# LECTURE II: THE ORDINARY REPRESENTATIONS OF THE SYMMETRIC GROUPS AND THE DEFINITION OF THE HECKE ALGEBRA

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## 1. Ordinary representations of the symmetric groups

We denote by $S_n$ the symmetric group on $n$ objects. Let $F$ be a field of characteristic 0 or of characteristic $p$ with $p > n$. In addition, we assume that $F$ is sufficiently large in the sense that we defined in the last lecture. For the symmetric groups $S_n$, the conjugacy classes consist of all elements with the same cycle structure. Each cycle structure can be represented by a partition of $n$, and thus there is a natural one-to-one correspondence between conjugacy classes and partitions. The conjugacy class representative with the integers $\{1, \ldots, n\}$ inserted in cycles of lengths $\lambda_1, \ldots, \lambda_k$ will be denoted by $g_\lambda$.

It is considerably more difficult, though standard, to show that the irreducible representations are also in one-to-one correspondence with the partitions of $n$. There are various classical proofs of this fact. We now leave the standard material which has been known in essence since the time of Frobenius 150 years ago, and describe a new approach to the representations of the symmetric group which goes back about 15 years. We will give a sketch of a more modern proof, based on what is called the "multiplicity-one" theorem. For the purposes of these lectures,we wish to focus on a particular set of generators and relations for the symmetric group,the so-called Serre relations. We take as generators the transpositions $s_i = (i\ i+1)$ These generate the symmetric group and, addition to satisfying $s_i^2 = e$, they satisfy the braid relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} s_i s_j = s_j s_i, |i - j| > 0.$$

For any $k < n$, we consider $S_k$ as the subgroup of $S_n$ consisting of all permutations fixing all but the first $k$ numbers. For each $k < n$, define $L_k \in FS_k$ by

$$L_k = \sum_{m<k} (m\ k)$$

Note that $L_k \in C_{FS_k}(S_{k-1})$, because conjugation by an element of $S_{k-1}$ leave the $k$ fixed in each transposition and permutes the transpositions in the sum. This implies that the $L_k$ commute with each other. We also have the following properties:

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(1) $L_1 = 0, L_2 = s1$,
(2) $L_{i+1} = s_i L_i s_i + s_i$

The commutative algebra $\mathcal{A}_n$ generated by $\{L_1, \ldots, L_n\}$ is called the *Gelfand-Zeitlin algebra*

Since $F$ is of generic characteristic, we have Maschke's theorem, which tells us that $FS_n$ is semisimple as a module over itself. All our modules will be left modules.

**Theorem 1.1.** *If $V \in Irr(FS_n)$, then the restriction of $V$ to $FS_{n-1}$ is multiplicity-free, i.e.*

$$res^n_{n-1} V = \bigoplus W_j, W_j \in Irr S_{n-1}, i \neq j \Rightarrow W_i \ncong W_j$$

**Sketch of proof:**

(1) Let $C = C_{FS_n}(FS_{n-1})$. Then $C$ is generated by $Z(FS_{n-1})$ and $L_n$. Therefore $C$ is a commutative subalgebra of $FS_n$.
(2) Let us consider a particular component $W$ of the restriction and let $k$ be its multiplicity.

$$res^n_{n-1} V = W^{\oplus k} \bigoplus X$$

It is possible to demonstrate that $C \supseteqq End_{FS_n}(W^{\oplus k})$. Since $C$ is a commutative and the endomorphism algebra is only commutative if $k = 1$, we find that $W$ has multiplicity 1, as we wished to show.

Since $L_n$ commutes with $FS_{n-1}$, $L_n$ acts as a scalar with eigenvalue $i_j$ on $W_j$. It will turn out that $i_j \in \mathbb{Z}$ and that if $j \neq k$, then $i_j \neq i_k$. If we continue decomposing the $W_j$, each one breaks up into irreducible $FS_{n-2}$ modules, each determined uniquely by a pair of eigenvalues, one for $L_n$ and one for $L_{n-1}$. Continuing in this fashion, we get down to $S_1$, for which all irreducible modules are one-dimensional. Thus we get a basis of $V$, called the *Gelfand-Zeitlin basis*, such that each basis element $v$ is a simultaneous eigenvector for the Gelfand-Zeitlin algebra defined above, and each simultaneous eigenvector $\underline{i} = (i_1, \ldots, i_n)$is different. We also get a graph $\mathbb{B}$ called the *branching graph*or *crystal*, an infinite graph in which the vertices correspond to irreducible modules for the various symmetric groups and an irreducible module $V$ for $S_n$ is connected to an irreducible module $W$ for $S_{n-1}$ if $W$ is a component of the restriction of $V$.

We define the *content* of a sequence $\underline{i}$ to be the two sided sequence $\gamma(\underline{i}) = (\ldots, \gamma_{-1}, \gamma_0, \gamma_1, \ldots)$ in with $\gamma_k$ is the number of time that $k$ occurs as an eigenvalue in the sequence $\underline{i}$. Thus is $\gamma(\underline{i})$ is a sequence of positive integers with only a finite number of nonzero terms. Not every sequence of integers can occur as a path $\underline{i}$. It must satisfy the following conditions:

(1) $i_1 = 0$
(2) $\{i_j - 1, i_j + 1\} \cap \{i_1, \ldots, i_i - 1\} \neq \emptyset$ for all $j, 1 \leq j \leq n$

(3) if $i_j = i_k = a$ for some $j < k$ then

$$\{a-1, a+1\} \subseteq \{i_{j+1}, \ldots, i_{j-1}\}$$

With considerable hard work, it can be shown that two different paths $\underline{i}$ and $\underline{i}'$ correspond to basis elements in the same irreducible $V$ if and only if they have the same content. Furthermore, with yet more hard work, one can demonstrate that the conditions on $\underline{i}$ given above are precisely what are needed to show that the path $\underline{i}$ defines a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of $n$.

**Example 1.1.** Let $n = 7$, and suppose the $\underline{i} = (0, 1, -1, 2, 3, -2, 0)$. It starts with 0 and never moves by steps of more than 1 from the previous numbers, so the first two conditions are fulfilled. The only two numbers with repeat are the 0, and there we have $-1$ and 1 in between the two copies of 0. So the third condition also holds.

We build a partition by considering a grid of boxes in which the numbers are the same on the diagonals and increase as one goes upward to the right. We then fill in the numbers from 1 to $n$ as we go through the sequence.

| | | | |
|---|---|---|---|
| 1 | 2 | 4 | 5 |
| 3 | 7 | | |
| 6 | | | |

We give the residues for the partition $(4, 2, 1)$. For compactness we use $a, b, \ldots$ to designate $-1, -2, \ldots$.

| | | | |
|---|---|---|---|
| 0 | 1 | 2 | 3 |
| $a$ | 0 | | |
| $b$ | | | |

The diagram

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | | |
| 7 | | | |

corresponds to a sequence $\underline{i} = (0, 1, 2, 3, -1, 0, -2)$.

The diagram

| | | | |
|---|---|---|---|
| 1 | 4 | 6 | 7 |
| 2 | 5 | | |
| 3 | | | |

corresponds to a sequence $\underline{i} = (0, -1, -2, 1, 0, 2, 3)$.

This diagram, called a Young diagram, represents a partition. For ordinary representations, each Young diagram is uniquely described by the lengths of its diagonals, which are given by its content, the list of the number of nodes of each residue. The partition is regarded as having been built up one node at a time from the empty partition, in such a way that each intermediate diagram is a partition. Each such path is recorded by placing the integers from 1 to $n$ in the nodes of the diagram, creating a *standard tableau* in which the integer are increasing on every row and column. The degree $r_\lambda$ of the representation $\rho_\lambda$ is the number of such paths and the set of standard tableaux may be taken as a basis. The defining property of this representation, in the classical theory, is that it is simultaneously induced from the trivial representation of the subgroup of $S_n$ stabilizing the rows and from the alternating representation of the subgroup of $S_d$ stabilizing the columns.

The character corresponding to a partition $\lambda$ will the denoted by $\chi_\lambda$. The representation corresponding to the partition with one part, $(n)$, is the trivial representation of degree 1, sending every group element to 1, and the representation corresponding to the partition with $d$ parts, $(1, 1, \ldots, 1)$ is the alternating representation, also of degree 1, which sends each group element to 1 if the permutation is even and to $-1$ if the permutation is odd.

## 2. Coxeter groups and Hecke algebras

The symmetric group is an example of a Coxeter group. A Coxeter groups is defined by a tree whose edges are labeled by integers greater than or equal to 3. In fact, we will suppress the label 3 and only label adges when the label is greater than or equal to 4. The generators are assumed to be elements of order 2, called reflections, in one-to-one correspondence to the vertices of the tree. If two generators are not connected by an edge,they are assumed to commute and we add the corresponding commutation relation to the list of relations. If $s_i$ and $s_j$ are connected by an edge of label $m$, then we add a relation

$$(s_i s_j)^m = e$$

When $m = 3$, this relation is usually rewritten in the braid form:

$$s_i s_j s_i = s_j s_i s_j$$

**Example 2.1.** The dihedral group $D_m$ is a Coxeter group defined by a tree with two vertices and one edge labeled by $m$. If $a$ and $b$ are the usual generators of the dihedral group, of order 2 and $m$ respectively, then we take the two Coxeter generators to be $a$ and $ab$.

The symmetric group on $n$ objects is the Coxeter group determined by a linear tree with $n - 1$ vertices and only labels 3.

We will be concerned in these lectures with Coxeter groups determined by two families of graphs and three exceptional graphs. (Draw: $A_n, D_n, E_6, E_7, and E_8$). These graphs are important as being graphs corresponding to finite Lie algebras. There are other graphs corresponding to finite Lie algebras (Types B, C, F, G). However, they do not fit in well with the Coxeter scheme and will not be included in our lectures.

To each of the Coxeter groups corresponding to one of these graphs, we can associate a group algebra, $FG$. This algebra has a deformation. These means that there is a flat family of algebras (flat basically saying that the vector space dimension does not jump in any fiber). The deformed algebra has the same number of generators but deformed relations. In our case, the relations that we will deform will be the relations declaring the generators to be transpositions, while the braid relations will remain fixed. We have a deformation parameter $q$, but, unlike most deformation, the default value is 1 rather than 0.

Thus we replace each $s_i$ by a generator $T_i$, and use the new relation $(T_i + 1)(T_i - q) = 0$.

It is easy to see that each of the $T_i$ is still invertible, the inverse being given by $T_i^{-1} = \frac{1}{q}(T_i + (1 - q))$.