TOPOLOGICAL SPACES AND RELATED CATEGORIES: A CATEGORICAL POINT OF VIEW

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Abstract. We discuss the category of topological spaces, as well as several related categories from the point of view of universal objects and universal constructions.

1. Introduction

1.1. **Aim and structure of this work.** The aim of this work is to introduce some categorical constructions in the category of topological spaces, and exhibit some connections to related categories. This work is the final project submitted in the course 88-806 in Bar-Ilan University.

More concretely, we start with investigating monomorphisms and epimorphisms, initial and terminal objects, products and coproducts, limits and colimits, fiber products and pushouts. Then we turn to investigate the connection with several other categories, in particular with **Set**, introducing right and left adjoints to the forgetful functor. We then show how topological spaces form small categories by themselves, and take as a case study the notion of compact objects and compactly generated categories to exhibit the meta-mathematical idea of generalizing properties of objects from specific categories in different mathematical contexts - to the wide and general framework of category theory. During the entire work we try to put examples that demonstrate in our opinion the points we try to show.

1.2. **Preliminary remarks.** In this work, the emphasis is on general understand of the categories under investigation. Proofs are given usually in full details, while in some repetative cases, where few technical details had already appeared (or quite similar details had) we omit them in order to put the emphasis on the new or crucial points - in our opinion.

As mentioned before, this work is the final project of the writer in the a course about categroy theory, number 88-806 in Bar-Ilan University, lectured by Prof. Malka Schaps, to whom I would like to thank for the wide vision on category theory (and hence, in a way, on the whole mathematics) provided during the lectures.

2. The Category **Top** and Related Categories

2.1. **Basics.** Let **Top** denote the category of topological spaces. Its objects are topological spaces and the morphisms are continuous maps. Clearly this is a locally small category. Recall that the topology consisting of all subsets of a given space is called the *discrete* topology on it, whereas the topology consisting only of the empty set and the full space is called the *trivial* topology. Recall also that isomorphisms in **Top** are called *homeomorphisms*. We say a map is *dominant* if its image is dense in the codomain.

Recall that topology can be essentially expressed in terms of nets. We remind the reader that a *directed set* is a poset in which every pair of elements has a common upper bound, i.e. $(D, \leq)$ is a directed set if for every $\alpha, \beta \in D$ there exists some $\gamma \in D$ for which $\alpha, \beta \leq \gamma$.

Provided a topological space X and a directed set $(D, \leq)$, a *net* is a function $\phi : D \to X$. We say the net ϕ converges to a point $x \in X$ if for every $x \in U$ open neighborhood there exists some $\alpha \in D$ such that $\phi(\beta) \in U$ for all $\alpha \leq \beta$. Indeed, a function f is continuous at a point x if and only if for every net ϕ converging to x we have that the net $f \circ \phi$ converges to $f(x)$; a point lies in the closure of a given set if and only if it is a limits of a net contained in the given set, and so on.

2.2. **Objects and morphisms.** We begin by providing a full description of monics and epics.

Proposition 2.1.
(1) *Let $f : X \to Y$ be a continuous map. Then f is monic if and only if it is injective.*
(2) *Let $f : X \to Y$ be a continuous map. Then f is epic if and only if it is surjective.*

Proof.
(1) It is clear that if f is injective then it is a monomorphism, as if $fg_1 = fg_2$ then by forgetting the topological structure we obtain that $g_1 = g_2$ pointwise.

For the other direction, assume f is a monomorphism. On the contrary, assume that there exist distinct $p, q \in X$ such that $f(p) = f(q)$. Let $\{*\}$ be a one point space and define $g_1(*) = p, g_2(*) = q$. Clearly $g_{1,2}$ are distinct morphisms, but $fg_1 = fg_2$.

(2) Assume f is surjective and $g_1, g_2 : Y \to Z$ are distinct. Then there exists $p \in Y$ such that $g_1(p) \neq g_2(p)$, and since f is onto there exists some $q \in X$ that is mapped by f to p; thus $g_1 f(q) = g_1(p) \neq g_2(p) = g_2 f(q)$ so $g_1 f, g_2 f$ are distinct morphisms.

For the other direction, assume f is an epimorphism. On the contrary, suppose that f is not surjective - so there exists $p \in Y$ such that no point in X is mapped by f to p. Let $\{*, \#\}$ be the topologically trivial two point spaces. Define $g_{1,2} : Y \to \{*, \#\}$ by: $g_1(y) = *$ for any $y \in Y$, and $g_2(z) = *$ for every $z \neq p$ and $g_2(p) = \#$. It is clear that g_1, g_2 are distinct morphisms, as the topology on $\{*, \#\}$ is trivial; but $g_1 f = g_2 f$, a contradication.

□

We present a slight modification of **Top** in which the situation is different.

Example 2.2 (Hausdorff Spaces)**.** *Let **T2** be the full subcategory of **Top** whose objects are topological spaces satisfying Hausdorff's separation axiom, namely for every two distinct points x, y there exist disjoint open sets $x \in U, y \in V$.*

We have an analoguous result to 2.1, that is (the same proof for monomorphisms holds):

Proposition 2.3. *Let $f : X \to Y$ be a morphism. Then it is dominant if and only if it is epic.*

Proof. Assume f is dominant, and let $g_{1,2} : Y \to Z$ such that $g_1 f = g_2 f$. Pick a point $y \in Y$. As f is dominant we can take a directed system Λ and a net $y_\lambda \to y$, such that $y_\lambda = f(x_\lambda)$ (convergence in λ). Then $g_1(y_\lambda) = g_1 f(x_\lambda) = g_2 f(x_\lambda) =$

$g_2(y_\lambda)$ and so $g_1(y), g_2(y)$ are limits of the same net, and since we work in the category of Hausdorff spaces they must coincide. Hence $g_1 = g_2$, so f is epic.

For the opposite direction, assume f is epic. Let $T \subset Y$ denote the closure of the image of f, and set $S = Y - T$ the complement. Define $Z = S \coprod \{*\}$ declaring that $O \subseteq Z$ is open if and only if either $O \subseteq S$ open in the topology inherited from Y or $* \in O$ and $O \cap S \cup T$ is open in Y. This is in fact the quotient topology, induced by identifying T with $*$. It is evident that Z is again Hausdorff, thus we may define morphisms $g_1 : Y \to Z$ by $g_1(T) = *, g_1(s) = s$ for every $s \in S$ (i.e. the quotient map) and $g_2 : Y \to Z$ by $g_2(Y) = *$. It is clear that $g_{1,2}$ are morphisms, and in addition $g_1 f = g_2 f$. Thus $g_1 = g_2$ so $S = \emptyset$ hence $\overline{Im\ f} = Y$ so f is dominant. □

Before passing to discuss some universal constructions, let us mention that the empty space is initial, while the one point set is terminal. (It is clear that the empty function and the constant map onto the one point set are both continuous.)

2.3. **Products and Coproducts.** We start by describing coproducts. Let $\{X_i\}_{i\in I}$ be a family of topological spaces, I an index set. The underlying set theoretic structure is nothing but the disjoint union of the spaces; this is clear, since we can apply the forgetful functor and pass to **Set**.

Regarding the topology on $\coprod_{i\in I} X_i$, we put $S \subseteq \coprod_{i\in I} X_i$ open if and only if $S \cap X_i$ is open in X_i for every index $i \in I$. This can be easily seen to form a topology on $\coprod_{i\in I} X_i$.

The set theoretic embeddings $j_i : X_i \to X_i$ are naturally defined and can be easily seen to be morphisms.

Proposition 2.4. *The topological space $\coprod_{i\in I} X_i$ constructed above is indeed the coproduct in the category **Top**.*

Proof. Let Y be a topological space with $k_i : X_i \to Y$. We define $f : \coprod_{i\in I} X_i \to S$ by $f(x) = k_i(x)$ where $i \in I$ is the unique index such that $x \in X_i$. Clearly $k_i = f j_i$ (and this is the only possible map with this property, even set theoretically).

To see that f is a continuous map, let $U \subseteq Y$ be open. Then to prove $f^{-1}(U)$ is open, it is necessary and sufficient to prove that $f^{-1}(U) \cap X_i$ is; but the latter is nothing but $k_i(U)$ (as $k_i = f j_i$), but this evidently follows from the assumption that k_i are continuous. □

We now describe products. Let $\{X_i\}_{i\in I}$ be a family of topological spaces, I an index set. The set theoretic description of $\prod_{i\in I} X_i$ is the usual cartesian product in the category **Set**: this is the set of functions $a : I \to \coprod_{i\in I} X_i$ (the coproduct in **Set** is the disjoint union) such that $a(i) \in X_i$ for every $i \in I$. The projections π_i are naturally defined: $\pi_i(a) = a(i)$.

We now turn to describe the topology on $\prod_{i\in I} X_i$. This is nothing but the Tychonoff's topology, whose non-empty open sets are unions of products of sets $(U_i)_{i\in I}$ where $U_i \subseteq X_i$ are open and for all but finitely many indices, $U_i = X_i$ is the full component.

Proposition 2.5. *The topological space $\prod_{i\in I} X_i$ constructed above is indeed the product in the category **Top**.*

Since we already dealt with some similar details while constructing the coproducts, and the construction of product spaces is basic and well known, let us provide a partial proof to this:

Proof. Modulo some technical details which we omit (since they are essentially the same as in the dual statement for coproducts), it remains to prove that Tychonoff's topology is the minimal such that π_i are continuous; since $\pi_i^{-1}(U_i)$ must be open whenever U_i are, it follows that sets which are the full components X_i in almost all indices are open (and arbitrary open subsets in the rest components) - must be open in order that the projections will be continuous, as the collection of open sets is stable under finite intersections. □

Remark 2.6. *We note that the product endows **Top** with a monoidal structure, the identity object being the one point set. However, this is not a preadditive category; no group structure can be placed for the hom-sets.*

2.4. **Limits and Colimits.** Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a directed system with index set Λ, and $f_{ij} : X_i \to X_j$ the associated compatible (in the sense that $f_{jk}f_{ij} = f_{ik}$) morphisms with respect to which we attempt to form a limit or a colimit. In general one deals with Λ a category, and morphisms $X_i \to X_j$ are associated with morphisms between the objects in the index category. (For this reason we write $i \to j$ even when working in posets where $i \leq j$). For now, we restrict ourselves to the case where all morphisms are 1:1 (constructing colimits) or onto (limits).

For the case of colimits, consider the union if all spaces, with a subset being open if and only if its intersection with every X_λ is open. Alternatively, one can take the coproduct defined in 2.3 and identify points $x_i \in X_i$ and $x_j \in X_j$ whenever there exists $i, j \to k$ such that $f_{ik}(x_i) = f_{jk}(x_j)$, resulting in a quotient space being the desired colimit.

We construct limits in the same spirit: using the construction of products from 2.3. Let $\lim X_\lambda$ be the subspace of $\prod X_\lambda$ consisting of all functions $a \in \prod X_\lambda$ satisfying $f_{ij}a(i) = a(j)$ whenever $i \to j$.

If X_λ are all from **T2** then we have the following:

Lemma 2.7. *The space* $\lim X_\lambda$ *is a closed subspace of* $\prod X_\lambda$.

Proof. We can present $\lim X_\lambda = \bigcap_{i\to j}\{a \in \prod X_\lambda : f_{ij}a(i) = a(j)\}$, an intersection of closed subsets. The latter claim can be proved as follows: assume $f_{ij}\pi_i(b) \neq \pi_j(b)$, then there exist $f_{ij}\pi_i(b) \in U$ open and disjoint from $\pi_j(b) \in V$ (hence also $f_{ij}^{-1}(U)$ is open). Taking the open subset of the product: $\pi_j^{-1}(V) \cap \pi_i^{-1}f_{ij}^{-1}(U)$. It contains b and disjoint from $\lim X_\lambda$. This finishes the proof. □

Remark 2.8. *A nice place where the limits/colimits duality appears is in Pontryagin duality: for a locally compact abelian group G, define $\hat{G} = \{\chi : G \to \mathbb{T}\}$ the group of continuous homomorphisms to the unit circle group.*

The assignment $G \mapsto \hat{G}$ defines an auto-antiequivalence of the category of locally compact abelian groups, fixing finite groups, and thus exchanging between limits of finite groups (that is, profinite groups) and colimitis of finite groups (that is, torsion groups).

We demonstrate the above remark in the following example:

Example 2.9. *Let p be a prime, and consider the groups $G_n = \mathbb{Z}/p^n\mathbb{Z}$ endowed with the discrete topology, and consider the epimorphisms $\phi_n : G_{n+1} \to G_n$ given by $\phi_n(k) = k_{(mod\ p^n)}$; they form a compatible system of morphisms, with respect to which we can take a limit, which can be described as the ring of p-adic integers, $\mathbb{Z}_p = \{\sum_{i=0}^{\infty} a_i p^i | a_i \in \{0, \ldots, p-1\}\}$. Another way to see it is by deforming p to* *an indeterminate* X*: consider* $H_n = \mathbb{Z}[X]/(X^n) \to G_n$ *by* $X \mapsto p$*. We thus recover* $\mathbb{Z}_p$ *as* $\mathbb{Z}[[X]]/(X-p)$*.*

For the dual morphisms $\psi_n : G_n \to G_{n+1}$ *let* $\psi_n(k) = kp$*, resulting in a colimit isomorphic to the group of all complex roots of the unity ehich are of the form* $e^{\frac{2\pi i}{p^n}}$ *for some* n*.*

2.5. **Fiber products and Pushouts.** We start by describing fiber products of spaces.

Proposition 2.10. *Let* $f_i : Y_i \to Z$ *for* $i = 1, 2$*. Then* $X = \{(a_1, a_2)|f_1(a_1) = f_2(a_2)\} \subseteq Y_1 \times Y_2$ *is the fiber product and it is a closed subspace of the product* $Y_1 \times Y_2$*.*

Proof. The fact that $X \subseteq Y_1 \times Y_2$ is closed is very similar to the proof of 2.7. It is also clear that X is the set theoretic fiber product.

The maps $\pi_i : X \to Y_i$ are the natural projections from the product restricted to X_i (for $i = 1, 2$), thus continuous. Let $g_i : W \to Y_i$ be another space with morphisms to Y_1, Y_2, such that $f_1 g_1 = f_2 g_2$. Define a map $\phi : W \to Y_1 \times Y_2$ by $\phi(w) = (g_1(w), g_2(w))$. Its image falls inside X. As this is already the fiber product in **Set**, it follows that this map is unique; what remains to prove is that ϕ is continuous.

Let $U_1 \times U_2 \subseteq Y_1 \times Y_2$ with $U_{1,2}$ open in the corresponding spaces. It suffices to prove that for such sets $\phi^{-1}(U_1 \times U_2) = \{w \in W|g_1(w) \in U_1, g_2(w) \in U_2\}$ are open; but they can be presented as $g_1^{-1}(U_1) \cap g_2^{-1}(U_2)$ which are clearly open, as $g_{1,2}$ are continuous. □

Remark 2.11. *Being a closed subspace of the product, many properties of* $Y_{1,2}$ *are inherited to the fiber product: compactness, local compactness, several separation axioms (such as* $T1, T2$*; but not* $T4$*).*

Let us now describe pushouts in terms of quotient spaces. Suppose we are given $f_{1,2} : X \to Y_{1,2}$, respectively. We define $Z = Y_1 \coprod Y_2/\mathcal{R}$ where $\mathcal{R}$ is the equivalence relation defined by $y_1 \approx y_2$ if and only if there exists some $x \in X$ such that $f_1(x) = y_1, f_2(x) = y_2$. Let $\pi : Y_1 \coprod Y_2 \to Z$ be the natural projection.

We now endow the pushout Z with a topology. This is nothing but the quotient topology: a subset $U \subseteq Z$ is open if and only if $\pi^{-1}(U) \subseteq Y_1 \coprod Y_2$ is.

2.6. **Related Categories and Fibers over Set.** In this subsection we discuss several categories related to **Top**.

We start with the forgetful functor $\mathcal{F}$: **Top** $\to$ **Set** that assigns to a topological space its underlying set of points, and continuous functions are mapped to themselves, 'forgetting' they were continuous.

A natural functor $\mathcal{D}$ from **Set** to **Top** is given by endowing a set with the discrete topology, declaring all subsets are open. It is well defined since every function between discrete spaces is continuous. We claim that there is an intimate connection between $\mathcal{F}$ and $\mathcal{D}$:

Claim 2.12. *The functors* $\mathcal{F}$ *admits a left adjoint which is* $\mathcal{D}$*:*

$$Hom_{\textbf{\textit{Top}}}(\mathcal{D}(X), S) \cong Hom_{\textbf{\textit{Set}}}(X, \mathcal{F}(S))$$

naturally in both X, S*.*

The map between these two sets is clear: it maps a continuous fuction to itself; as every function from a discrete space to another space is obviously continuous the result follows. In fact, it is the case that

$$\mathcal{F}\mathcal{D} \cong 1_{\mathbf{Set}}$$

embedding **Set** fully faithfully into **Top**.

We can think of the discrete topology over a set X as a 'point' in the fiber of the forgetful functor $\mathcal{F} : \mathbf{Top} \to \mathbf{Set}$ above the set X. Actually the set of topologies over X (to which we refer as the 'fiber' over X) forms a lattice with respect to inclusion, the maximal element being the discrete topology mentioned above, and the minimal being the trivial topology. Let $\mathcal{T} : \mathbf{Set} \to \mathbf{Top}$ be the functor assigning to a set the trivial topology over it. As every function (from an arbitrary space) with codomain a trivial topological space is continuous, we have the following:

$$\mathrm{Hom}_{\mathbf{Top}}(S, \mathcal{T}(X)) \cong \mathrm{Hom}_{\mathbf{Set}}(\mathcal{F}(S), X)$$

proving $\mathcal{T}$ is the right adjoint of $\mathcal{F}$.

We turn to discuss some additional categories related to **Top**. Namely, let X be a topological space and denote by X_{cl} the category whose objects are the open subsets of X with morphisms the inclusions. We note that this category is small. We mention that the category of functors from X_{cl} to the category of commutative rings (or abelian groups, or modules) is precisely the category of presheaves on X (with coefficients in one of the mentioned categories).

In this context, there is a good example of how properties of objects appearing 'in nature' - that is, in concrete and well studied categories - generalize to the wide context of category theory.

Definition 2.13. *We say an object $c \in \mathcal{C}$ is compact if the hom-functor preserves filtered colimits:*

$$\mathrm{Hom}_{\mathcal{C}}(c, \mathrm{colim}_{\Lambda} X_{\lambda}) \cong \mathrm{colim}_{\Lambda} \mathrm{Hom}_{\mathcal{C}}(c, X_{\lambda})$$

the colimit taken over some filtered category Λ, which we refer to, at least in the following discussion as a poset.

We now deal with the question: which objects are compact in X_{cl}?

Proposition 2.14. *A subset U of X is a compact object in X_{cl} if and only if it is a compact subset.*

Proof. Let U be compact, and $\{X_{\lambda}\}_{\lambda \in \Lambda}$ a collection of open sets, ordered by inclusion with respect to a poset Λ. In this case, the colimit is nothing but the union $\bigcup_{\lambda \in \Lambda} X_{\lambda}$.

Pick a morphism $U \to \bigcup_{\lambda \in \Lambda} X_{\lambda}$ namely U is a subset of the union. As U is compact, there is some $\mu \in \Lambda$ such that the morphism we picked factors through $U \to X_{\mu}$; conversely, it is clear that every morphism $U \to X_{\mu}$ gives rise to a umique morphism $U \to \bigcup_{\lambda \in \Lambda} X_{\lambda}$.

For the converse, assume U is a compact object. Let $\{X_{\lambda}\}_{\lambda \in \Lambda}$ be an open covering of U (we can easily define an appropriate Λ in this case, simply by assigning to every open subset in the covering an index, and declaring $\lambda \le \mu$ whenever $X_{\lambda} \subseteq X_{\mu}$). Then there exists some morphism $U \to \bigcup_{\lambda \in \Lambda} X_{\lambda}$ and thus a morphism $U \to X_{\mu}$ for some $\mu \in \Lambda$ (as the colimit of empty sets is again empty). □

We mention that X_{cl} is a category in which every morphism is both monic and epic. This is because for any two objects either there exists a unique morphism between them (the natural inclusion) or there does not exist any morphism.

We call a category in which every object is a colimit of compact objects a *compactly generated* category. It is clear that a σ-compact space X gives rise to X_{cl} a σ-compactly generated category; namely, every object is a colimit over a countable index set of compact objects.

On the other hand, recall that there is a countable version of compactness: a space is Lindelof if for every covering by open sets there exists some countable subcovering. It is clear that if X_{cl} is a σ-compactly generated category then X is Lindelof (by intersecting with every compact subset from the countable covering by compact subsets). We introduce the following example, showing that the converse does not necessarily hold:

Example 2.15. *Let $X = \mathbb{R}$ with the co-$\aleph_0$ topology: the non-empty sets are precisely those with complement (at most) countable. We note that this space is Lindelof (pick one open subset in the covering, and other countably many open sets containing the complement. This is indeed possible as the complement is countable).*

However, X_{cl} is not σ-compactly generated. If it would be, we could have present X as a union of countably many compact subsets, so at least one of them would be uncountable. But every uncountable subset of X is homeomorphic to X itself, and clearly non-compact.

Let us (briefly) go over some universal construction in the context of X_{cl}. As we mentioned, the colimit can be though of as a union, as well as the pushouts and the coproduct, while the limit is just the intersection; the fiber products and products are again formed by intersection.

Before ending this summary, we briefly mention that there are some natural continuations of the categorical research of the category of topological spaces. We refer to [1] for information about projective and injective objects in various categories of topological spaces.

For discussion of algebaic topology, in particular homotopy and homology theory, there are many references. A relatively elementary one is [4].

References

[1] T. G. Raghavan, I. L. Reilly, "On projective and injective objects in some topological categories". Lecture Notes in Mathematics Volume 719, 1979, pp 308-315

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[4] Allen Hatcher, "Algebraic Topology", 2001. Free online (http://www.math.cornell.edu/ hatcher/AT/AT.pdf)