LECTURE III: THE ORDINARY AND MODULAR REPRESENTATIONS OF THE SYMMETRIC GROUPS

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1. ORDINARY REPRESENTATIONS OF THE SYMMETRIC GROUPS

The action of L_n on W_j is scalar, with eigenvalue i_j on W_j. It will turn out that $i_j \in \mathbb{Z}$ and that if $j \neq k$, then $i_j \neq i_k$. If we continue decomposing the W_j, each one breaks up into irreducible FS_{n-2} modules, each determined uniquely by a pair of eigenvalues, one for L_n and one for L_{n-1}. Continuing in this fashion, we get down to S_1, for which all irreducible modules are one-dimensional. Thus we get a basis of V, called the *Gelfand-Zeitlin basis*, such that each basis element v is a simultaneous eigenvector for the Gelfand-Zeitlin algebra defined above, and each simultaneous eigenvector $\underline{i} = (i_1, \ldots, i_n)$is different. We also get a graph $\mathbb{B}$ called the *branching graph*or *crystal*, an infinite graph in which the vertices correspond to irreducible modules for the various symmetric groups and an irreducible module V for S_n is connected to an irreducible module W for S_{n-1} if W is a component of the restriction of V.

We define the *content* of a sequence $\underline{i}$ to be the two-sided sequence $\gamma(\underline{i}) = (\ldots, \gamma_{-1}, \gamma_0, \gamma_1, \ldots)$ in with γ_k is the number of time that k occurs as an eigenvalue in the sequence $\underline{i}$. Thus is $\gamma(\underline{i})$ is a sequence of positive integers with only a finite number of nonzero terms. Not every sequence of integers can occur as a path $\underline{i}$. It must satisfy the following conditions:

(1) $i_1 = 0$
(2) $\{i_j - 1, i_j + 1\} \cap \{i_1, \ldots, i_i - 1\} \neq \emptyset$ for all $j, 1 \leq j \leq n$
(3) if $i_j = i_k = a$ for some $j < k$ then
$$\{a - 1, a + 1\} \subseteq \{i_{j+1}, \ldots, i_{j-1}\}$$

With considerable hard work, it can be shown that two different paths $\underline{i}$ and $\underline{i}'$ correspond to basis elements in the same irreducible V if and only if they have the same content. Furthermore, with yet more hard work, one can demonstrate that the conditions on $\underline{i}$ given above are precisely what are needed to show that the path $\underline{i}$ defines a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n.

Example 1.1. Let $n = 7$, and suppose the $\underline{i} = (0, 1, -1, 2, 3, -2, 0)$. It starts with 0 and never moves by steps of more than 1 from the

previous numbers, so the first two conditions are fulfilled. The only two numbers with repeat are the 0, and there we have -1 and 1 in between the two copies of 0. So the third condition also holds.

We build a partition by considering a grid of boxes in which the numbers are the same on the diagonals and increase as one goes upward to the right. The underlying grid of residues, where we use $a, b, c, \ldots$ for negative numbers, is

0	1	2	3	4	5
a	0	1	2	3	4
b	a	0	1	2	3
c	b	a	0	1	2

We then fill in the numbers from 1 to n as we go through the sequence.

1	2	4	5
3	7		
6			

We give the residues for the partition $(4, 2, 1)$. For compactness we use $a, b, \ldots$ to designate $-1, -2, \ldots$.

0	1	2	3
a	0		
b			

The diagram

1	2	3	4
5	6		
7			

corresponds to a sequence $\underline{i} = (0, 1, 2, 3, -1, 0, -2)$.

The diagram

1	4	6	7
2	5		
3			

corresponds to a sequence $\underline{i} = (0, -1, -2, 1, 0, 2, 3)$.

This diagram, called a Young diagram, represents a partition. For ordinary representations, each Young diagram is uniquely described by the lengths of its diagonals, which are given by its content, the list of the number of nodes of each residue. The partition is regarded as having been built up one node at a time from the empty partition, in such a way that each intermediate diagram is a partition. Each such path is recorded by placing the integers from 1 to n in the nodes of the diagram, creating a *standard tableau* in which the integer are increasing on every row and column. The degree r_λ of the representation ρ_λ is the number of such paths and the set of standard tableaux may be

taken as a basis. The defining property of this representation, in the classical theory, is that it is simultaneously induced from the trivial representation of the subgroup of S_n stabilizing the rows and from the alternating representation of the subgroup of S_d stabilizing the columns.

The character corresponding to a partition λ will the denoted by χ_λ. The representation corresponding to the partition with one part, (n), is the trivial representation of degree 1, sending every group element to 1, and the representation corresponding to the partition with d parts, $(1, 1, \ldots, 1)$ is the alternating representation, also of degree 1, which sends each group element to 1 if the permutation is even and to -1 if the permutation is odd.

2. Modular representations of the symmetric groups and Iwahori-Hecke algebras at root of unity

We now turn to the representation theory of the symmetric group over a field whose characteristic e does divide $d!$, the so-called modular case. Dipper and James discovered that in this case the representation theory is similar to that of an Iwahori-Hecke algebra in which ξ is an e-th root of unity in F (which has characteristic different from e). The symmetric group case is called the degenerate case, and the remaining values of ξ give non-degenerate Hecke algebras, but the two cases can be treated together with only minor modifications.

We now define the *Iwahori-Hecke* algebra $H_d(F, \xi)$. Let ξ be an element of F^*. We define an algebra H_d with generators $T_1, \ldots, T_{n-1}$ over the field F, using relations

$$(T_i + 1)(T_i - \xi) = 0, 1 \le i \le n - 1$$
$$T_{i+1} T_i T_{i+1} = T_i T_{i+1} T_i 1 \le i \le n - 2$$
$$T_i T_j = T_j T_i, 1 \le i < j \le n - 1, |i - j| \ge 2$$

If ξ is not a root of unity, then the representation theory of H_d is essentially the same as the ordinary representation theory of S_d, with the algebra being semi-simple and the irreducible representations parameterized by the partitions of d. If the field F is generic for S_n and $\xi = 1$, then we get the symmetric group. There is an analog of this construction for the so-called *cyclotomic Hecke algebras* H_d^Λ, in which the blocks correspond to weights with highest weight Λ but we will not go into that theory in these lectures.

The main difference is in the meaning of the residues label the nodes of a partition. These are now taken to be elements of $I = \mathbb{Z}/e\mathbb{Z}$, which we will represent by numbers in the set $\{0, 1, \ldots, e - 1\}$. In the degenerate case they represent elements of the base field of F, and in the nondegenerate case they represent powers of ξ. More formally, we define

$$\nu(i) \equiv \begin{cases} i & (\text{if } \xi = 1) \\ \xi^i & (\text{if } \xi \neq 1). \end{cases}$$

These elements $\nu(i)$ enter the theory as the eigenvalues of the following elements of H_d, known as *Jucys-Murphy* elements:

$$L_r \equiv \begin{cases} (1,r) + (2,r) + \cdots + (r-1,r) & (\text{if } \xi = 1) \\ \xi^{1-r} T_r T_{r-1} \ldots T_2 T_1 T_2 \ldots T_{r-1} T_r & (\text{if } \xi \neq 1); \end{cases} \quad 1 \leq r \leq d.$$

Note that in the degenerate case, $L_1 = 0$ and $L_2 = (1,2)$. In either case, the L_r commute with each other and form a commutative subalgebra Z of H_d known as the Gelfand-Zetlin subalgebra. The restriction of any H_d module M to this subalgebra breaks up into a direct sum of linear representations, with eigenvalues of the form $\nu(i)$. The module induced from an irreducible module for H_{d-1} will be a direct sum of indecomposable projective modules for H_d, each with irreducible socle and each socle with a single eigenvalue of the Jucys-Murphy element L_d. It is this choice of idempotents which distinguishes among various components in the induced modules.

Example 2.1. Let $n = 3$. When F was a field of characteristic 0, there were three irreducible representations, corresponding to three distinct partitions:

$$\boxed{0}\;\boxed{1}\;\boxed{2} \,, \quad \begin{array}{l}\boxed{0}\\\boxed{a}\\\boxed{b}\end{array} \,, \quad \begin{array}{l}\boxed{0}\;\boxed{1}\\\boxed{a}\end{array} \;.$$

However, if F is a field of characteristic 3, all of these young diagrams have the same content, that is, they all have the same residues

$$\boxed{0}\;\boxed{1}\;\boxed{2} \,, \quad \begin{array}{l}\boxed{0}\\\boxed{2}\\\boxed{1}\end{array} \,, \quad \begin{array}{l}\boxed{0}\;\boxed{1}\\\boxed{2}\end{array}$$

This means that they all collapse into a single block, of dimension 6, since the blocks are determined by the content.

References

[1]   A. Kleshchev, *Linear and Projective Representations of Symmetric Groups*, Cambridge Tracts in Mathematics, Cambridge Univ. Press(2005).