NORM, STRONG, AND WEAK OPERATOR TOPOLOGIES ON $B(H)$

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In this lecture we will consider 3 topologies on $B(H)$, the space of bounded linear operators on a Hilbert space H.
In sections 1 and 2, we shall be reminded of some definitions and basic properties and also see some new ones, that we shall use in what follows.
In section 3, we shall be aquainted with $C^* algebras$ and some very basic properties of them.
Then, after these preparations, we shall see in section 4 some nice theorems concerning the 3 topologies mentioned above (in section 2) on $B(H)$.

1. Topologies on a Hilbert space H.

A Hilbert space has two useful topologies, which are defined as follows:

Definition 1.1. (1) The strong or norm topology:
Since a Hilbert space has, by definition, an inner product $<,>$, that inner product induces a norm, and that norm induces a metric.
So our Hlilbert space is a metric space.
The strong or norm topology is that metric topology.
A subbase, as always in a metric space, is the collection of all sets of the form:
$O(x_0; \varepsilon) := B_\varepsilon(x_0)$
Which is, in fact, a base for the metric topology.

(2) The weak topology:
A subbase for the weak topology is the collection of all sets of the form:
$O(x_0; y, \varepsilon) := \{x \in H : \ | < x - x_0, y > | < \varepsilon\}$

The names "strong" and "weak" are not arbitrary, as the following shows:

Proposition 1.2. *The weak topology is weaker then the strong topology.*

Proof. It is enough to prove that a weakly closed set is strongly closed.
Then, using nets ([3]Theorem 1.13 which says that a subset of a topological space is closed $\Leftrightarrow$ it is "netly" closed) ,and the Cauchy-Schwartz inequality,
the proof is done. □

The corresponding concepts of convergence for sequences (and nets)
can be described easily:

Proposition 1.3. *For a net $x_\lambda \subseteq H$:*
(1) In the strong topology: $x_\lambda \to x \Leftrightarrow \|x_\lambda - x\| \to 0$.
(2) In the weak topology: $x_\lambda \to x \Leftrightarrow < x_\lambda, y > \to < x, y >$ for each $y \in H$.

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Proof. An easy exercise.

Proposition 1.4. *Strong convergence implies weak convergence.*

Proof. Immediate from Proposition 1.2.

2. Topologies on $B(H)$, the space of bounded linear operators on a Hilbert space H.

Now let H be a Hilbert space.
Let B(H)=all bounded linear operators on H.
It is known that B(H) is a normed space.
Moreover, it is complete- so it is a Banach space.
Still moreover, B(H) is a $C^*algebra$, definition of which will be found in section 3.

Remark 2.1. $A \in B(H)$ is bounded $\Leftrightarrow$ it is continuos (where the topology on H is the strong topology). A proof can be found in [4] page 54.

Now, let's define 3 different topologies on B(H).

Definition 2.2. (1) Norm Topology:
Since B(H) is a normed space, the given norm induces a metric, so B(H) is a metric space.
So the norm topology is just defined to be the metric topology.
(2) Strong Operator Topology (sometimes abbreviated S.O.T.):
A subbase for the Strong Operator Topolgy is the collection of all sets of the form $O(A_0; x, \varepsilon) := \{A \in B(H) : \|(A - A_0)x\| < \varepsilon\}$

As always, a base is the collection of all finite intersections of such sets.
Notice that it follows that a base is the collection of all sets of the form

$O(A_0; x_1, \cdots, x_k, \varepsilon) := \{A \in B(H) : \|(A - A_0)x_i\| < \varepsilon \; i = 1, \cdots, k\}$
We shall use this form in Theorem 4.3.
3) Weak Operator Topology (sometimes abbreviated W.O.T.):
A subbase for the Weak Operator Topolgy is the collection of all sets of the form $O(A_0; x, y, \varepsilon) := \{A \in B(H) : \; | < (A - A_0)x, y > | < \varepsilon\}$

As always, a base is the collection of all finite intersections of such sets.

The names "strong" and "weak" are not arbitrary, as the following shows:

Proposition 2.3. *The weak topology is weaker then the strong topology and the strong topology is weaker then the norm topology.*

Proof. An exercise. □

The corresponding concepts of convergence for sequences (and nets) can be described easily:

Proposition 2.4. *(1) In the norm topology:* $A_\lambda \to A \Leftrightarrow \|A_\lambda - A\| \to 0$*, where the norm is that of* $B(H)$*.*
(2) In the strong topology: $A_\lambda \to A strongly \Leftrightarrow$ *for each* $x \in H$ $A_\lambda x \to Ax$ *strongly which means that for each* $x \in H$ $\|A_\lambda x - Ax\| \to 0$*.*
(3) In the weak topology: $A_\lambda \to A$ *weakly* $\Leftrightarrow$ *for each* $x \in H$ $A_\lambda x \to Ax$ *weakly which means that for each* $x \in H$ *and for each* $y \in H$ $< A_\lambda x, y > \to < Ax, y >$*.*

Proof. An exercise.

□

Proposition 2.5. *Norm convergence implies Strong convergence and Strong convergence implies weak convergence.*

Proof. An exercise. □

3. C^* ALGEBRAS

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A Banach algebra R *is a Banach space which is also an (associative) algebra with a unit.*
That means that in addition of being a Banach space, it also has an associative multiplication, a neutral element wrt multiplication and the two distibutive laws hold.
Also the following holds:
$\|AB\| \leq \|A\|\|B\|$ *for all* $A, B \in R$

Now let's define what an involution is:
An involution is a function from a Banach algebra to itself,
such that for all $A, B \in R$*,* $\alpha \in \mathbb{C}$
1. $(A + B)^* = A^* + B^*$
2. $(AB)^* = B^* A^*$
3. $(\alpha A)^* = \bar{\alpha} A^*$
4. $(A^*)^* = A$

A C^* *algebra is a Banach algebra, with an involution, which also satisfies:*
$\|A^* A\| = \|A\|^2$ *for all* $A \in R$

4. SOME NICE THEOREMS CONCERNING THE 3 OPERATOR TOPOLOGIES MENTIONED ABOVE ON B(H).

In this section we will omit the word "operator" when relating to the strong and weak operator topologies, and just call them strong and weak topologies.
No confusion with the strong and weak topologies on H should occur.

We have seen in proposition 2.5:
norm convergence $\longrightarrow$ strong convergence$\longrightarrow$weak convergence.
If we impose additional conditions, then the reverse is also true, in the following sense:

Theorem 4.1. *The following is true in $B(H)$:*

1. *If $< A_nx, y > \to < Ax, y >$ uniformly for $\|y\| = 1$, then $\|A_nx - Ax\| \to 0$*
2. *If $\|A_nx - Ax\| \to 0$ uniformly for $\|x\| = 1$, then $\|A_n - A\| \to 0$*

Proof. (1)$wlog$, $A = 0$:
$< A_nx, y > \to < Ax, y >$ whenever $\|y\| = 1$
$\Rightarrow < (A_n - A + A)x, y > \to < Ax, y >$
$\Rightarrow < (A_n - A)x + Ax, y > \to < Ax, y >$
$\Rightarrow < (A_n - A)x, y > + < Ax, y > \to < Ax, y >$
$\Rightarrow < (A_n - A)x, y > \to 0$
So taking $B_n = A_n - A$, we have $< B_nx, y > \to 0$ whenever $\|y\| = 1$
So assuming we have proved the new claim
(which is: $< A_nx, y > \to 0$ uniformly for $\|y\| = 1$, then $\|A_nx\| \to 0$),
we get $\|B_nx\| \to 0$, then $\|(A_n - A)x\| = \|A_nX - Ax\| \to 0$, as requiered.

Proving the new claim is left as an exercise.

(2) $wlog$, A=0. (explain!)
The assumption now is $\|A_nx\| \to 0$ whenever $\|x\| = 1$,
which means that for every $\varepsilon > 0$ there exists N *s.t.* $for\ all\ n \geq N$ $\|A_nx\| < \varepsilon$ whenever $\|x\| = 1$
The uniformity is in the sense that N does not depend on x.
It follows that $for\ all\ n \geq N$ $\|A_n(\|x\|^{-1}x)\| < \varepsilon$ whenever $x \neq 0$
and hence that, $for\ all\ n \geq N$ $\|x\|^{-1}\|A_nx\| < \varepsilon$ whenever $x \neq 0$
so $for\ all\ n \geq N$ $\|A_nx\| < \|x\|\varepsilon$ whenever $x \neq 0$
so $for\ all\ n \geq N$ $\|A_nx\| \leq \|x\|\varepsilon$ for all $x \in H$
This implies that $for\ all\ n \geq N$ $\|A_n\| \leq \varepsilon$, which says that $\|A_n\| \to 0$.

□

Remark 4.2. The above argument is general; it applies to all nets, not only to sequences.

Theorem 4.3. *multiplication is continuos with respect to the norm topology and discontinuos with respect to the strong and weak topologies.*

Proof. Norm topology:
The proof for the norm topology is contained in the inequalities
$\|AB - A_0B_0\| \leq \|AB - AB_0\| + \|AB_0 - A_0B_0\| \leq \|A\|\|B - B_0\| + \|A - A_0\|\|B_0\| \leq (\|A - A_0\| + \|A_0\|)\|B - B_0\| + \|A - A_0\|\|B_0\|$

Strong and weak topologies:

Caution! In the theorem about the norm, we will see strong discontinuity of the norm, and will deduce, by an easy topological consideration (which we shall give immediately before proving the norm theorem) , the weak discontinuity of the norm. That topological consideration is not applicable here: there we change only the topology of the domain of the norm function (make it larger) and remain the topology of the codomain, $\mathbb{R}$, unchanged. But here we change both the topology of the domain of the multiplication function (make it larger) and also change the topology of the codomain (make it larger)-so there is not telling what will happen to continuity -and we shall see in theorem 4.7 that everything can happen (theorem 4.7 is about the involution function= the adjoint).

Discontinuity of multiplication in the strong topology:
Step 1:
The set N of all nilpotent operators of index 2 (i.e.,the set of all operators A such that $A^2 = A$), is strongly dense.
To prove this, suppose that
$O(A_0; x_1, \cdots, x_k, \varepsilon) := \{A \in B(H) : \|(A - A_0)x_i\| < \varepsilon \ i = 1, \cdots, k\}$
is an arbitrary basic strong nbd.
There is no loss of generality in assuming that the x's are linearly independent or even orthonormal (think why! a hint: otherwise, replace them by a linearly independent or even orthonormal set with the same span, and, at the same time, make ε as much smaller as is necessary).
For each i ($i = 1, \cdots, k$) find a vector y_i
such that $\|A_0x_i - y_i\| < \varepsilon$ and such that the span of the y's has only 0 in common with the span of the x's;
So as long as the underlying Hilbert space is infinite dimensional, this is possible.
Actually, we will always assume that the underlying Hilbert space is infinite dimensional, because otherwise all the topologies coincide.

Let A be the operator such that
$Ax_i = y_i \ and \ Ay_i = 0 \ (i = 1, \cdots, k)$
and $Az = 0 \ < z, x_i >= 0 \ and \ < z, y_i >= 0 \ (i = 1, \cdots, k)$.
Clearly A is nilpotent of index 2, and, just as clearly, A belongs to the prescribed nbd.

Step 2:
If multiplication were strongly continuos, then ,in particular,
it were strongly continuos in pairs of the form (A,A).
But strong continuity in (A,A)$\longrightarrow$ net/ sequential continuity in (A,A),
which means: $if\ (A_n, A_n) \to (A, A)\ then\ A_n^2 \to A^2$
Now take an arbitrary $A \in B(H)$.
$A \in B(H) = cl_s(N)$, so there exists a net $A_\lambda \subseteq N$ s.t. $A_\lambda \to A$,
so $(A_\lambda, A_\lambda) \to (A, A)$ (by definition of convergence in the product topology)
If multiplication were net continuous, then $A_\lambda^2 \to A^2$.
But $A_\lambda \subseteq N$, so for each λ we have $A_\lambda^2 = 0$.
So, $0 \to A^2$, and from uniqueness of the strong limit ,if it exists (prove it),
we get $A^2 = 0$.
But A was arbitrary, in particular we could have taken A to be the identity map on H, getting a contradiction.

Discontinuity of multiplication in the weak topology:
Step 1 accomodated to the weak topology:
Since the strong topology is larger than the weak topology, so that a strongly dense set is necessarily weakly dense,
the set N of all nilpotent operators of index 2 is weakly dense.
Step 2 accommodated to the weak topology:
Just replace everywhere strong by weak, and it will work, too.

□

One can ask the following:

Theorem 4.4. *(1) Right multiplication is both strongly and weakly continuous.*
This means that, for a fixed, B the mapping $B(H) \to B(H)$ defined by $A \mapsto AB$ is both strongly and weakly continuous.
(2) Left multiplication is both strongly and weakly continuous.
(for a fixed A, the mapping $B(H) \to B(H)$ defined by $B \mapsto AB$ is both strongly and weakly continuous).

Proof. Let's use convergence. Although sequential continuity does not imply continuity, net continuity does imply continuity (see [3]).

Strong continuity:
1.for right multiplication:
Suppose that $A_\lambda \to A$ strongly i.e., that $A_\lambda x \to Ax\ strongly\ for\ each\ x \in H$.
It follows, in particular, that $A_\lambda Bx \to ABx\ for\ each\ x \in H$, and this settles strong continuity in A.
2.for left continuity:
Suppose that $B_\lambda \to B$ i.e., that $B_\lambda x \to Bx\ strongly\ for\ each\ x \in H$.
Then, since A is continuous (remember bounded equals continuous, see remark 2.1), if we apply A we get $AB_\lambda x \to ABx\ strongly\ for\ each\ x \in H$, and this settles strong continuity in B.

Weak continuity:
1.for right multiplication:

If $A_\lambda \to A$ weakly i.e., that $A_\lambda x \to Ax\ weakly\ for\ each\ x \in H$,
i.e., that $< A_\lambda x, y > \to < Ax, y >\ \ for\ each\ x, y \in H$.
Then, in particular, $< A_\lambda Bx, y > \to < ABx, y >\ \ for\ each\ x, y \in H$,
and this settles weak continuity in A.

2.for left continuity:
If $B_\lambda \to B$ weakly i.e., that $B_\lambda x \to Bx\ weakly\ for\ each\ x \in H$,
i.e., that $< B_\lambda x, y > \to < Bx, y >\ \ for\ each\ x, y \in H$.
Then, in particular, $< AB_\lambda x, y > = < B_\lambda x, A^* y > \to < Bx, A^* y > = < ABx, y >$ $for\ each\ x, y \in H$, this settles strong continuity in B.

□

Now, as we promised earlier:

Remark 4.5. An easy topological consideration: If a function from one space to another is continuous, then it remains so if the topology of the domain is made larger, and it remains so if the topology of the codomain is made smaller.

Theorem 4.6. *The norm is continuous wrt the norm topology, and discontinuous wrt the strong and weak topologies.*

Proof. Norm topology:
The proof for the norm topology is contained in the inequality
$|\ \|A\| - \|B\|\ | \leq \|A - B\|$.

Explanation: For continuity in $A_0 \in B(H)$:
We should prove:
for every $\varepsilon > 0$ there exists $\delta > 0$ s.t. if $\|A - A_0\| < \delta$ then $|\ \|A\| - \|A_0\|\ | < \varepsilon$.
Let $\varepsilon > 0$, and just take $\delta = \varepsilon$.
Then if $\|A - A_0\| < \varepsilon$ then $|\ \|A\| - \|A_0\|\ | \leq \|A - A_0\| < \varepsilon$.

Notice: The above argument works for any normed space, not only $B(H)$, because the inequality $|\ \|A\| - \|B\|\ | \leq \|A - B\|$ is true in any normed space.

Strong and weak topologies:
Using the above remark:
discontinuity wrt the strong topology $\longrightarrow$ discontinuity wrt the weak topology.
So,it suffices to show discontinuity of the norm wrt the strong topology.
We shall see an example where the norm is not seqentially continuous $\longrightarrow$
not continuous.

Take an infinite dimentional Hilbert space H. Build a decreasing sequence of non-zero subspaces with intersection 0 (this is impossible for a finite dimensional space, but we have already said that we are dealing only with infinite dimensional spaces), and let P_n be the corresponding sequence of (orthogonal) projections.
The sequence P_n converges to 0 strongly.
The sequence of the images $\|P_n\|$ does not converges to $\|0\| = 0$,
because the sequence of the images $\|P_n\|$ is the constant sequence which equals 1, since for any orthogonal projection we have:

$\|P\| = \|P^2\| = \|PP\| = \|P^*P\| = \|P\|^2 \Rightarrow \|P\| = \|P\|^2 \Rightarrow \|P\| = 1.$

Finally, let's see what happens with the involution. We mentioned in theorem 4.3 that anything can happpen when changing both the topology of the domain and codomain (for example, both made larger), so now it's time to see it:

Theorem 4.7. *The involution is continuous wrt the norm and weak topologies and discontinuous wrt the strong topology.*

Proof. Norm continuity:
Just use $\|A^* - B^*\| = \|A - B\|$ (and take $\delta = \varepsilon$).

Weak continuity:
Weak continuity is implied by the identity
$| < A^*x, y > - < B^*x, y > | = | < (A^* - B^*)x, y > | = | < x, (A^* - B^*)^*y > | =$
$| < x, (A - B)y > | = | < x, Ay > - < x, By > | = | < Ay, x > - < By, x > |$
where in the last equality we used the known fact that $|z| = |\bar{z}|$ for all $z \in \mathbb{C}$.

Strong discontinuity:
To prove the strong discontinuity of the adjoint, consider B(l_2).
Let U be the unilateral shift (one shift of coordinates to the right), explicitely:
U: B(l_2)$\longrightarrow B(\mathrm{l}_2)$
$U(\xi_0, \xi_1, \xi_2, \cdots) = (0, \xi_0, \xi_1, \cdots)$
and define
A$_k = U^{*k}$, $k = 1, 2, 3, ...$
Notice that U$^*(\xi_0, \xi_1, \xi_2, \cdots) = (\xi_1, \xi_2, \xi_3 \cdots)$
(one shift of coordinates to the left).

Assertion:
$A_k \to 0$ strongly, but the sequence A_k^* is not strongly convergent to $0^* = 0$.
Indeed,
$\|A_k(\xi_0, \xi_1, \xi_2, \cdots \|^2 = \|(\xi_k, \xi_{k+1}, \xi_{k+2}, \cdots\|^2 = \sum |\xi_n|^2$
so that $\|A_k x\|^2$ (where $x = (\xi_0, \xi_1, \xi_2, \cdots)$) is, for each x, the tail of a convergent series, so $\|A_k x\|^2 \to 0$ for each x, then $\|A_k x\| \to 0$ for each x,
therefore $A_k \to 0$ strongly.

A_k^* is not strongly convergent to $0^* = 0$:
Otherwise, for an arbitrary $0 \neq x \in H$ A$_k^* x \to 0$ strongly, which means $\|A_k^* x\| \to 0$.
But $\|A_k^* x\| \to 0$ is not true, since it is not a cauchy sequence:

$\|A_{m+n}^* x - A_n^* x\|^2 = \|U^{m+n}x - U^n x\|^2 = \|U^n(U^m x) - U^n(x)\|^2 = \|U^n(U^m x - x)\|^2 =$
$\|U^m x - x\|^2 = \|(U^m x)\|^2 - 2\text{Re} < U^m x, x > + \|x\|^2 = \|x\|^2 - 2\text{Re} < U^m x, x > + \|x\|^2 =$
$2(\|x\|^2 - \text{Re} < U^m x, x >) = 2(\|x\|^2 - \text{Re} < x, U^{*m} x >)$

We saw above that $\|A_m x\| \to 0$ which is by definition $\|U^{*m} x\| \to 0$.
It follows that $\|A_{m+n}^* x - A_n^* x\|^2 \to 2\|x\|^2$, as m and n become large.
So $\|A_{m+n}^* x - A_n^* x\| \to \sqrt{2}\|x\| \neq 0$

$\square$

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