INTERVAL MAPS

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ABSTRACT. In this chapter we shall concentrate on the special case of continuous maps on the closed interval I.
This level of specialization allows us to prove some particularly striking results on periodic points and topological entropy.

1. ORBITS

1.1. **Iteration.** There are many kinds of problems in science and involve *iteration.* Iteration means to repeat a process over and over.In dynamics ,the process that is repeated is the application.
We wiil consider only functions of one variable as encountered in elementary calculus. We wiil spend quite a bit of time discussing the *quadratic* functions $Q_c(x) = x^2 + c$ where $c \in R$ (a real number) is a constant.Other functions that will arise often are the *logistic* functions $F_\lambda(x) = \lambda x(1-x)$, the *exponentials* $E_\lambda(x) = \lambda e^x$, and the sine functions $S_\mu(x) = \mu sinx$. Here λ and μ are constant . The constants c, μ , and λ are called *parameters.* One of the important questions we will address later is how the dynamics of these functions change as these parameters are varied.
To iterate a function means to evaluate the function over and over , using the output of the previous application as the input for the next .This is the same process as typing a number into a scientific calculator.We write this as follows. For a function $F, F^2(x)$, is the second iterate of F, namely $F(F(x)), F^3(x)$ is the third iterate $F(F(F(x)))$, and ,in general , $F^n(x)$ is the n-fold composition of F with itself.For example, if $F(x) = x^2 + 1$ then $F_2(x) = (x^2+1)^2 + 1$ and $F^3(x) = ((x^2+1)^2+1)^2 + 1$. Similarly , if $F(x) = \sqrt{x}$, then $F^2(x) = \sqrt{\sqrt{x}}$and $F^3(x) = \sqrt{\sqrt{\sqrt{x}}}$. It importent to realize that $F^n(x)$ does not mean raise $F(x)$ to the nth power(an operation we wiil never use).Rather ,$F^n(x)$ is the iterate of F evaluated at x.

1.2. **Orbit.** Given $X_0 \in R$, we define the *orbit* of x_0 under F to be the sequence of points $x_0, x_1 = F(x_0)$, $x_2 = F^2(x_0)$, ... , $x_n = F^n(x_0)$, ... The point x_0 us called the *seed* of the orbit.
For example, if $F(x) = \sqrt{x}$ and $x_0 = 256$, the first few points on the orbit of x_0 are : $x_0 = 256$,$x_1 = \sqrt{256} = 16$,$x_2 = \sqrt{16} = 4$,$x_3 = \sqrt{4} = 2$,$x_4 = \sqrt{2} = 1.41\ldots$

An another example,if $S(x) = sinx$ the orbit of $x_0 = 123$ is $x_0 = 123$,$x_1 = -0.4599\ldots$,$x_2 = -0.4438\ldots \quad \cdots \quad x_{300} = -0.0975\ldots$,$x_{301} = -0.0974\ldots\cdots$. Slowly, ever so slowly ,the points on this orbit tend to 0 .If $C(x) = cosx$,then orbit of $x_0 = 123$ is $x_0 = 123$,$x_1 = -0.8879\ldots$, $\cdots \quad x_{50} = 0.739085$, $x_{51} = 0.739085$, $x_{52} = 0.739085, \cdots$.

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1.3. **Types of Orbits.** There are many different kinds of orbits in a typical dynamical system.Undoubtedly the most important kind of orbit is a *fixed point.*A fixed point is a point x_0 that satisfies $F(x_0) = x_0$. Note that $F^2(x_0) = F(F(x_0)) = F(x_0) = x_0$ and , in general,$F^n(x_0)$.So the orbit of a fixed point is the constant sequence $x_0, x_0, x_0 \ldots$A fixed point never moves.As its name implies , it is fixed by the function . For example, 0 , 1 , and -1 , are all fixed points for $F(x) = x^3$,while only 0 and 1 are fixed points for $F(x) = x^2$.Fixed points are found by solving the equation $F(x) = x$.Thus ,$F(x) = x^2 - x - 4$ has fixed points at solutions of $x^2 - x - 4 = x$,which are $1 \pm \sqrt{5}$ as determined by the qurdratic formula.Fixed points may also be found geometrically by examining the intersection of the graph with the diagonal line $y = x$.For example $S(x) = sinx$ is $x_0 = 0$, since that is the only point of intersection of the graph of S with the diagonal $y = x$.Similarly, $C(x) = cosx$ has a fixed point at $0.739085\ldots$

Another important kind of orbit is the *periodic orbit or cycle.*The point x_0 is periodic if $F^n(x_0) = x_0$ for some $n > 0$.The least such n is called the *prime periodic* of the orbit. Note that if x_0 is periodic with prime period n, then the orbit of x_0 is just a repeating sequence of numbers $x_0, F(x_0), \ldots, F^{n-1}(x_0), x_0, F(x_0), \ldots, F^{n-1}(x_0), \ldots$ For example, 0 lies on a cycle of prime period 2 for $F(x) = x^2 - 1, sinceF(0) = -1, andF(-1) = 0$ .Thus the orbit of 0 is simply 0,-1,0,-1,0,-1,... We also say that 0 and -1 from a 2-cycle. Similarly , 0 lies on a periodic orbit of prime period 3 or a 3-cycle for $F(x) = \frac{-3}{2}x^2 + \frac{5}{2}x + 1$,since $F(0) = 1, F(1) = 2, andF(2) = 0$.So the orbit is 0,1,2,0,1,2,... We will see much later that the appearance of this seemingly harmless 3-cycle has surprising implications for the dynamics of this function.In general,it is very difficult to find periodic points exactly.For example to find cycles of period 5 for $F(x) = x^2 - 2$,we would have to solve the equation $F^5(x) - x = 0$ Note that if x_0 has prime period k,then x_0 is also fixed by F^{2k}.Indeed $F^{2k}(x_0) = F^k(F^k)(x_0) = F^k(x_0) = x_0$Similarly x_0 is fixed by F^{nk},so we say that $x_0$has period nk for any positive integer n.We reserve the word prime period for the case n=1.Also,$x_0$lies on a periodic orbit of period k,then all points on the orbit of x_0 have period k as well.Indeed, the orbit of x_1 is $x_1, x_2, \ldots, x_{k-1}, x_0, x_1, \ldots, x_{k-1}, x_0, x_1 \ldots$which has period k.A point x_0 is called *eventually* fixed or eventually periodic if x_0 itself is not fixed or periodic, but some point on the orbit of x_0 is fixed or periodic.For example, -1 is eventually fixed for $F(x) = x^2, sinceF(-1) = 1$,which is fixed.Similarly,1 is eventually periodic for $F(x) = x^2 - 1 sinceF(1) = 0$,which lies on a cycle of period 2.The point$\sqrt{2}$is also eventually periodic for this function,since the orbit is $\sqrt{2}$,1,0,-1,0,-1,0,-1,...

In a typical dynamical system, most orbits are not fixed or periodic.For example,for the linear function $T(x) = 2x$,only 0 is a fixed point.All other orbits of T get larger and larger(in absolute value)under iteration since $T^n(x_0) = 2^n x_0$ Indeed,if $x_0 \neq 0$,$|T^n(x_0)|$ tends infinity as n approaches infinity .We denote this by $|T^n(x_0)| \to \infty$. the situation is reversed for the linear function $L(x) = \frac{1}{2}x$.For L,only 0 is fixed,but for any $x_0 \neq 0$,$L^n(x_0) = \frac{x_0}{2^n}$.we have $L^n(x_0) \to 0$.We say that the orbit of x_0 converges to the fixed point 0.As another example,consider the squaring function $F(x) = x^2$.If $|x_0| < 1$,it is easy to check that $F^n(x_0) \to 0$.For example if $x_0 = 0.1$,then the orbit of x_0 is $0.1, 0.01, 0.0001, \ldots, 10^{-2^n}, \ldots$, which clearly tends to zero.

2. Graphical Analysis

In this section we introduce a geometric procedure that will help us understand the dynamics of one-dimensional mappings.This procedure,called *graphical analysis*,enables us to use the graph of a function to determine the behavior of orbits in many cases.Suppose we have the graph of a function F and wish to display the orbit of a given point x_0.We begin by superimposing the diagonal line y=x on the graph of F.As we saw the points of intersection of the diagonal with the graph give us the fixed points of F.To find the orbit

of x_0,we begin at the point (x_0, x_0) on the diagonal directly above x_0 on the x-axis.We first draw a vertical line to the graph of F.When this line meets the graph,we have reached the point $(x_0, F(x_0))$.We then draw a horizontal line from this point to the diagonal.We reach the diagonal at the point whose y-coordinateis $F(x_0)$,and so the x-coordinate is $F(x_0)$,the next point on the orbit of x_0.Now we continue this procedure.Draw a vertical line from $(F(x_0), F(x_0))$ on the diagonal to the graph:this yields the point $(F(x_0), F^2(x_0))$. Then a horizontal line to the diagonal reaches the diagonal at $(F^2x_0), F^2(x_0))$,directly above the next point in the orbit.To display the orbit of x_0 geometrically,we thus continue this procedure:we first draw a vertical line from the diagonal to the graph,then a horizontal line from the graph back to the diagonal.The resulting "staircase" or "cobweb" provides an illustrative picture of the orbit of x_0. In * we shows a typical application of graphical analysis.This procedure may be used to describe some of the dynamical behavior we saw in the previous section.In * For example,we sketch graphical analysis of $F(x) = \sqrt{x}$.Note that any positive x_0 gives a staircase which leads to the point of intersection of the graph of F with the diagonal.This is,of course,the fixed point at x=1.* We depict graphical analysis of $C(x) = cosx$.Note that any orbit in this case tends again to the point of intersection of th!
e graph

of C with the diagonal.As we observed numerically in the previous section,this point is given approximately by $0.73908\ldots$ *As we saw,periodic points for F satisfy $F^n(x_0) = x_0$.This means that the line segments generated by graphical analysis eventually return to (x_0, x_0) on the diagonal,thus yielding a closed "cicuit" in the graphical analysis.we can shows * that $F(x) = x^2 - 1.1$ a 2-cycle as illustarted by the square generated by graphical analysis.and we can shows * that many orbits tend to the cycle.We cannot decipher the behavior of all orbits by means of graphical analysis.For example,we have applied graphical analysis to the quadratic function $F(x) = 4x(1 - x)$.Note how complicated the orbit of x_0 is! This is another glimps of chaotic behavior.*

2.1. **Orbits Analysis.** Graphical analysis sometimes allows us to describe the behavior of all orbits of a dynamical system.For example,consider the function $F(X) = x^3$.The graph of F shows that there are three fixed points : at 0,1and -1.Graphical analysis then allows us to read off the following behavior.If * $|x_0| < 1$ then the orbit of x_0 tends to zero.On the other hand,If $|x_0| > 1$,then the orbit of x_0 tends to $\pm\infty$.

2.2. **phase portrait.** One succinct method for all orbits of a dynamical system is the *phase portrait* of the system.In the phase portrait,we represent fixed points by solid dots and the dynamics along orbits by arrows.For example,as we saw above,for $F(x) = x^3$,the fixed point occur at ± 1.If $|x_0| < 1$,then $F^n(x_0) \to 0$,whereas if $|x_0| > 1$,$F^n(x_0) \to \pm\infty$. * As another example,$F(x) = x^2$ has two fixed point,at 0 and 1,and an eventually fixed point at -1.Note that if $x_0 < 0$,then $F(x_0) > 0$ and all subsequent points on the orbit of x_0 are positive. *

3. A Fixed Point Theorem

3.1. **The Intermediate Value Theorem.** Suppose $F : [a, b] \to R$ is continuous. Suppose y_0 lies between $F(a)$ and $F(b)$. Then there is an x_0 in the interval [a,b] with $F(x_0) = y_0$.

3.2. **Fixed Point Theorem.** . Suppose $F : [a, b] \to [a, b]$ is continuous. Then there is a fixed point for F in [a,b]

Definition. Suppose x_0 is a fixed point for F. Then x_0 is an *attracting fixed point* if $|F'(x_0)| < 1$. the point x_0 is a *repelling fixed point* if $|F'(x_0)| > 1$. Finally if $|F'(x_0)| = 1$,the fixed point is called *neutral or indifferent.* The geometric rationale for this terminology is supplied by graphical analysis. Consider the graphs in the Figure *,Both of these functions

have fixed points at x_0. The slope of the tangent line at x_0,$F'(x_0)$, is an both cases less than 1 in magnitude: $|F'(x_0)| < 1$. Note that this forces nearby orbits to approach x_0, just as the linear cases above. If $-1 < F'(x_0) < 0$, the orbit hops from one side to the other as it approaches x_0. The phase portraits in the two cases $0 < F'(x_0) < 1$ and $-1 < F'(x_0) < 0$ are sketched in the Figure*. On the other hand if $|F'(x_0)| > 1$, graphical analysis shows that nearby points have orbits that move farther away, that is are repelled. Again, if if $|F'(x_0)| < -1$, orbits oscillate from side to side of x_0 as they move away. As an example, consider the function $F(x) = 2x(1-x) = 2x - 2x^2$. Clearly,0 and $\frac{1}{2}$ are fixed points for F. We have $F'(x) = 2 - 4x$,so $F'(0) = 2$ and $F'(\frac{1}{2}) = 0$. Thus 0 is a repelling fixed point, while $\frac{1}{2}$ is attracting.Graphical analysis confirms this.*
Why is true?The answer is provided by the theorems *The Mean Value Theorem*.Suppose F is a differentiable function on the interval $a \leq x \leq b$.Then there exists c between a and b for which the following equation is true:$F'(c) = \frac{F(a)-F(b)}{b-a}$. The content of this theorem is best exhibited geometrically.The quantity $M = \frac{F(a)-F(b)}{b-a}$ is the slope of the straight line connecting the two points $(a, F(a))$ and $(b, F(b))$ on the graph of F.So the theorem simply says that,provided F is differentiable on the interval $a \leq x \leq b$,there is some point c between a and b at which the slope of the tangent line,$F'(c)$,is exactly equal to M.*

3.3. ***Attracting Fixed Point Theorem.*** Suppose x_0 is an *attracting fixed point* for F.Then there is an interval I that contains x_0 in its interior and in which the following condition is satisfied :$if x \in I$, then $F^n(x) \in I$ for all n and,moreover,$F^n(x) \to x_0$ as $n \to \infty$
PROOF.Since $|F'(x_0)| < 1$,there is a number $\lambda > 0$ such that $|F'(x_0)| < \lambda < 1$.We may therefore choose a number $\delta > 0$ so that $|F'(x)| < \lambda$ provided x belongs to the interval $I = [x_0 - \delta, x_0 + \delta]$.Now let p be any point in I.By the *Mean Value Theorem* $\frac{F(p)-F(x_0)}{p-x_0} < \lambda$,so that $|F(p) - F(x_0)| < \lambda|p - x_0|$.Since x_0 is a fixed point,it follows that $|F(p) - x_0| < \lambda|p - x_0|$.This means that the distance from $F(p)$ to x_0 is smaller than the distance from p to x_0,since $0 < \lambda < 1$.In particular,$F(p)$ also lies in the interval I.Therefor we may apply the same argument to $F(p)$ and $F(x_0)$,finding $|F^2(p) - x_0| = |F^2(p) - F^2(x_0)| < \lambda|F(p) - F(x_0)| < \lambda^2|p - x_0|$. Since $\lambda < 1$,we have $\lambda^2 < \lambda$.This means that the points $F^2(p)$ and x_0 are even closer toghether than $F(p)$ and x_0.Thus we may continue using this argument to find that,for any $n > 0$, $|F^n(p) - x_0| < \lambda^n|x - x_0|$ Now $\lambda^n \to 0$ as $n \to \infty$. Thus,$F^n(p) \to x_0$ as $n \to \infty$.This completes the proof.

3.4. Repelling Fixed Point Theorem Suppose x_0 is a repelling fixed point for F.Then there is an interval I that contains x_0 in its interior and in which the following condition is satisfied :$if x \in I$and $x \neq x_0$, then there is an integer $n > 0$ such that $F^n(x) \notin I$.These two theorems combined justify our use of the terminology "attracting"' and "repelling" to describe the corresponding fixed points.In particular,they tell us the "local" dynamics near any fixed point x_0 for which $|F'(x_0)| \neq 1$. One major difference between attracting and repelling fixed points is the fact that attracting points are "visible" on the computer,whereas repelling fixed points generally are not.We can often find an attracting fixed point by choosing an initial seed randomly and computing its orbit numerically.If this orbit ever enters the interval I about an attracting fixed point,then we know the fate of this orbit-it necessarily converges to the attracting fixed point.On the other hand,in the case of the repelling fixed point, the randomly chosen orbit would have to land exactly on the fixed point in order for us to see it.This rarely happens,for even if the orbit comes very close to a repelling fixed point,roundoff error will throw us off this fixed point and onto an orbit that moves away.The situation for a neutral fixed point is not nearly as simple as the attracting or reppeling cases.For example,the identity function $F(x) = x$ fixes all points,but none are attracting or repelling.Also,$F(x) = -x$ fixes zero, but this is not an attracting or repelling fixed point

since all other points lie on cycles of period 2.Finally, as in * ,$F(x) = x-x^2$ has a fixed point at zero,which is attracting from the right but repelling from the left.Note that $|F'(0)| = 1$ .in all three cases.On the other hand,neutral fixed points may attract or repel all nearby orbits.For example,graphical analysis shows that $F(x) = x - x^3$ has a fixed point that attracts the orbit of any x with $|x| < 1!$, wher
eas $F(x) = x + x^3$ repels all orbits away from 0.These fixed points are sometimes called *weakly* attracting or *weakly* repelling,since the converegence or divergenceis quite slow.

4. Fixed point and periodic points

Let $T : I \to I$ be a continuous map of the interval I=[0,1] to itself.Recall that a fixed point $x \in I$ satisfies $Tx = x$ and that a periodic point (of periodic n) satisfies $T^n x = x$. We say that x has prime period n if n is the smallest positive integer with this property (i.e. $T^k x \neq x$ for $k = 1, ..., n-1$).
For interval maps a very simple visualization of fixed points exists. We can draw the graph G_T of $T : I \to I$ and the digonal $D = (x, x) : x \in I$.

Lemma 4.1. *The fixed points Tx=x occur at the intersection points $(x, x) \in G_T \bigcap D$ (see figure 4.1). Similarly, if for $n \geq 2$ we look for intersections of the graph G_{T^n} (of n compositions $T^n : I \to I$) with the diagonal D then the intersection points $(x, x) \in G_{T^n} \cap D$ are periodic points of period n.*

Lemma 4.2. *Assume that we have an interval $J \subset I$ with $T(J) \supset J$, then there exists a fixed point $Tx = x \in J$.*

PROOF.We see that showing the restriction of the graph G_T to the portion above J intersects the diagonal D.
This is obvious by the intermediate value theorem and figure 4.1.

Lemma 4.3. *If $T : I \to I$ is a continuous map and $J_1, J_2 \in I$ are (closed) sub-intervals with $T(J_1) \supset J_2$ then we can choos a sub-interval $J_0 \subset J_1$ with $T(J_0) = J_2$.*

Proof. Let $J_2 = [a, b]$ and introduce the disjoint closed sets $A = \{x \in J_1 : T(x) = a\}$.and $B = \{y \in J_1 : T(Y) = b\}$. Choose $a' \in A, b' \in B$ .such that $|a' - b'| = inf\{|x - y| : x \in A, y \in B\}$. then with $J_0 = [a', b'] or J_0 = [b', a']$ the results follows. □

Theorem 4.4. *Let $T : I \to I$ be a continuous map and suppose there exists a periodic point x of prime period 3.Then for all $n \geq 1$ there exists a periodic point of prime period n (i.e $\forall n \geq 1, \exists z \in I$ with $T^n z = z$).*

Proof. We shall do the simpler case $n = 1$ and the trickier case $n \geq 2$ separately . □

(I) Existence of a fixed point (i.e $n = 1$). Let x, Tx, T^2x be the three distinct points in the orbit of x . Let us assume (for simplicity) that $x < Tx < T^2x$ (The five other permutations are easy to derive from this case either by replacing x by Tx or T^2x , or by reversing the horizontal axis of the graph of T.). Let $J = [x, T^2x]$,then we can write $J = J' \bigcup J''$ with $J' = [x, Tx]$ and $J'' = [Tx, T^2x]$.
With these choices we have

(1) $T(J') \supset J''$,and

(2) $T(J'') \supset J$

since the endpoints of the intervals J' and J'' are mapped to the endpoints of J' and J ,respectively , and the continuous image of an interval is again an interval . The existence of a fixed point in J'' now follows immediately from Lemma 4.2 and (1) since $T(J'') \supset J \supset J''$.
(II) Existence of a point of period $n \geq 2$. Since by the hypothesis of the theorem we already have a point x of prime period 3 we shall assume henceforth that $n \neq 3$.
SUBLEMMA 4.4.1. There exists a nested sequence of intervals

$$J" = I_0 \supset I_1 \supset I_2 \supset \ldots \supset I_{n-2} \supset I_{n-1}$$

with the following properties :
$(i) I_k = T(I_{k+1})$ for $k = 0, \ldots, n-2$.
$(ii) T^{n-1}(I_{n-1}) = J'$, and
$(iii) T^n(I_{n-1}) \supset J"$.
To see that Sublemma 4.4.1implies Theorem 4.4 we first observe that by part (iii) we have that $T^n(I_{n-1}) \supset J" \supset I_{n-1}$ and so applying Lemma 4.2 (with T replaced by the n-fold composition T^n) shows the existence of a fixed point $z = T^n z \in I_{n-1}$ for T^n (i.e. z is a point of period n for $T : I \to I$). However, we still have to show that this is a periodic point of prime period n.We see from Sublemma 4.4.1 that
$z, Tz, T^2z, \ldots, T^{n-2}z \in J"$(by part (i)since $T^i z \in T^i(I_{n-1}) = I_{n-i-2}$),$T^{n-1}z \in J'$ (by part (ii)) .
To proceed we want to eliminate the possibility that $z = Tx(\in J' \cap J")$. Assume for a contradiction that $z = Tx$, then we immediately have $T^3z = z$ (since $T^3x = x$). However, this contradicts our assumption that $n \neq 3$. In particular , this means that in (4.1) we can " improve " the second conclusion to $T^{n-1}z \notin J"$ We are now in a position to see that n is the prime period of z . If this were not the case , then $T^k z = z$ for some $1 \leq k \leq n-1$ (which must divide n).But this would mean , in particular, that $T^{k-1}z = T^{n-1}z \notin J"$ which is inconsistent with the first line of (4.1)
The only thing that remains in order to complete the proof of theorem 4.4 is to prove Sublemma 4.4.1 .
PROOF OF SUBLEMMA 4.4.1 .
(i) We know from (b) that $T(J") \supset J \supset J"$ and so by Lemma 4.3 we can choose $I_1 \subset J"$ with $T(I_1) = J"$ Similarly , since $T(I_1) = J" \supset I_1$ we can apply Lemma 4.3 again to choose $I_2 \subset I_1$ with $T(I_2) = I_1$.
Proceeding inductively , we can construct a sequence $J" \supset I_1 \supset I_2 \supset \ldots \supset I_{n-2}$ with $T(I_k) = I_{k-1}$ for $k = 1, 2, \ldots, n-2$ (which , in particular , implies that $T^k(I_k) = J"$ for $k = 1, 2, \ldots, n-2$).
(ii) To construct I_{n-1} , observe that $T^{n-1}(I_{n-2}) = T(J") \supset J'$ by (b). Applying Lemma 4.3 we can find $I_{n-1} \subset In-2$ with $T^{n-1}(I_{n-1}) = J'$.
(iii) Finally,we observe that $T^n(I_{n-1}) = T(J') \supset J"$ (by (a)).
This completes the proof of the sublemma (and consequently of Theorem 4.4).
This result that a point of prime period 3 implies points of all possible prime periods is a special case of a more general result due to Sharkovski .We can introduce a new ordering on the natural numbers N by

$$\begin{array}{l}
\quad 3 \prec 5 \prec 7 \prec 9 \prec \ldots 2m+1 \prec \ldots \\
\ldots \prec 6 \prec 10 \prec 14 \prec 18 \prec 22 \prec \ldots \prec 2(2m+1) \prec\prec \ldots \\
\ldots \prec 12 \prec 20 \prec 28 \prec 36 \prec 44 \prec \ldots \prec 4(2m+1) \prec \ldots \\
\quad \ldots \quad \ldots \quad \ldots \\
\ldots \prec 2^r \cdot 3 \prec 2^r \cdot 5 \prec 2^r \cdot 7 \prec 2^r \cdot 9 \prec 2^r \cdot 11 \prec \ldots \prec 2^r(2m+1) \prec \ldots \\
\quad \ldots \quad \ldots \quad \ldots
\end{array}$$

$\ldots \prec 2^{r+1} \prec 2^r \prec 2^{r-1} \prec \ldots \prec 16 \prec 8 \prec 4 \prec 2 \prec 1$
This ordering is clearly somewhat different from the usual ordering on the natural numbers . For example , the ordering of the first dozen natural numbers becomes $3 \prec 5 \prec 7 \prec 9 \prec 11 \prec 6 \prec 10 \prec 12 \prec 8 \prec 4 \prec 2 \prec 1$.

Theorem 4.5. *SHARKOVSKI.Let $T : I \to I$ be a continuous map and assume that T has a point of prime period n.Then for each $m > n$ (with respect to the above ordering)there exist periodic points of prime period m.*
The proof of Sharkovski's theorem runs along similar lines to that of Theorem 4.4.

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