a metric ρ on a group G is called *left invariant* if $\rho(ax, ay) = \rho(x, y)$ for each elements $a, x, y \in G$.

Theorem 2.1. *Let (G, T) be a topological group satisfying the first axiom of countability. Then there exists a left invariant metric ρ which generates the topology T.*

Proof. Define by induction a fundamental system $\{U_n\}_{n\in\mathbb{N}}$ of symmetric neighborhoods of e such that $U_1 = G$, $U_{n+1} \subseteq U_n$, for every $n \in \mathbb{N}$.
Consider the function: $f : G \times G \to \mathbb{R}$,

$$f(x,y) = \begin{cases} 0, & \text{if x = y;} \\ 2^{1-n}, & \text{if } n \text{ is the greatest natural number for which } x^{-1}y \in U_n \\ & \text{(equivalently, } x^{-1}y \in U_n \setminus U_{n+1}). \end{cases}$$

Put $\rho(x,y) = inf\{\sum_{i=0}^{n} f(x_i, x_{i+1}) : x_0 = x, \quad x_{n+1} = y, \quad n \in \mathbb{N}\}$.

Evidently, $\rho(x,y) \geq 0$ for each $x, y \in G$. Since U_n are symmetric, we obtain that $\rho(x,y) = \rho(y,x)$ for all $x, y \in G$.

We shall prove that $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for all $x, y, z \in G$.

Let $\rho(x,y) = a, \quad \rho(x,z) = b, \quad \rho(y,z) = c$. If $\epsilon > 0$, then there exist $n, m \in \mathbb{N}, x_0, \dots, x_{n+1}, y_0, \dots, y_{m+1} \in G$ such that $a \leq f(x_0, x_1) + \dots + f(x_n, x_{n+1}) < a + \frac{\epsilon}{2}$, $b \leq f(y_0, y_1) + \dots + f(y_m, y_{m+1}) < b + \frac{\epsilon}{2}$, $x_0 = x$, $x_{n+1} = y$, $y_0 = y$, $y_{n+1} = z$. Then $c \leq f(x_0, x_1) + \dots + f(x_n, x_{n+1}) + f(y_0, y_1) + \dots + f(y_m, y_{m+1}) < a + b + \epsilon \Rightarrow c \leq a + b$

Since $ax = ay$ implies that $x = y$ and $x^{-1}y = (ax)^{-1}(ay)$ we obtain that $\rho(ax, ay) = \rho(x,y)$ for all $a, x, y \in G$.

Claim: If $f(x_1, x_2) + f(x_2, x_3) + \dots + f(x_k, x_{k+1}) < 2^{1-n}$, then $x_1^{-1}x_{k+1} \in U_n$.

Induction on k. If $k = 1$, then $f(x_1, x_2) \leq 2^{1-n}$. Indeed, $x_1 = x_2$ the claim is obvious. If $x_1 \neq x_2$, then $f(x_1, x_2) = 2^{1-p} < 2^{1-n}$ for some $p \in \mathbb{N}$. Then $x_1^{-1}x_2 \subseteq U_p \subseteq U_n$.

Let $k \geq 2$ and $s = f(x_1, x_2) + f(x_2, x_3) + \dots + f(x_k, x_{k+1}) < 2^{1-n}$. Without loss of generality we may assume that $x_i \neq x_{i+1}$, $i = 1, \dots, k$.

Assume that $f(x_1, x_2) + f(x_2, x_3) + \dots + f(x_k, x_{k+1}) < 2^{-n} = 2^{1-(n+1)}$. Then $f(x_1, x_2) < 2^{-n}$ and $f(x_2, x_3) + \dots + f(x_k, x_{k+1}) < 2^{-n} \Rightarrow x_1^{-1}x_2 \in U_{n+1}$, $x_2^{-1}x_{k+1} \in U_{n+1} \Rightarrow x_1^{-1}x_{k+1} = x_1^{-1}x_2x_2^{-1}x_{k+1} \in U_{n+1} \subseteq U_n$.

Therefore we may assume that $2^{-n} \leq s < 2^{1-n}$.

Case I. There exists an i such that x_i is the last element with the property $f(x_1, x_2) + \dots + f(x_{i-1}, x_i) < 2^{-n}$, and $f(x_i, x_{i+1}) + \dots + f(x_k, x_{k+1}) < 2^{-n}$. According to the hypothesis $x_1^{-1}x_i \in U_{n+1}$, $x_i^{-1}x_{k+1} \in U_{n+1} \Rightarrow x_i^{-1}x_{k+1} = x_i^{-1}x_ix_i^{-1}x_{k+1} \in U_{n+1} \subseteq U_n$.

Case II. There exists an i, j such that x_i is the last element with the property $f(x_1, x_2) + f(x_2, x_3) + \dots + f(x_{i-1}, x_i) < 2^{-n}$, the number j is the last element for which $f(x_i, x_{i+1}) + \dots + f(x_{j-1}, x_j) < 2^{-n}$ and $f(x_j, x_{j+1}) + \dots + f(x_k, x_{k+1}) < 2^{-n}$.

According to the hypothesis $x_1^{-1}x_i \in U_{n+1}$ & $x_1^{-1}x_j \in U_{n+1}$ & $x_j^{-1}x_{k+1} \in U_{n+1} \Rightarrow x_1^{-1}x_{k+1} \in U_n$.

We affirm that there are no other cases.

Assume that $s = s_1 + s_2 + \ldots + s_m + \delta$, where numbers $s_1, \ldots, s_m, \delta$ are defined as it follows: $s_1 = f(x_1, x_2) + \ldots + f(x_{i-1}, x_i)$ and x_i is the last element for which this sum is less than 2^{-n}; $s_2 = f(x_i, x_{i+1}) + \ldots + f(x_{j-1}, x_j)$ and x_j is the last element for which this sum is $< 2^{-n}$. At least, if $s_m = f(x_t, x_{t+1}) + \ldots + f(x_{q-1}, x_q)$, then $\delta = f(x_q, x_{q+1}) + \ldots + f(x_k, x_{k+1}) < 2^{-n}$.

if $m = 3$ and $\delta = 0$, then this case is included in case II. Hence we may assume that $m = 3$ and $\delta > 0$ or that $m > 3$. In both cases $s_1 + s_2 \geq 2^{-n}, s_3 + \ldots s_m + \delta \geq 2^{-n}$. Then $s \geq 2 \cdot 2^{-n} = 2^{1-n}$, a contradiction.

We affirm that $\{x \in G : \rho(e, x) < 2^{1-n}\} \subseteq U_n$. Indeed, let $\rho(e, x) < 2^{1-n}$. Then there exist $x_1, \ldots, x_{k+1} \in G, k \in \mathbb{N}$ such that $x_1 = e, x_{k+1} = x$ and $f(x_1, x_2) + \ldots + f(x_k, x_{k+1}) < 2^{1-n}$. By the above assertion $e^{-1}x = x \in U_n$.

It is obvious that $e \in \{x \in G : \rho(e, x) < 2^{1-n}\}$. If $x \neq e, x \in U_{n+1}$, then $x = e^{-1}x \in U_{n+1} \Rightarrow f(e, x) = 2^{1-t}$, $t \geq n + 1$. But $2^{1-t} < 2^{1-n}$, hence $\rho(e, x) < 2^{1-n}$. We proved that $U_{n+1} \subseteq \{x \in G : \rho(e, x) < 2^{1-n}\}$.

□

3 Sorgenfrey Line - A Counter Example

For non topological groups, the opposite is not always true. Let us give a counterexample to illustrate this.

3.1 definition of Sorgenfrey Line

Definition 3.1 (Sorgenfrey Line). A topological space $(\mathbb{R}, \tau_s)$ on the set of all reals, where the family of the intervals $[a, b), \quad a < b \in \mathbb{R}$ forms the bases of the topology τ_s. We will denote that space as R_s.

It is easy to see that R_s is first countable. Indeed, for every real number a we shall take the intervals collection $\{[a, a + 1/n)\}_n$ to be a countable neighborhood base of a.

Theorem 3.1. *R_s is not metrizable.*

3.2 first proof

Proof. In order to prove that theorem we need some preliminaries.

Lemma 3.2. *If a space X is metrizable then $X \times X$ is also metrizable.*

Proof. (sketch) If d is the metrization of X, then

$$\hat{d}(< x_1, x_2 >, < y_1, y_2 >) := d(< x_1, y_1 >) + d(< x_2, y_2 >)$$

would be the metrization of $X \times X$ □

Definition 3.2 (Normal Space). A topological space is called *normal* if and only if every two disjoint closed sets can be separate by disjoint open sets. In a more formal manner, for any closed $A, B \quad A \cap B = \varnothing$ there exist open U_A and U_B such that $U_A \cap U_B = \varnothing, B \subseteq U_B, A \subseteq U_A$.

Theorem 3.3. *The product space $R_s \times R_s$ is not normal.*

Proof. For any $a =< a_1, a_2 >$ in $R_s \times R_s$, the half open squares

$$[a_1, a_1 + \epsilon) \times [a_2, a_2 + \epsilon)$$

are open neighborhood base of a.

Let us look at the subspace E over the secondary diagonal of $\mathbb{R}^2$, i.e. the line $y = -x$.

For any $x =< x, -x >\in E$, the half open square

$$[x, x + 1) \times [-x, -x + 1)$$

is an open neighborhood of x in $R_s \times R_s$ and its intersection with E is the singleton $\{x\}$, therefore every singleton in E is open in the subspace E, and therefore this is a discrete subspace: Every subset of E is closed in E. E itself is a closed subspace of $R_s \times R_s$ (since it is easy to see that E^c is open in $R_s \times R_s$). Therefore every subset of E is closed in $R_s \times R_s$.

Define P_s and Q_s as follows:

$$P_s := \{\mathbf{p} =< p, -p >: p \in P\}$$

$$Q_s := \{\mathbf{q} =< q, -q >: q \in Q\}$$

(Q and P are the real rational and irrational numbers)

P_s and Q_s are disjoint and closed in E and therefore also closed in $R_s \times R_s$.

Claim: P_s and Q_s don't have disjoint neighborhoods in $R_s \times R_s$.

Proof. Let us take two open sets U and V in $R_s \times R_s$ s.t. $P_s \subseteq U$ and $Q_s \subseteq V$. We will show that necessarily $U \cap V \neq \varnothing$.
For each $p \in P_s$ let's choose ϵ_p s.t. $[p, p + \epsilon_p) \times [-p, -p + \epsilon_p) \subseteq U$.
For every $n \in \mathbb{N}$ let $P_n \subseteq P$ be: $P_n := \{p \in P : \epsilon_p \geq 1/n\}$.
Evidently, $P = \cup_{n=1}^{\infty} P_n$.

P is second category in the complete space $\mathbb{R}$, therefore there is a natural k s.t. P_k isn't nowhere dense in $\mathbb{R}$. Let choose such k and let (a, b) be an open interval in which P_k is dense. Let choose a rational point q in this interval. For $\mathbf{q} =< q, -q >$, $\mathbf{q} \in Q_s \subseteq V \subseteq R_s \times R_s$. Therefore, there exist an ϵ for which $M_q(\epsilon) := [q, q+\epsilon) \times [-q, -q+\epsilon) \subseteq V$. In the interval (a, b) there are points p from P_k as closed as we want to q for which the square $M_p(1/k)$:

$$M_p(1/k) := [p, p+1/k) \times [-p, -p+1/k) \subseteq [p, p+\epsilon_p) \times [-p, -p+\epsilon_p) \subseteq U$$

If we shall get close enough to q we will get using this way a square $M_p(1/k)$ which meets $M_q(\epsilon)$.
Therefore $U \cap V \neq \varnothing$. □

Therefore there exist two closed disjoint sets which can't be separate and it yields that $R_s \times R_s$ is not normal. □

Having proved the above we can now finish the proof of our counter example.
$R_s \times R_s$ is not normal and therefore is not metrizable. As a result from this and from lemma 3.2 R_s is not metrizable. □

3.3 second proof

There is another simple and elegant proof for theorem 3.1. The following theorem is well known:

Theorem 3.4. *A metric space is separable if and only if it is a second countable space.*

Theorem 3.5. *R_s is not second countable.*

Proof. Assume that R_s is a second countable space. Let $\mathscr{B} = \{[a, b) : a < b\}$ be its base. By our assumption one can choose a sequence $< [a_k, b_k) >_k$ from $\mathscr{B}$ which is also a base of R_s's topology. Now choose $a \in \mathbb{R}$ s.t. for every $k, a \neq a_k$ and let $b > a$. We show that the open set $[a, b)$ is not a union of sets from the base $< [a_k, b_k) >_k$.
Assume that there exist a $K \subseteq N$ s.t.
(*) $[a, b) = \cup_{k \in K}[a_k, b_k)$, then there exist a $k \in K$ s.t. $a \in [a_k, b_k)$, i.e. $a_k \leq a < b_k$. Since $a \neq a_k, a_k < a$, therefore there exist a real number c s.t. $a_k < c < a$ but $c \in [a_k, b_k)$ contrary to (*). □

Corollary 3.6. R_s is not metrizable.

Proof. it is separable since the rational numbers are dense in it. Now apply Theorem 3.4. □

Corollary 3.7. The *sorgenfrey line* is not a topological group for any group operation on $\mathbb{R}$.

4 A complete monothetic non locally compact group example

Example 1. The circle group $\mathbb{T}$ is a compact metric and monothetic.

Hint: for every α such that α/π is irrational the complex number $z := cis\alpha$ generates an infinite cyclic subgroup $H := \{z^n\}_{n\in Z}$ which is dense in $\mathbb{T}$.

In order to construct an example of a complete monothetic non locally compact group we recall the following result of Kronecker: Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be independent over rationals. Then for the real numbers $x_1, x_2, \dots, x_n$ and $\epsilon > 0$ there exist $q \in \mathbb{Z}$ and n numbers $p_j \in \mathbb{Z}$ such that $|q\lambda_j - p_j - x_j| \leq \epsilon (1 \leq j \leq n)$.
Let G be the sequence $z = \{z_n\}$, where $z_n \in \mathbb{R}/\mathbb{Z}$ with $z_n \to 0$.

Recall that if $\bar{x} = a + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$, then $||\bar{x}|| = inf\{|a+n| : n \in \mathbb{Z}\}$.

Define the addition $\{x_n\} + \{y_n\} = \{x_n + y_n\}$ and the distance between two elements $\{x_n\}$ and $\{y_n\}$ by $d(x,y) = max||x_n - y_n||$.

Then d is an invariant metric. We will prove that G is complete.

Let $\{z^{(i)}\}$ be a Cauchy sequence in G, where $\{z_1^{(i)}, \dots, z_k^{(i)}, \dots\}$. It follows immediately from the definition of the metric d that for each k the sequence $z_k^{(i)}$ is a Cauchy sequence. Denote $z_k = \lim {z_k}^{(i)}$.

We note that $z \in G$: Let $\epsilon > 0$. There exist $n_0 \in \mathbb{N}$ s.t. $d(z^{(i)}, z^{(j)}) < \epsilon$ for $i, j \geq n_0$. Choose $m_0 \geq n_0$ s.t. $||z_s^{(n_0)}|| < \epsilon$ for each $s \geq m_0$. Since $d(z^{(l)}, z^{(n_0)}) < \epsilon$ for each $l \geq m_0$, we obtain $||z_q^{(l)} - z_q^{(n_0)}|| < \epsilon$ for each $q \in \mathbb{N}$. If $l \to \infty$, we get $||z_q - z_q^{(n_0)}|| \leq \epsilon \Rightarrow ||z_q|| \leq 2\epsilon$ for each $q \geq m_0$, hence $z_n \to 0 \Rightarrow z \in G$.

We affirm that $z = \lim z^{(i)}$. Indeed, let $\epsilon > 0$. There exists $n_0 \in \mathbb{N}$ s.t. $d(z^{(i)}, z^{(j)}) < \epsilon$ for $i, j \geq n_0$. Therefore for each $q \in \mathbb{N}$, $||z_q^{(i)} - z_q^{(j)}|| < \epsilon \Rightarrow ||z_q^{(i)} - z_q|| \leq \epsilon \Rightarrow$ for each $i \geq n_0 \Rightarrow d(z^{(i)}, z) \leq \epsilon$ for each $i \geq n_0 \Rightarrow z = \lim z^{(i)}$.

Now we shall construct by induction integers $k_1, k_2, \dots$ and numbers $\lambda_1, \lambda_2, \dots$ linearly independent over $\mathbb{Q}$. Denote $\bar{\lambda_i} = \lambda_i + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}, i \in \mathbb{N}$.

If $n \in \mathbb{N}, y = \{y_i\} \in G$, put $P_n y = \{y_1, \dots, y_n, 0, \dots\}$. Put $k_1 = 1$ and choose λ_1 s.t. $|\lambda_1| \leq \frac{1}{2}, \lambda_2$ is non-zero. Assume that we constructed

$\lambda_1, \ldots, \lambda_{n-1}$ and $k_1, \ldots, k_{n-1}$.

Define k_n to be the smallest integer s.t. $d(i\{\bar{\lambda_1}, \ldots, \bar{\lambda_{n-1}}, 0, \ldots\}, P_{n-1}y) \leq \frac{1}{2^{n-1}}$ is satisfied for each $y \in G$ and suitable $i \in [0, k_n]$. the existence of k_n follows from the theorem of Kronecker. Choose $\lambda_n \in \mathbb{R}$ s.t. $\{\lambda_1, \ldots, \lambda_n\}$ is linearly independent over $\mathbb{Q}$ and $|\lambda_n| \leq \frac{1}{2^n k_n}$. Put $x = \{\lambda_1, \bar{\lambda_2}\}$.

We observe that $k_n \leq k_{n+1}$ for all $n \in \mathbb{N}$. Assume the contrary: then for each $y \in G$ there exist $i \in [0, k_n]$ s.t. $d(iP_n x, P_n y) \leq \frac{1}{2^n} < \frac{1}{2^{n-1}}$. Since $d(iP_{n-1}x, P_{n-1}y) \leq d(iP_n x, P_n y) \leq \frac{1}{2^{n-1}}$ we obtain a contradiction with the definition of k_n.

We affirm that x is topological generator of G. Let y be an element of G and $\epsilon > 0$. There exist $n_0 \in \mathbb{N}$ s.t. $d(y, P_n y) < \frac{\epsilon}{3}$ for all $n > n_0$. Choose $n > n_0$ so as to have $\frac{1}{2^n} < \frac{\epsilon}{3}$. There exists $i \in [0, k_{n+1}]$ s.t. $d(iP_n x, P_n y) \leq \frac{1}{2^n}$. For $m > n ||i\bar{\lambda_m}|| \leq i||\bar{\lambda_m}|| \leq i|\lambda_m| \leq \frac{k_{n+1}}{2^m k_m} \leq \frac{k_m}{2^m k_m} = \frac{1}{2^m} < \frac{1}{2^n}$. It follows that $d(ix, P_n x) \leq \frac{1}{2^n}$. therefore $d(ix, y) \leq d(ix, iP_n x) + d(iP_n x, P_n y) + d(P_n y, y) < \epsilon$. Now we note that G is not locally compact. Since G is monothetic, it suffices to show that G is neither compact nor discrete. Obviously, G is not discrete. On the other hand G is not compact. Indeed, consider the topological product $\Pi_{n=1}^{\infty} G_n, G_n = \mathbb{R}/\mathbb{Z}$ for each $n \in \mathbb{N}$. Then G as abstract group is a dense subgroup of $\Pi_{n=1}^{\infty} G_n$ and the topology induced from $\Pi_{n=1}^{\infty} G_n$ is weaker that the topology defined above. Therefore G is not compact.

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