EXPLICIT TILTING COMPLEXES FOR THE BROUÉ CONJECTURE ON 3-BLOCKS

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Abstract. The Broué conjecture, that a block with abelian defect group is derived equivalent to its Brauer correspondent, has been proven for blocks of cyclic defect group and verified for many other blocks, mostly with defect group $C_3 \times C_3$ or $C_5 \times C_5$. In this paper, we exhibit explicit tilting complexes from the Brauer correspondent to the global block B for a number of Morita equivalence classes of blocks of defect group $C_3 \times C_3$. We also describe a database with data sheets for over a thousand blocks of abelian defect group in the ATLAS group and their subgroups.

1. Introduction

Let G be a finite group and let k be a field of characteristic p, where p divides $|G|$. Let $kG = \oplus B_i$ be a decomposition of the group algebra into blocks, and let D_i be the defect group of the block B_i, of order p^{d_i}. By Brauer's Main Theorems (see [A] for an accessible exposition) there is a one-to-one correspondence between blocks of kG with defect group D_i and blocks of $kN_G(D_i)$ with defect group D_i. Let b_i be the block corresponding to B_i, called its Brauer correspondent.

Broué [B2] has conjectured that if D_i is abelian and B_i is a principal block, then B_i and b_i are derived equivalent, i.e., the bounded derived categories $D^b(B_i)$ and $D^b(b_i)$ are equivalent. In fact, it is generally believed by researchers in the field that the hypothesis that B_i be principal is unnecessary.

By a fundamental theorem of Rickard, if two algebras A and B are derived equivalent, then there are "two-sided" *tilting complexes*, complexes ${}_AX'_B$ and ${}_BY'_A$ of bimodules, projective as A- and B-modules, such that ${}_AX'_B \hat{\otimes}_B \, {}_BY_A \xrightarrow{\sim} {}_AA_A$ in $D^b(A)$, and ${}_BY_A \hat{\otimes}_A \, {}_AY_B \xrightarrow{\sim} {}_BB_B$ in $D^b(B)$. Each two-sided tilting complex induces a "one-sided" tilting complex ${}_AT$ by ignoring one of the algebra actions. The one-sided complexes, after removing superfluous modules and maps, can be quite simple to describe. In all the examples we will bring, the ${}_AT$ are complexes of projective modules and, with one exception, each indecomposable projective occurs in a unique degree. A one-sided complex ${}_AT$

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satisfies

$$\mathrm{End}_A({}_AT) \xrightarrow{\sim} B.$$

In Sections 2–3, we will give explicit tilting complexes of a number for blocks. We are particularly interested in cases where one tilting complex can be obtained from another by an automorphism, as in the passage from the principal block of A_6 to that of S_6 and from the principal block of A_7 to that of S_7.

In some cases, the proof of the Broué conjecture for the block is due to Okuyama [O] and our contribution is to compute the tilting complex from his algorithm by taking mapping cones. In other cases, the experimental work to discover a conjectural tilting complex was done by G. Chereshnya and the proof by M. Schaps.

The blocks given here were chosen from a database of blocks of abelian defect group which will be described in Section 4. Using the sorting into Morita equivalences in the database, the examples in Sections 2–3 actually provide tilting complexes for numerous other blocks. The basic approach guiding our research is that outlined in [S1], in which the column operation induced on the decomposition matrix by the tilting are taken as providing a first approximation to the tilting complex, up to a choice of folding. Thus whenever the Brauer character table and the decomposition matrix were available in GAP, we included the decomposition matrix on the data sheet for the block in the database. We also included the generalized decomposition matrices where we could generate them. These were useful in determining the permutations of the rows and columns of the decomposition matrix which occur when the rows and columns are reordered by degrees after tilting.

The verification of an explicit tilting complex permits machine calculation of the indecomposable projectives. We have written a program in MAGMA, based on Holloway's program "homotopy", to make these calculations for a few of the Brauer correspondents, as described in [HS]. It is currently being tested and debugged.

2. Automizers C_4 and D_4

This section includes most of the blocks of the alternating and symmetric groups with defect group $C_3 \times C_3$. We assume a first ordering of the irreducibles of the character table of the group, as given in the ATLAS or the GAP Character Table library. The B_1 will always be the principal block, B_2 will contain the first irreducible which is not in the principal block, and so forth.

For the automizer C_4, the four projective modules $Q_1, \ldots, Q_4$ are of dimension 9, and each is diamond-shaped. We give the simples of Q_1, the structure of Q_2, Q_3 and Q_4 being obtained from that of Q_1 by cyclic permutation of {1,2,3,4}.

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 4 & \\ 3 & & 1 & & 3 \\ & 2 & & 4 & \\ & & 1 & & \end{array}$$

For the automizer D_4, there is one projective module P_1 of dimension 18 corresponding to the simple module of dimension 2, and there are four projective modules P_2, P_3, P_4, P_5 of dimension 9. We give the composition factors for P_1 and P_2. The exact maps are given by the theorems in [HS].

P_1 :

$$\begin{array}{ccccc} & & 1 & & \\ & 2 \oplus 3 & & 4 \oplus 5 & \\ 1 & & 1 & & 1 \\ & 2 \oplus 3 & & 4 \oplus 5 & \\ & & 1 & & \end{array}$$

P_2 :

$$\begin{array}{ccc} 2 & & \\ & 1 & \\ 2 & & 3 \oplus 4 \\ & 1 & \\ 2 & & \end{array}$$

The principal blocks B_1 of A_7 and S_7.

The tilting complex for B_1 of A_7 was given by Okuyama [O] as

$$\begin{array}{ll} Q_1' : Q_3 \oplus & Q_3 \to Q_1 \\ Q_2' : & Q_3 \to Q_2 \\ Q_3' : & Q_3 \\ Q_4' : & Q_3 \to Q_4 \end{array}$$

The tilting complex B_1 for S_7 was also used by Okuyama [O], but not identified as belonging to S_7. It is

$$
\begin{aligned}
P_1' &: P_4 \oplus P_5 \to P_1 \\
P_2' &: P_4 \oplus P_5 \to P_2 \\
P_3' &: P_4 \oplus P_5 \to P_3 \\
P_4' &: P_4 \\
P_5' &: P_5
\end{aligned}
$$

Note that P_2' and P_3' are obtained by splitting Q_1', P_4' and P_5' are the splittings of P_3', and Q_2', Q_4' each give P_1'.

The principal blocks B_1 of A_6 and S_6

The algorithm for the principal blocks in these groups was given by Okuyama. In each case, one takes the elementary tilting complex giving B_1 of A_7 or S_7 and applies a further elementary tilting.

B_1 of A_6.

$$
\begin{array}{lll}
Q_1'' = Q_2' \oplus Q_4' \to Q_1' : & & Q_2 \oplus Q_4 \to Q_1 \\
Q_2'' = Q_2'[1] : & Q_3 \to Q_2 & \\
Q_3'' = Q_2' \oplus Q_4' \to Q_3' : & Q_3 \to Q_2 \oplus Q_4 & \\
Q_4'' = Q_4'[1] & Q_3 \to Q_4 &
\end{array}
$$

B_1 of S_6.:

$$
\begin{array}{lll}
P_1'' = P_1'[1] : & P_4 \oplus P_5 \to P_1 & \\
P_2'' = P_1' \to P_2' : & & P_1 \to P_2 \\
P_3'' = P_1' \to P_3' : & & P_1 \to P_3 \\
P_4'' : P_1' \to P_4' : & P_5 \to P_1 & \\
P_5'' : P_1' \to P_4' : & P_4 \to P_1 &
\end{array}
$$

If we analyze what has happened here, we see that Q_2'' or Q_4'' each give P_1'', that Q_1' splits in P_2'' and P_3'', while Q_3'' splits into P_4'' and P_5''.

$B2$ of $2.A_6$, with automizer C_4:

This block is isomorphic to $2.SL(2,7)$, and thus the Broué conjecture was proven for it in [Hol]. The explicit tilting complex was given in [S1]. The first step, with $I = \{3,4\}$, produces the complex

$$
\begin{array}{ll}
Q_1' : & Q_3 \oplus Q_4 \to Q_1 \\
Q_2' : & Q_3 \oplus Q_4 \to Q_2 \\
Q_3' : & Q_3 \\
Q_4' : & Q_4
\end{array}
$$

The endomorphism ring of this complex in the derived category has the same decomposition matrix as $2.A_7$. A difficulty with the stable equivalence has prevented us from demonstrating that they are Morita equivalent, but we believe it to be true. Since, by [KS], the blocks $2.A_7$ and $2.S_8$ are Morita equivalent, this would also establish an explicit tilting complex for $2.S_8$.

The second step is an elementary tilting with $I = \{1, 2\}$.

$$\begin{aligned}
Q'_1 &: Q_3 \oplus Q_4 \to Q_1 \\
Q'_2 &: Q_3 \oplus Q_4 \to Q_2 \\
Q'_3 &: \qquad\quad Q_4 \to Q_1 \\
Q'_4 &: \qquad\quad Q_3 \to Q_2
\end{aligned}$$

In this case the action of the automorphism giving $2.A_6.2$ does not act on the tilting complex by permutation.

Note that in all three cases involving A_6, the block is a result of two elementary tiltings of which the first gives the corresponding block for A_7.

3. Automizer Q_8

As with the automizer D_4, we have four projective modules of dimension 9 and one of dimension 18. The projective indecomposables look quite similar, but there is one major difference: all simples appear in each projective indecomposable. Thus for P_2, we have now

$$\begin{array}{ccc}
2 & & \\
& 1 & \\
3 & & 4 \oplus 5 \\
& 1 & \\
2 & &
\end{array}$$

The composition factors for P_1 are the same as in the case of D_4 , though the maps are slightly different. The exact maps can be obtained from the theorems in [HS].

<u>The principal block B_1 of M_{22}.</u>

The sequence of elementary tilting complexes was determined by Okuyama [O]: $I_0 = \{4, 5\}$. The tilting complexes are:

$$\begin{aligned}
P_1^{(1)} &: P_4 \oplus P_5 \to P_1 \\
P_2^{(1)} &: P_4 \oplus P_5 \to P_2 \\
P_3^{(1)} &: P_4 \oplus P_5 \to P_3 \\
P_4^{(1)} &: P_4 \\
P_5^{(1)} &: P_5
\end{aligned}$$

Then $I_1 = \{1\}$. Here we can determine that the multiplicity of $P_t^{(1)}$ into the other $P_j^{(1)}$ is always 1, because this gives the correct decomposition matrix when we apply column operations. Since the maps from $P_4 \oplus P_5$ into P_2 and P_3 factor through P_1, we get

$$\begin{aligned}
P_1^{(2)} &: P_4 \oplus P_5 \to P_1 \\
P_2^{(2)} &: \qquad\qquad P_1 \to P_2 \\
P_3^{(2)} &: \qquad\qquad P_1 \to P_3 \\
P_4^{(2)} &: \qquad P_5 \to P_1 \\
P_5^{(2)} &: \qquad P_4 \to P_1
\end{aligned}$$

Another block with the same invariants is B_5 of $2.L3(4).2_1$, treated in [HS]. It has the same tilting complex.

The principal block B_1 of $PSL(3,4)$.

The algorithm follows Okuyama. The first two steps are as for the principal block of M_{22} and give the above tilting complex. At this point, the decomposition matrix is

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

In this case, the calculation of the elementary tilting complex was not straightforward. The next stage is the elementary tilting complex determined by the third column, $I_2 = 3$, followed by an elementary tilting complex given by the last two columns. There are two separate maps of $P_3^{(2)}$ into $P_2^{(2)}$, neither of which factors through the other. Using a variant of Holloway's MAGMA homotopy computation program developed at Bar-Ilan, we chose representatives; there are only zero maps from P_3 to P_2 in $P_2^{(3)}$, and both maps $P_1 \oplus P_1 \to P_1$ map into the radical squared. Thus when we take mapping cones, we get:

$$\begin{aligned}
P_1^{(3)} &= P_3^{(2)} \to P_1^{(2)} : && P_4 \oplus P_5 \to P_3 \\
P_2^{(3)} &= P_3^{(2)} \oplus P_2^{(2)} \to P_1^{(2)} : && P_1 \oplus P_1 \to P_1 \oplus P_3 \oplus P_3 \to P_2 \\
P_3^{(3)} &= P_3^{(2)}[1] : && P_1 \to P_3 \\
P_4^{(3)} &= P_3^{(2)} \to P_4^{(2)} : && P_5 \to P_3 \\
P_5^{(3)} &= P_3^{(2)} \to P_5^{(2)} : && P_4 \to P_3
\end{aligned}$$

The final stage in the algorithm is given by an elementary tilting with $I_3 = \{4, 5\}$. The resulting mapping cones give the correct decomposition matrix and the final tilting complex

$$
\begin{array}{lll}
P_1^{(4)} = P_4^{(3)} \oplus P_5^{(3)} \to P_1^{(3)} : & & P_3 \\
P_2^{(4)} = P_4^{(3)} \oplus P_5^{(3)} \to P_3^{(3)} : & P_4 \oplus P_5 \to & P_1 \oplus P_1 \to P_1 \to P_2 \\
P_3^{(4)} = P_4^{(3)} \oplus P_5^{(3)} \to P_3^{(3)} : & P_4 \oplus P_5 \to & P_1 \oplus P_3 \\
P_4^{(4)} = P_4^{(3)}[1] : & P_5 \to & P_3 \\
P_5^{(4)} = P_5^{(3)}[1] : & P_4 \to & P_3
\end{array}
$$

Note that in this example, unlike all the previous explicit tilting complexes, we have a projective, P_1, which appears in two different degrees.

The faithful blocks of $4.M_{22}$. This extremely interesting example is treated at length in [MS]. The action of Q_8 on $C_3 \times C_3$ is a Frobenius action, and thus the results in [P2] imply the existence of a stable equivalence given by restricting, tensoring with a 16-dimensional endopermutation module, and cutting to the correct block. The indecomposables of maximal vertex $C_3 \times C_3$ lead, using Okuyama's theorems, to the following explicit tilting complex.

$$
\begin{array}{lll}
P_2 \to & P_1 & \\
& P_1 \to & P_3 \\
& P_1 & \\
& P_4 & \\
& P_5 &
\end{array}
$$

This is the first non-cyclic defect group example of a split elementary tilting complex as defined in [S1].

4. A database of blocks of abelian defect group

For experimental mathematics one must generate data either bit by bit as needed or *en masse*. For an investigation of blocks of abelian defect group, we chose the second option, creating a database of blocks of over a thousand blocks of abelian defect group for the 1129 groups in the GAP 4.2 Character Table Library [G]. The database was intended to help researchers working on the Broué conjecture. Since Rickard proved the conjecture for blocks of cyclic defect [R2], the blocks of cyclic defect were treated separately. Blocks with non-cyclic abelian defect group existed only for primes 2, 3, 5, 7, 11, and 13. Since the case of prime 2 was well understood, we included only the odd primes in this range. For cyclic blocks of defect 1, the cases of primes 2 and 3 are trivial because there is only one Morita equivalence class of blocks, so we included only primes between 5 and 31.

The database also has some relevance for Donovan's conjecture and Puig's conjecture that for a given defect group there are only a finite number of equivalence classes under Morita or Puig equivalence [P1].

The organization of the data

The first division of the data was by the prime p which was the characteristic of the field. For each prime p, the program IndexOfBlocks.gap, written in GAP 4.2, generated hypertext indices to a collection of data sheets for individual blocks. The hypertext indices for the various primes are as follows:

(1) Elementary abelian defect group, $d \geq 2$, for $p = 3, 5, 7, 11, 13$, BLOCKS/SortAt_p.html.
(2) Non-elementary abelian defect group, $d \geq 2$, for $p = 3, 5, 7$, BLOCKS/CyclicSortAt_p.html. Mostly defect group C_{p^2}.
(3) Cyclic defect group, $d = 1$, for $p = 5, 7, 11, 13, 17, 19, 23, 29, 31$, CYCLIC/SortAt_p.html.
(4) Nonabelian defect group, $d \geq 3$, for $p = 3, 5, 7$, BLOCKS/NonabSortAt_p.html.

For each block of a group in the Character Table Library, the program generated a text file data sheet containing various invariants of the groups and the block, including the decomposition matrix where known. Where the decomposition matrix was not known, the program at least calculated its dimensions $k(B)$ and $\ell(B)$. The invariant $k(B)$, the number of ordinary characters of the block, was available. The invariant $\ell(B)$ was calculated by using a method taken from [LP] to get an integral matrix, restricting the character table to the given rows and the elements of order prime to p, and calculating the rank of the matrix. The first sorting of the data was by $(k(B), \ell(B))$ pairs.

The next problem was to sort the data into Morita equivalence classes. From the decomposition matrix one computes the diagonal of the Cartan matrix by taking the scalar product of each column with itself, a procedure for which the chosen ordering of the rows is irrelevant. If one sorts the Cartan diagonal in ascending order, the ordering of the columns is also irrelevant. The list of blocks was sorted by these sorted Cartan diagonals.

For blocks with noncyclic elementary abelian defect group, the sorted Cartan diagonals generally classed together blocks with the same decomposition matrix up to permutations of the rows and columns. A notable exception was the case $p = 3$, $k(B) = 6$, and $\ell(B) = 4$, which contained two sets of decomposition matrices, corresponding to the principal and non-principal blocks the $2.A_6$, the double cover of the alternating group on six numbers. Another interesting case was $p = 3$, $k(B) = 6$, and $\ell(B) = 2$. There are three possible Brauer correspondents, with differing shapes for their generalized decomposition matrices. For each sorted Cartan diagonal there is a unique decomposition matrix, but there may be up to three different Morita equivalence classes of blocks, which can be distinguished by the shapes of their generalized decomposition matrices.

For blocks with cyclic defect group, the Cartan diagonal is not very helpful in distinguishing Morita equivalence classes. In the cyclic case the Morita equivalence class are determined by the Brauer trees. For the sporadic groups the Brauer trees of the cyclic blocks are calculated in [HL]. For the other blocks in our database the trees were calculated by Tal Kadaner [Ke] in her thesis, up to some uncertainty about the cyclic ordering of branches in complicated cases.

Finally, for a given Cartan diagonal, the blocks were sorted by the list of character degrees, after reduction by dividing out by the g.c.d of the degrees. Ruth Leabovich undertook the study of blocks with the same reduced character degrees in the database which were neither Morita equivalent to the Brauer correspondent nor derived from blocks of cyclic defect. She was able to show in almost all cases that there was a Morita equivalence derived from Clifford theory.

Even for blocks for which the decomposition matrix was not available in GAP 4.2, we listed the blocks under the Cartan diagonal of a block with the same reduced degress and indicated by "*" that the Cartan diagonal was only conjectural. For many of these blocks the Morita equivalence to a block in that group was established by other means.

Based on her work, the hypertext index was generated again with links to show the connections among the blocks leading back to a "root" block, usually in a group of minimal size. This was largely a matter of filtering out "noise", in order to reduce the totality of blocks to a nontrivial subset. The links were categorized into six possibilities:

(1) (IN) Inflation, where the block is linked to a block of a quotient group from which it is derived by inflation;
(2) (MU) The group is a direct product, and a block in one factor is "multiplied" by a defect zero block in the other factor, with the link given to the corresponding block of the first factor;
(3) (CL) More complicated Clifford theory, e.g., two blocks conjugate under an automorphism producing a single block in the extension by the automorphism;
(4) (MO) Application of Morita's theorem for grouping blocks by conjugacy classes of blocks of the maximal normal p' subgroup N, where the link is to the stabilzer of a representative, divided by N.
(5) (ISO) Isomorphism between blocks;
(6) (SC) Scopes reduction or some extension thereof [Sc].

In many cases the links were obvious from the group names, but in case (4) an analysis of the character table was required and (6) was more advanced.

We have not extended the system of links to nilpotent blocks, and certain other classes of blocks which were particularly problematical. Ruth Leabovich expects to make this extension as part of the work for her Ph.D. thesis.

Some of the root blocks were cases for which the Broué conjecture has been solved, and for these cases Mikhal Cohen supplied the references. Other cases which we believe to be open and interesting are marked with a "?".

Normalizers and Centralizers

The basic conception of the database, as a tool for working on the Broué conjecture, was to match up blocks with their Brauer correspondent, calculating the decomposition matrices for each, and comparing the centralizers of representatives of various conjugacy classes of non-trivial elements of the defect group.

One difficulty with working in the GAP Character Table Library is the lack of the group. This makes it difficult to calculate the Brauer correspondent. The calculation was done in two stages:
(1) *Via invariants.*

The normalizer $N = N_G(D)$ of the defect group D of a block B is a group with a normal subgroup. By a theorem of Külshammer, such a block is of the form $M_t(K^\alpha[D \rtimes H'])$, where H' is a p'-group and α is a cocycle in $H^2(H', K^*)$. Using a trick of Reynolds [Re], we note that this is Morita equivalent to a block of a group $D \rtimes H$, where H is a central extension with cyclic kernel C_m and quotient H' with $D \xrightarrow{\sim} C_p \times C_p$. For defect 2, we can find the candidate blocks by considering p'-subgroups of $GL(2,p)$ and their central extensions by C_2, a list of which are given at [S2]. These groups produced $(k(b), \ell(b))$ pairs corresponding to all the global blocks B. For defect $d \geq 3$ the computation was more complicated and generally required determining the normalizer by one of the methods described below. Leabovitch [L] also showed that in many cases there were linear characters extending the central characters, and thus the various blocks of the normalizer were isomorphic.

The minimal group $D \rtimes H$ having a block Morita equivalent to the Brauer correspondent b was called the *Reynolds group* of B and tables of the Reynolds groups for the blocks with $p = 3, k(B) \leq 24$ and $p = 5, 7$ are given in [HS].

Had there been a global blocks B for which we could not find a block with the same invariants in some group with normal subgroup D, this would have been a counter-example to the Broué conjecture, but in fact no such block was found. The next step, then, was to verify that the block of the Reynolds group was indeed Morita equivalent to a block of the normalizer of B, and choose among Reynolds groups when there was a choice.
(2) *Explicit calculation of normalizers and centralizers.*

For the root blocks, those not readily available from smaller groups by Clifford theory, many of the normalizers and centralizers were calculated explicitly by Ayala Bar-Ilan [Ba] and and Tzviya Berrebi [Be]. This was done in one of four possible ways, depending on the group.

(a) *Symmetric and alternating groups and their covers.* In this case the groups were known from the combinatorial considerations. Let $c = (1 \dots p)$ and $d = (p+1 \dots 2p)$, and let D be the elementary abelian group $< c, d >$.

(a.1) Assume $n \geq 2p$. Then
  (a) $N_{S-n}(D) = (C_p \rtimes C_{p-1}) \wr C_2 \times S_{n-2p}$,
  (b) $C_G(D) = D \times S_{n-2p}$,
  (c) $C_G(c) \xrightarrow{\sim} C_p \times S_{n-p}$, and
  (d) $C_G(cd) \xrightarrow{\sim} (D \rtimes C_2) \times S_{n-2p}$.

(a.2) For the alternating group A_n, we get the intersection of the above normalizers and centralizers with A_n. The cases $n = 2p, 2p+1$ are special. Thereafter, every odd permutation in the normalizer of D in S_{2p} can be multiplied by a transposition in S_{n-2p}. The case of particular interest is the centralizer $C_{A_n}(cd) = (D \times A_{n-2p}) \rtimes C_2$, where the action of the nontrivial element of C_2 in D is to interchange c and d, while its projection onto S_{n-2p} is a transposition. This is actually isomorphic to $C_3 \times (C_3 \times A_{n-2p}) \rtimes C_2$,

(a.3) The covering groups $\tilde{A}_n$ and $\tilde{S}_n$. Here we get a central extension with kernel C_2 of the groups in (a.1) and (a.2). Let N denote the normalizer of the defect group D in S_{2p} and N_A the normalizer of D in A_{2p}. Let $\tilde{N}$ and $\tilde{N}_A$ be the preimages in $\tilde{S}_{2p}$ and $\tilde{A}_{2p}$, each a central extension by C_2 of the corresponding group N or N_A, using the rules given in [C]. Similarly, we can consider $\tilde{C}$, the preimage of the centralizer of cd in S_{2p}. If $p \equiv 1 \pmod 4$, we get $\tilde{C} \xrightarrow{\sim} D \rtimes C_2$, and if $p \equiv 3 \pmod 4$, then $\tilde{C} \xrightarrow{\sim} D \rtimes C_4$. In [HH], Hoffman and Humphreys define an amalgamated product of covering groups, which is a direct product modulo the identified kernel, and which we will denote by "$\bar{Y}$". With this notation,
  (a) $N_{\tilde{S}_n}(D) = \tilde{N} \bar{Y} \tilde{S}_{n-2p}$.
  (b) $C_{\tilde{S}_n}(c) = C_p \times \tilde{S}_{n-p}$.
  (c) $C_{\tilde{S}_n}(cd) = \tilde{C} \bar{Y} \tilde{S}_{n-2p}$.
  (d) $N_{\tilde{A}_n}(D) = \tilde{N}_A \bar{Y} \tilde{A}_{n-2p}$.
  (e) $C_{\tilde{A}_n}(c) = C_p \rtimes \tilde{A}_{n-p}$.
  (f) $C_{\tilde{A}_n}(cd) = (D \times \tilde{A}_{n-p}).C_2$.

(b) *Special linear groups.* One common source of blocks with elementary abelian defect group are the special linear groups $SL(2, q)$, where $q = p^d$ is a non-trivial power of the odd characteristic p. The defect group is given by the upper triangular unipotent matrices and the normalizer is $N = (C_p)^d \rtimes C_{q-1}$. The library of groups contains various quotients and extensions, for which the group itself is not available in GAP. To determine the exact extension or quotient of N required an analysis of the character table and a study of the possible groups of the correct order with the proper invariants. Because the number of simples $\ell(B)$ in a block of normal defect group is the number of characters of the p' complement, there were usually few candidates. For this work, the advanced search capabilities in our database of groups of order up to 2000 [MSV] were useful, because we could filter out by number of characters and then compare the structure description of the groups with that of N. This database, like all the others, was created in GAP 4.2 and should be used subject to the GAP copyright.

(c) *Other groups for which suitable representations were known.* The first problem was to obtain the group; this was done whenever possible by downloading permutation representations from the ATLAS of finite group representations. (For calculating normalizers of small p-groups inside large groups, the matrix representations generally were too slow.)

Normalizers and centralizers are standard functions in GAP, so if the defect group was the p-Sylow subgroups it was straightforward to calculate them. When the defect groups was not a p-Sylow group, Bar-Ilan and Berrebi used a program of T. Breuer to calculate the defect classes, i.e., the conjugacy classes of p' order such that the p-Sylow subgroup of the centralizer of a representative is isomorphic to D. Then by comparing orders and centralizer sizes, the program located a corresponding conjugacy class representative in the group, and calculated the p-Sylow subgroup of its centralizer. The representations were downloaded by hand, but the program calculated the desired results for all the groups at once and stored them in a file to be read by the program which generated the data sheets. For these root blocks, the following subgroups were described:

1) $H = N_G(D)$;
2) $C_G(D)$;
3) For each conjugacy class $[u]$ of p-elements in D, $C_G(u)$ and $C_H(u)$.

(d) *Other groups.* When the normalizer and the centralizer were too large, they could at times be identified from the list of maximal subgroups in the ATLAS and from the centralizer orders of elements.

Example 1. For the sporadic Held group (He) and prime $p = 3$, the normalizer of D can be identified as $((C_4 \times C_2) \rtimes C_4) \rtimes (S_3 \times C_2)$, but the centralizer of an element u of order 3 is 7560, too large for the Small Groups Library. However, the list of maximal subgroups contains a group of order 15120, identified as the normalizer of an element of order 3, with structure $C_3.S_7$, from which we deduce that the centralizer is $C_3.A_7$.

Example 2. Consider the principal 3-block of defect $d = 4$ in the fifth maximal subgroup G of the O'Nan group, a non-solvable group of order 25920. This is a block with invariants $k(B) = 24$, $l(B) = 14$. The sixth maximal subgroup N of the O'Nan group is a solvable group of the same order, which, according to the information in the ATLAS, is the normalizer of the 3-Sylow subgroup D, the defect group of the block. Thus $N_G(D) = G \cap N$. The structure of G as given in the ATLAS is $((C_3 : C_4) \times A_6).C_2$, so the structure of $N_G(D)$ must be $(C_3)^4 : (C_4 \times C_4).C_2$. In order to find the exact group, we downlowded a permutation representation of the O'Nan group, calculated the Sylow-3 group and then its normalizer N. The Sylow-2 subgroup of N had order 64. It was necessary to find a subgroup of order 32 with the proper structure, which, together with D, generated the desired normalizer, and this we did. It was of order 2592, too large for the GAP library of small groups.

For algebraic groups in non defining characteristic q, for which $p \mid q-1$, it occasionally happened that the centralizer of an element was of the form $C_p \times G'$, where G' was a related algebraic group of smaller rank. For simple groups, centralizers of elements of low order play a role in the classification . So for some primes and some groups, we may be able to fill in the remaining gaps from the literature.

Generalized Decomposition Numbers

There is an extension of Broué's conjecture by Rickard [R2], in which the two-sided tilting complex X is *splendid*, i.e.,can be made up of modules which are direct summands of p-permutation modules. Let us suppose that X is a tilting complex from a symmetric algebra kGe to a symmetric algebra kHf, both of which are group blocks with a common abelian defect group D. Let Q be a non-trivial p-subgroup of D. The point to this is to ensure that the various restrictions X_Q of X as complexes of $C_G(Q)e$-$C_H(Q)f$-modules will also be tilting complexes. This means that the decomposition matrices of the corresponding blocks of the centralizers would also be also obtained, one from the other, by column operations, permutations and multiplication of rows by ± 1.

To understand the significance of this claim, let us consider, for each non-trivial u in D, the submatrix M_u of the character table consisting of rows of the block and columns which are conjugacy classes with representatives of the form uy for a p-regular element y, i.e., elements y of p' order. The part of the generalized decomposition matrix of the block corresponding to u describes how the rows of this matrix are generated by the Brauer characters of the corresponding blocks of $C_G(Q)$ for $Q =< u >$ ([Th],§43). The column operations taking the decomposition matrix of $C_G(Q)e$ to $C_H(Q)f$ determine a change of basis on the Brauer characters. Thus we expect corresponding parts of the generalized decomposition matrix to differ by column operations. Were we to find a case in which this did not hold, it would be a potential counterexample to the claim that there is a splendid tilting complex. So far we have not found any such example.

We have other reasons for wishing to know the complete generalized decomposition matrix. Our attempts to find column operations transforming the regular decomposition matrix is often hampered by difficulties in matching up the rows, which have undergone a permutation. Since the generalized decomposition matrix of a block is nonsingular, and often some of the sections of the generalized decomposition matrix have a single column, the information from the generalized decomposition matrix is often sufficient to determine the permutation.

Ayala Bar-Ilan wrote a program to construct the generalized decomposition matrix, but it requires character tables for the centralizers which are not currently available for each block. One of the most complicated parts of the program was an algorithm for matching up the conjugacy classes in the large group with the conjugacy classes in the centralizer of u.

Since we did not have the character tables of the centralizers for most of the groups in our database, this would have been quite complicated to implement as part of the program which constructs the database.

The remaining problem was to find a basis for the set of Brauer characters of $C_G(u)$. From the examples calculated by Ayala Bar-Ilan, it became clear that these Brauer characters can almost always be found among the rows of the section of u, up to a possible multiple by -1 to make the value at u positive. (This is quite different from the decomposition matrix, where the values of the p-regular elements rarely contain a complete basis for a positive integral decomposition.) Taking such a basic set of vectors for each section, we calculated the resulting decomposition of the rows of the section according to this basis. Occasionally there were failures involving arithmetic with algebraic integers. The function available in GAP was intended originally for rows of character tables. It attempts to convert the vector of algebraic integers into an integral vector using an integral basis as in [LP]. Unfortunately, when u is not conjugate to all of its powers, one does not have all the necessary columns in M_u.

When the decomposition matrix was known, and the matrix obtained by adjoining the columns of this matrix to the columns of the decomposition matrix was square, we reproduced the total matrix at the bottom of the data scheet.

Example 3. Let G be S_{10} and let B be the first of the two blocks of defect 2. The centralizer of the element (123) is isomorphic to $C_3 \times S_7$. The set of Brauer characters is the same as the set of Brauer characters for S_7. The submatrix M_u of the character table of S_{10} consisting of rows of the block and conjugacy classes with respresentatives of the form uy for y a p-regular element commuting with u is given by the rows of the following matrix:

$$\begin{bmatrix} 6 & 2 & 0 & 1 & -1 & 4 & 0 & 2 & -1 \\ 15 & -1 & -1 & 0 & 1 & -5 & 3 & -1 & 0 \\ 6 & 2 & 0 & 1 & -1 & 4 & 0 & 2 & -1 \\ 21 & 1 & -1 & 1 & 0 & -1 & 3 & 1 & -1 \\ 15 & -1 & -1 & 0 & 1 & -5 & 3 & -1 & 0 \\ -21 & -1 & 1 & -1 & 0 & 1 & -3 & -1 & 1 \\ -6 & -2 & 0 & -1 & 1 & -4 & 0 & -2 & 1 \\ 21 & 1 & -1 & 1 & 0 & -1 & 3 & 1 & -1 \\ -15 & 1 & 1 & 0 & -1 & 5 & -3 & 1 & 0 \end{bmatrix}$$

The first two rows form a basis for the row space, and an examination of the character table of S_7 shows that they are also the Brauer characters of the centralizer.

When the inertial quotient acts freely on the defect group, e.g., a Frobenius action, the blocks of the centralizer are nilpotent and the sections of the generalized decomposition matrix for nontrivial u have dimension one [P2]. Since the generalized decomposition matrix of a block is nonsingular [Th], one of the rows must be a multiple by 1 or -1 of the unique Brauer character. Where, as in these cases of Frobenius action or other cases where we know the exact centralizers, we have succeeded in establishing that our matrix is identical with the generalized decomposition matrix, we label it as such. Mikhal Cohen has been working on increasing the number of blocks for which the generalized decomposition matrix is provided.

Applications

The database has so far been known only to our local research group. The applications to date have been as follows.

Explicit tilting complexes. In [S1], we suggested an approach to the blocks of defect $C_p \times C_p$ using decomposition matrices to build up explicit tilting complexes using a sequence of elementary steps and simplifying the resulting mapping cones to get explicit one-step tilting complexes from the Brauer correspondent to the block B. The complexes which have been verified are given in the first part of this paper. The project has foundered on the difficulty with constructing stable equivalences for non-principal blocks. However, as part of the project, Genadi Chereshaya did find tilting complexes which, when applied to the Brauer correspondent b, produce blocks with the same decomposition matrix as B. This was done for various non-principal blocks with defect group $C_3 \times C_3$ and for blocks with defect group $C_5 \times C_5$ that have a maximal number of exceptional characters (i.e., characters producing multiple rows in the decomposition matrix). These have not yet been included in the database because they are still conjectural.

It is particularly easy to construct the conjectural tilting complexes for blocks with a maximal number of exceptional characters, i.e., characters whose multiplicity when restricted to the p' elements is greater than one. For example, when $p = 5$, all the blocks with $k(B) = 14$, $\ell(b) = 6$ have four pairs of exceptional characters. In these cases there is an actual algorithm to calculate the conjectural tilting complex [Che]. When this tilting complex is stable under an outer automorphism, then it may also provide a conjectural tilting complex for a block of the group extended by the automorphism.

Morita equivalent families. In addition to the Clifford theory-type Morita equivalences, the categorization in the database has brought to light other structurally determined Morita equivalences, including a whole set of such equivalences between blocks of $\tilde{S}_n$ and blocks of $\tilde{A}_n$, where these are, respectively, the Schur covering groups of the symmetric and alternating groups [KS].

The databases

The home page for the entire collection of databases, including a database of character tables, the p' subgroups of $GL(2, p)$ and some extensions, and the database of abelian non-cyclic blocks for 3, 5, 7, 11, and 13:
`http://www.cs.biu.ac.il/ mschaps/math.html`

In view of the results of Chuang-Rouquier on derived equivalent families of non-abelian defect group, there has been an extension for non-abelian defect groups, but it is not well developed as of this writing. Although the Broué conjecture has been solved for the cyclic blocks, other questions have arisen about the tilting complexes, so we have added a database of cyclic blocks [SZ],[RS].

We would appreciate it if any further results using these databases would include a reference and that we be informed.

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