LECTURE I: THE ORDINARY REPRESENTATIONS OF GROUPS

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The symmetric group S_n, the group of all permutations of a finite set with n elements, is one of the most pervasive objects in in mathematics, and, for that matter, in physics. Every finite group, for example, is a subgroup of a symmetric group. When we are considering the representation theory of the symmetric group, then we are concerned with the algebra generated by group, whose representation theory has many important combinatorial properties. The group algebra has a deformation known as the Hecke algebra, which we intend to discuss together with the symmetric group itself. In addition, the symmetric group is simplest example of the important class of groups appearing in the theory of Lie algebras under the name of "Weyl groups". Some of these other Weyl groups also have a Hecke algebra theory, and we hope to explain this in the course of these lectures.

1. Ordinary Representations of Groups

For any finite groups $G = \{g_1, \ldots, g_s\}$ and field F, a *representation* of G over F is a group homomorphism

$$\rho : G \to GL_r(F),$$

and r is called its *degree*. When we wish to describe the representation in a basis-free fashion, then we will give it as a homomorphism into $GL(V)$, where V is an F-vector space.

Let FG be the corresponding group algebra, with F-vector space basis $\langle g_1, \ldots, g_r \rangle$ and algebra multiplication induced by the group multiplication law and linearity. Then the representation ρ induces an algebra homomorphism

$$\tilde{\rho} : FG \to GL_r(F),$$

where $\sum a_i g_i \in FG$ is mapped to $\sum a_i \rho(g_i)$. The representation ρ can be recovered from $\tilde{\rho}$ by restricting to G, and we will consider them interchangeable. The most compact description of ρ or $\tilde{\rho}$ is given by specifying the images of a set of generators of G, as matrices satisfying the defining relations of the group. Each representation

$$\tilde{\rho} : FG \to GL(V)$$

determines a FG-module structure on V, given by

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$$g \cdot v = \tilde{\rho}(g)(v),$$

and every module gives a representation, so that we use the two terms almost interchangeably, depending on which viewpoint is more convenient for the purpose at hand. A subrepresentation, for example, is most easily defined as a submodule W of V. Since the action of the generators of the group maps W into itself, then in fact, the homomorphism $\tilde{\rho}$ determines a representation $\tilde{\tau} : FG \to GL(W)$.

Two representations ρ and τ are considered equivalent if they differ only by a change of basis in the underlying vector space F^r on which the matrices act. More formally, $\rho \sim \tau$ if there exists an invertible $r \times r$ matrix P such that for each $G \in G$, $\tau(g) = P^{-1}\rho(g)P$. The "atomic particles" of representation theory are the *irreducible* representations, which have no subrepresentations at all. The "molecules" of representation theory are the *indecomposable* representations, those which are not equivalent to a nontrivial direct sum of representations.

We assume, for the remainder of this section, that F satisfies the following two conditions, in which case we say that we are studying the *ordinary* representations of the finite group G.

- (generic characteristic): F is a field of characteristic 0, or of positive characteristic p such that $p \nmid |G|$.
- (F sufficiently large): F contains a primitive root of unity ζ for $|G| = s$.

If these two conditions hold, then the indecomposable representations are, in fact, irreducible, having no subrepresentations, and then by Maschke's theorem, we can write FG as a direct sum of simple algebras and decompose FG in the form

$$FG \cong \bigoplus_{i=1}^{t} M_{n_i}(F).$$

Each component determines a unique irreducible representation ρ_i and these give a complete set of representatives of irreducible representations. The size n_i of the matrix block is the degree of the corresponding irreducible representation. The inverse images e_i of the identity matrices I_{n_i} of the t components form a basis for the center $Z(FG)$ of the group algebra FG.

A representation ρ is completely determined by its character χ_ρ, a map from G into F given by

$$\chi_\rho(g) = tr(\rho(g)).$$

Since the trace is invariant on conjugacy classes of matrices, χ_ρ is a class function on G, that is, fixed on conjugacy classes. The character of the irreducible representation ρ_i will be denoted by χ_i for brevity. The addition of characters corresponds to direct sums of representations, and the componentwise product of characters corresponds to tensor products of representations.

The conjugacy class sums form an alternative basis for the center $Z(FG)$, and thus there are exactly t conjugacy classes. We will order the elements of G so that g_1 is the identity element e of the group and g_j, for $j = 1, \ldots, t$ are representatives of the distinct conjugacy classes. The conjugacy class $[g_j]$ will be denoted by C_j and its class sum $\sum_{g\ g_j} g$ will be denoted by $\widehat{C}_j$.

We recall that we have assumed that F is of generic characteristic and is sufficiently large. The character table X is the $t \times t$ square matrix $[\chi_i(g_j)]$. For any i, j the character value $\chi_i(g_j)$ is the sum of the eigenvalues of $\rho_i(g_j)$, and since g_j is of finite order dividing the order s of G, the entries in the character table can be expressed as powers of ζ, making them more or less independent of the actual choice of the field F, as long as it satisfies our assumptions. If N is the diagonal matrix with the degrees along the diagonal, and M is the diagonal matrix with conjugacy class sizes $m_j = |C_j|$ along the diagonal, then $W = N^{-1}XM$, the matrix of central characters, gives a base change matrix between the two bases of $Z(FG)$, in the sense that

$$\hat{C}_j = \sum w_{ij} e_i.$$

Since W, N and M are all invertible, so is X.

Example 1.1. Consider the symmetric group S_3. There are three conjugacy classes: The class of the identity is centralized by the entire group,the class of acycle of length 3 ia centralized by the unique normal subgroup that it generates, and the class containing all three transpositions,each of which is centralized by the subgroup that it generates.

There are two linear irreducible representations, the trivial representations which sends every group element to 1, and the alternating representation, which sends the even permutations to 1 and the odd permutations to -1. There there is a representation of degree 2.

$$X = \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{matrix}, N = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix}, M = \begin{matrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{matrix}.$$

We then get

$$W = \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{matrix}.$$

If we calculate

$$W^{-1} = \frac{1}{6} \begin{matrix} 1 & 1 & 4 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{matrix}$$

we can find the central idempotent corresponding to each irreducible representation.