7. C. S. Smith, "Grains, Phases and Interfaces—An Interpretation of Micro-structure", *Transactions,* American Institute of Mining and Metallurgical Engineers, Vol. 175, 1948, p. 15-51.

8. E. Scheil, "Statistical Investigations of the Structures of Alloys", *Zeitschrift für Metallkunde,* Vol. 27, 1935, p. 199-208. II, *ibid,* 1936, Vol. 28, p. 340-343.

9. W. A. Johnson, "Estimation of Spatial Grain Size", METAL PROGRESS, Vol. 49 (1), 1946, p. 89-92.

10. J. B. Rutherford, R. H. Aborn and E. C. Bain, "Relation of Grain Area on a Plane Section and the Grain Size of a Metal", *Metals and Alloys,* Vol. 8, 1937, p. 345-348; ASM METALS HANDBOOK, 1948, p. 405.

11. M. Brüchner, *Proceedings,* Mathematics Congress, Bologna, IV, 1930, II.

12. W. C. Graustein, "On the Average Number of Sides of Polygons of a Net", *Annals of Mathematics,* Vol. 32, 1932, p. 149-153.

13. J. Plateau, *Statique Experimentale et Theorique des Liquides Soumis aux Seules Forces Moleculaires,* 2 Volumes, Ghent, 1873.

14. Lord Kelvin, "On the Division of Space With Minimum Partitional Area", *Philosophical Magazine,* Vol. 24, 1887, p. 503-514; *Collected Papers,* Vol. 5, p. 333.

15. H. C. H. Carpenter and C. F. Elam, "Crystal Growth and Recrystallization in Metals", *Journal,* Institute of Metals, Vol. 24, 1920, p. 83-131.

16. J. K. Senior (University of Chicago), private communication.

17. C. S. Smith, "Interphase Interfaces", *Conference on Imperfections in Almost Perfect Crystals,* National Research Council, 1950 (in press).

18. Lord Kelvin, "On Homogeneous Division of Space", *Proceedings,* Royal Society of London, Vol. 55, 1894, p. 1; *Collected Papers,* Vol. 6, p. 333.

19. P. C. Grignon, *Memoires de Physique sur l'art de Fabriquer le fer . . . ,* Paris, 1775.

20. William M. Williams (E.C.A. Scholar, University of Chicago, Institute for the Study of Metals), to be published.

21. M. Goldberg, "The Isoperimetric Problem for Polyhedron", *Tohoku Mathematical Journal,* Vol. 40, 1934, p. 226-236.

22. W. H. Zachariasen, "The Atomic Arrangement in Glass", *Journal,* American Chemical Society, Vol. 54, 1932, p. 3841-3851.

23. C. S. Smith, Discussion of paper by Paul A. Beck, *Transactions,* American Institute of Mining and Metallurgical Engineers, Vol. 185, 1949, p. 312-313.

24. J. E. Burke, "Fundamentals of Recrystallization and Grain Growth", chapter in *Grain Control in Industrial Metallurgy,* American Society for Metals, Cleveland, 1949.

25. A. Guinier, "Substructures in Crystals", *Conference on Imperfections in Almost Perfect Crystals,* National Research Council, 1950 (in press).

26. "Symposium on Polygonization" in *Progress in Metal Physics* (Bruce Chalmers, editor), Vol. II, 1950, p. 151-202.

DISCUSSION

Written Discussion: By John von Neumann, Institute for Advanced Study, Princeton, N. J.

The considerations that follow deal with the changes of bubble-volume due to diffusion, that occur in a two-dimensional bubble-froth (a "Flat Cell"), as shown in Fig. 12 of Dr. C. S. Smith's paper (to be quoted as "G.L."). As pointed out on page 79, G.L., such changes of volume are due to the diffusion of the gas that fills the bubbles, through the liquid film that forms the (separating) bubble-walls. This diffusion is caused

by the pressure difference between adjacent cells (cf. loc. cit. above). In first approximation it is proportional to this pressure difference. To be more precise: The diffusion-flow across a particular bubble-wall is (in the approximation referred to above) proportional to the pressure difference between the two bubbles adjacent to this wall, multiplied by the length of the wall.

The pressure difference of the two adjacent bubbles, at a given point P of a wall, is $2\gamma/R$, where γ is the surface tension of the liquid forming the froth, and R is the radius of curvature of the wall at P (cf. page 77, G.L.). γ is constant throughout the froth. Let P move over one wall-side, i.e., one side separating two given bubbles. Then these two bubbles and their respective pressures are fixed, hence the pressure difference between them is fixed, and so $2\gamma/R$ must be constant. R is therefore constant, i.e., the side in question is a circular arc. Each bubble is bounded by a polygon formed of circular arcs.

Consider such an arc, of radius R and angular aperture α. The pressure difference across it is $2\gamma/R$, the length of the arc is $R\alpha$. Hence the diffusion-flow across this arc (wall-side) is proportional to $2\gamma/R \cdot R\alpha = 2\gamma\alpha$, i.e., to α.

Consider next a bubble in its entirety. Let the bounding circular-arc-polygon have n sides. These sides are circular arcs; let their angular apertures be $\alpha_1, \ldots, \alpha_n$, respectively. Let the angle between sides i and $i+1$ (side $n+1$ is side 1 !) be ϕ_i, i.e., the angles are $\phi_1, \ldots, \phi_n$, respectively. Actually each $\phi_i = 120° = 2\pi/3$ (cf. page 75, G.L.), but this is not relevant yet.

Replace each arc by its chord, then an ordinary (rectilinear) polygon obtains. The replacement of arc i by its chord increases each adjacent polygonal angle by $\frac{1}{2}\alpha_i$. Hence ϕ_i is increased by $\frac{1}{2}\alpha_{i-1} + \frac{1}{2}\alpha_i$. The corresponding external angle of the rectilinear polygon obtains by complementing this to $180° = \pi$, i.e., it is $\pi - \phi_i - \frac{1}{2}\alpha_{i-1} - \frac{1}{2}\alpha_i$. The sum of all external angles of the rectilinear polygon is $360° = 2\pi$, i.e.,

$$\Sigma\left(\pi - \phi_i - \frac{1}{2}\alpha_{i-1} - \frac{1}{2}\alpha_i\right) = 2\pi,$$
$$n\pi - \Sigma\phi_i - \Sigma\alpha_i = 2\pi,$$

and, using $\phi_i = \frac{2\pi}{3}$ (cf. above),

$$\frac{n}{3}\pi - \Sigma\alpha_i = 2\pi,$$
$$\Sigma\alpha_i = \frac{6-n}{3}\pi.$$

Now it was pointed out, above, that the diffusion-flow across the side is proportional to α_i. Note that α_i may be >0 as well as <0. $\alpha_i > 0$ (<0) means that the side in question is convex (concave); hence the bubble in question loses (gains) gas across this side by diffusion. (Cf. Fig. 10, G.L.) Thus the diffusion-flow's proportionality to α_i holds even with respect to the signs (i.e., the coefficient of proportionality is positive), if the flow is interpreted as a rate of loss of gas.

In this sense, then, the total diffusion flow of a bubble, i.e., its total gas-loss-rate, is (positively) proportional to $\Sigma\alpha_i = \frac{6-n}{3}\pi$ (cf. above), i.e., to $6-n$. Or, equivalently:

In a two-dimensional bubble-froth the total gas-gain-rate of any bubble is (positively) proportional to n — 6, where n is the number of sides of the bubble (i.e., of its bounding circular-arc-polygon). The (positive) coefficient of proportionality depends only on the general properties of the froth and of its containing "Flat Cell".

Thus every hexagonal bubble (irrespective of further details of shape!) has a constant-gas-content, every pentagonal bubble loses gas at the same rate; every heptagonal bubble gains gas at the same rate as the pentagonal ones lose it; every tetragonal (octagonal) bubble loses (gains) gas at twice the rate at which pentagonal (heptagonal) bubbles lose (gain) it, etc.

Note, that these results apply only to the continuous changes of gas-content due to diffusion. There remains the problem of finding a comparably simple characterization of the total changes of bubble-shapes due to these changes of gas-content. There remains, also, the problem of doing the same for the discontinuous changes that occur when a side disappears (cf. pages 78, 79, G.L.). Finally, these results are valid in two, but not in three, dimensions.

Written Discussion: By Edwin B. Matzke, professor of botany, Columbia University, New York.

From a biological standpoint, Dr. Smith's article on "Grain Shapes and Other Metallurgical Applications of Topology" serves to bring into focus the close interrelationship, within limits, of concepts of space partition in such diverse fields as mathematics, metallurgy, and cellular configurations in plant and animal tissues.

The significance of $5\frac{1}{7}$ as the average number of sides per face has not been appreciated in biological literature. It is obvious in the Kelvin tetrakaidecahedron with 6 quadrilateral and 8 hexagonal faces, and Lewis showed in 1925 how quadrilateral and hexagonal faces might be replaced by pairs of pentagons.

Although, as Dr. Smith says (page 97), topological principles have not been invoked in the Columbia Laboratory, the "(6 — n) rule" has been applied since the assembling of data was begun. The sampling of bubbles, referred to on page 97, was as complete as possible, within a given core, under the experimental conditions of study and tabulation, using a binocular stereoscopic dissecting microscope. The number and kinds of faces were tabulated for as many bubbles as could be seen with the instrument used; this included most, but not necessarily all, of the bubbles within a given core.

Many of Dr. Smith's topological demonstrations seem, upon reflection, almost as obvious as they are inevitable. It is now not often realized that, until Lewis's work in 1923, cells in undifferentiated plant and animal tissues were almost universally considered to be rhombic dodecahedra, and that in spite of convincing experimental evidence to the contrary, the Kelvin tetrakaidecahedra have been erroneously accepted as the shapes of bubbles in foam in scientific literature, even within the present decade. In helping to sweep away these and other misconceptions, and in clearing the air generally, many of Dr. Smith's demonstrations and conclusions will be of marked importance. Among such the following may be cited: "It should be apparent . . . . . . that there is considerable

freedom in space filling under the given conditions, and that no one shape can possibly be regarded as that of the 'typical' grain unless the grains are arranged with the symmetry of a lattice"; "Moreover, random contacts of bodies of approximately uniform size would tend to give faces more nearly of equal area than those of the 14-sided body."

Written Discussion: By W. M. Williams, Institute for the Study of Metals, University of Chicago, Chicago.

In his discussion of grain shapes, Dr. Smith mentions the use of stereoscopic microradiography in examining the true structure of a metal in three dimensions. I think it is worthwhile giving one or two preliminary observations which have been made on annealed and quenched aluminum-tin alloys (2 atomic per cent Sn) using this method.

The technique itself is very simple; a beam of "white" radiation from a tungsten target (50 kv., 20 m.a.) is directed at a specimen of convenient thickness (about 1 mm., say) and the X-ray shadow of the metal structure is registered on a fine-grained photographic plate placed in close contact behind the specimen. Eastman 548-0 spectroscopic plates
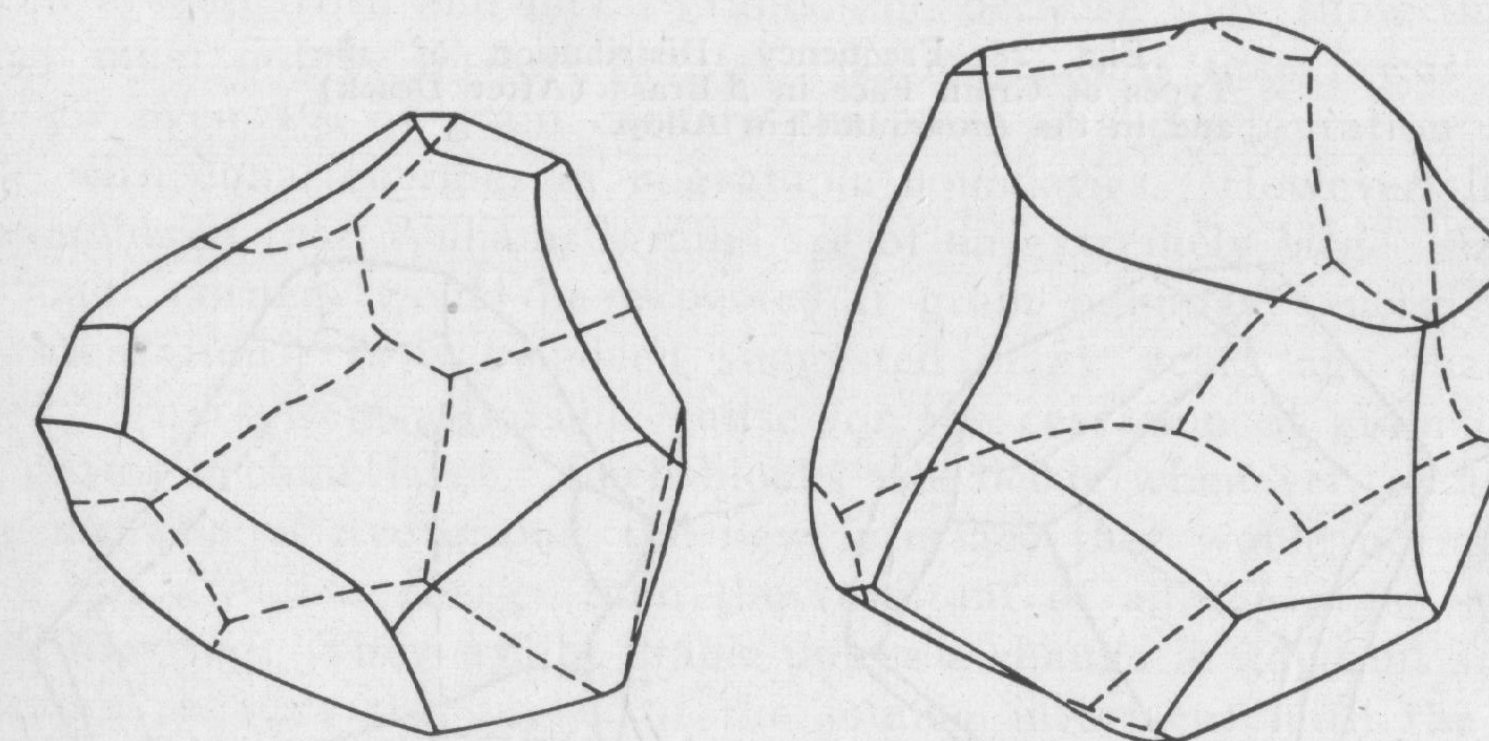

Fig. 27—Typical Grains in Annealed and Quenched Al-Sn Alloy.

are very suitable for this purpose. Two such microradiographs are taken, at two different angles of rotation around a vertical axis. The angle of rotation between the two microradiographs is usually about 4 to 6 degrees. The microradiographs are then enlarged and viewed in a stereoscope in the usual way. In the negative enlargements the once-liquid tin phase appears black, and is seen to have spread continuously along the grain edges, thus outlining the almost transparent aluminum grains.

The close resemblance of the grain structure to an ordinary soap froth is very evident, with four edges and four grains usually meeting at a point, with three grains meeting along a common edge, and so on. The two grains shown in Fig. 27 are typical grains in the well-annealed alloy, and illustrate the normal case of three edges to every vertex, and the preponderance of 4-, 5-, and 6-sided polygons. Fifty such grains occupying a continuous volume have been examined, and the types of faces classified. Fig. 28 shows the frequency distribution for the 50 aluminum-tin alloy grains and a re-plot of Desch's results on β-brass. The results are