LECTURE IV: THE MODULAR REPRESENTATIONS OF THE SYMMETRIC GROUPS AND THE REPRESENTATIONS OF HECKE ALGEBRAS AT ROOTS OF UNITY

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We review the definition of the *Iwahori-Hecke* algebra $H_d(F, \xi)$ from the previous lecture. Let ξ be an element of F^*. We define an algebra H_d with generators $T_1, \ldots, T_{n-1}$ over the field F, using relations

$$(T_i + 1)(T_i - \xi) = 0, 1 \leq i \leq n-1$$

$$T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i 1 \leq i \leq n-2$$

$$T_iT_j = T_jT_i, 1 \leq i < j \leq n-1, |i-j| \geq 2$$

If ξ is not a root of unity, then the representation theory of H_d is essentially the same as the ordinary representation theory of S_d, with the algebra being semi-simple and the irreducible representations parameterized by the partitions of d. If the field F is generic for S_n and $\xi = 1$, then we get the symmetric group. There is an analog of this construction for the so-called *cyclotomic Hecke algebras* H_d^Λ, in which the blocks correspond to weights with highest weight Λ but we will not go into that theory in these lectures.

The main difference is in the meaning of the residues label the nodes of a partition. These are now taken to be elements of $I = \mathbb{Z}/e\mathbb{Z}$, which we will represent by numbers in the set $\{0, 1, \ldots, e-1\}$. In the degenerate case they represent elements of the base field of F, and in the nondegenerate case they represent powers of ξ. More formally, we define

$$\nu(i) \equiv \begin{cases} i & (\text{if } \xi = 1) \\ \xi^i & (\text{if } \xi \neq 1). \end{cases}$$

These elements $\nu(i)$ enter the theory as the eigenvalues of the following elements of H_d, known as *Jucys-Murphy* elements:

$$L_r \equiv \begin{cases} (1,r) + (2,r) + \cdots + (r-1,r) & (\text{if } \xi = 1) \\ \xi^{1-r} T_r T_{r-1} \ldots T_2 T_1 T_1 T_2 \ldots T_{r-1} T_r & (\text{if } \xi \neq 1); \end{cases} 1 \leq r \leq d.$$

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In either case, the L_r commute with each other and form a commutative subalgebra Z of H_d known as the Gelfand-Zetlin subalgebra. Let us just consider L_2 and L_3 in the nondegenerate case:

$$L_2L_3 = \xi^2 T_1T_1(T_2T_1T_1T_2) = \xi^2 T_1(T_1T_2T_1)T_1T_2) = \xi^2 T_1(T_2T_1T_2)T_1T_2) = \xi^2 T_1T_2T_1(T_1T_2T_1)$$

$$L_3L_2 = \xi^2 (T_2T_1T_1T_2)T_1T_1 = \xi^2 T_2T_1(T_1T_2T_1)T_1 = \xi^2 T_2T_1(T_2T_1T_2)T_1 = \xi^2 (T_1T_2T_1)T_1T_2T_1$$

The restriction of any H_d module M to this subalgebra breaks up into a direct sum of linear representations, with eigenvalues of the form $\nu(i)$. The module induced from an irreducible module for H_{d-1} will be a direct sum of indecomposable projective modules for H_d, each with irreducible socle and each socle with a single eigenvalue of the Jucys-Murphy element L_d. It is this choice of idempotents which distinguishes among various components in the induced modules.

The residues which appear in the nodes of the Young diagram correspond to the nodes of a graph, called a Dynkin diagram, which is a cycle of length e. Associated to this diagram is a matrix, called a Cartan matrix, which has 2 along the diagonal, -1 on the off-diagonals, and an additional -1 at the corners of the anti-diagonal. The j-column is a projection of an object denoted by α_j and called the j-th simple root. The sum of the simple roots is called the null-root δ and its projection is 0. There is actually a great deal of Lie algebra theory hiding behind the scenes here. Thus for $e = 5$ the Cartan matrix would be

$$\begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

In the modular case, it is no longer true that every indecomposable representation is irreducible. The blocks now correspond to collections of partitions which can all be reduced to a common a common core partition, obtained by removing a fixed number w of rim hooks of length e. Each rim hook will contain a complete set of residues.

The irreducible modules correspond to those partitions which are e-restricted, i.e., do not have e consecutive columns of the same length. (This is dual to the more traditional notion of e-regular, under the duality which flips the partitions across the main diagonal.) There is a diagram called the crystal graph, due originally to Kashiwara, which describes the way that the irreducible representations are built up by induction.

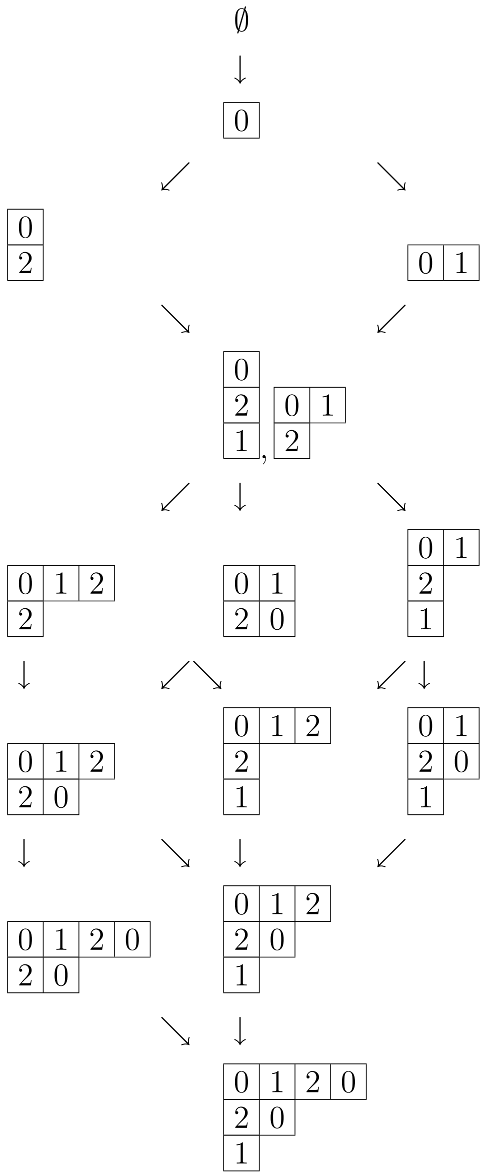
Example 0.1. Let us consider a Young tableau for $e = 3$.

$$\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$$

We can remove two rim hooks, which corresponds to adding two copies of δ, giving $\Lambda_0 - \alpha_0$.

$$\begin{array}{|c|c|c|c|}\hline - & - & - & - \\ \hline 2 & 0 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}\, , \; \begin{array}{|c|c|c|c|}\hline - & 1 & 2 & 0 \\ \hline\end{array}\, , \; \begin{array}{|c|}\hline 0 \\ \hline\end{array}\, .$$

We will now give that part of the crystal graph containing all the paths leading to $(4,2,1)$.



Each i-string, for $i \in I$, has a structure. As one goes down an i-string, the i-nodes are added starting at the bottom left and working along to the upper right. We give a few strings of this type. Note that the second 0-string is not contained in the diagram because it does not lead to the partition $(4,2,1)$.

$$i=1: \begin{array}{|c|}\hline 0\\\hline 2\\\hline\end{array} \to \begin{array}{|c|}\hline 0\\\hline 2\\\hline 1\\\hline\end{array} \to \begin{array}{|c|c|}\hline 0&1\\\hline 2\\\cline{1-1} 1\\\cline{1-1}\end{array} .$$

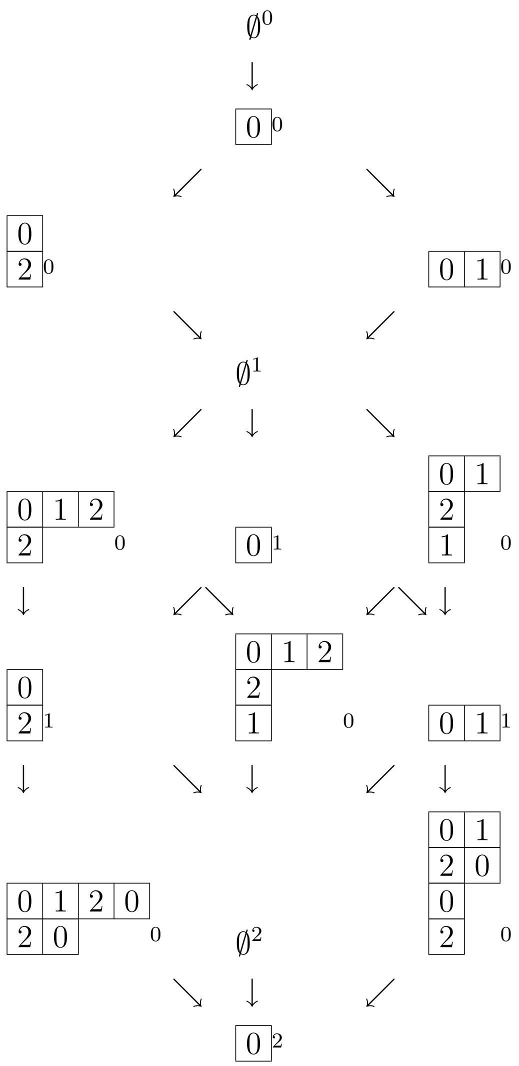
$$i=2: \begin{array}{|c|c|}\hline 0&1\\\hline\end{array} \to \begin{array}{|c|c|}\hline 0&1\\\hline 2\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 0&1&2\\\hline 2\\\cline{1-1}\end{array} .$$

$$i=0: \begin{array}{|c|c|}\hline 0&1\\\hline 2\\\cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline 0&1\\\hline 2&0\\\hline\end{array}.$$

$$i=0: \begin{array}{|c|}\hline 0\\\hline 1\\\hline 2\\\hline\end{array} \to \begin{array}{|c|}\hline 0\\\hline 1\\\hline 2\\\hline 0\\\hline\end{array}.$$

There is also a reduced version of the crystal graph in which the vertices correspond to blocks of the symmetric groups instead of to irreducible representations. The reduced crystal graph has a geometric representation in which the vertices are the integer lattice points determined by the b contents.

Example 0.2. If we redo the diagram in Example 0.1 as a part of the reduced crystal graph, letting a block with core ν and weight w be represented by ν^w, we get the following, where we have now included all blocks for $d \leq 7$, including one which was not in the previous diagram. Note the symmetry between left and right, which is just taking the dual partition and multiplying the residues by -1.



One can also see that the exponents are symmetric along the strings, increasing toward the center.

The *hub* of a block is the set of coordinates with respect to the basis for the rows of the Cartan matrix.

$$
\begin{tikzcd}
 & (1,0,0)^0 \arrow[d] & \\
 & (-1,1,1)^0 \arrow[dl] \arrow[dr] & \\
(0,2,-1)^0 \arrow[dr] & & (0,-1,2)^0 \arrow[dl] \\
 & (1,0,0)^1 \arrow[dl] \arrow[d] \arrow[dr] & \\
(2,1,-2)^0 \arrow[d] \arrow[dr] & (-1,1,1)^1 \arrow[dl] \arrow[dr] & (2,-2,1)^0 \arrow[dl] \arrow[d] \\
(0,2,-1)^1 \arrow[d] \arrow[dr] & (3,-1,-1)^0 \arrow[d] & (0,-1,2)^1 \arrow[dl] \arrow[d] \\
(-2,3,0)^0 \arrow[dr] & (1,0,0)^2 \arrow[d] & (-2,0,3)^0 \arrow[dl] \\
 & (-1,1,1)^2 &
\end{tikzcd}
$$

The blocks correspond to the weights in a highest weight space for Λ_0 for the affine Lie algebra $\widetilde{A}_{e-1}$, where e is the order of q in F. Chuang and Rouquier proved that the symmetrically-placed blocks along an i-string have equivalent derived categories [1].

References

[1] [CR] J. Chuang and R. Rouquier, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification, Annals of Mathermatics*, **167** (2008), 245-298.