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A Separation Principle for the Control of a Class of Nonlinear Systems

A. N. Atassi and H. K. Khalil

***Abstract*—In this note, we extend the separation results of a previous work to a case where a globally bounded state feedback controller renders a certain compact set positively invariant and asymptotically attractive. The extension covers a wide range of control tasks that arise in adaptive control, servomechanisms, and practical stabilization. It is shown that by implementing the control law using a high-gain observer, we can recover the performance of the state feedback controller.**

***Index Terms*—Nonlinear control, observers, output feedback control, separation principle, singular perturbation.**

I. Introduction

In [3], we present separation results for the stabilization of a class of nonlinear systems having a chain or more of integrators in their structure. Therein, we consider feedback controllers that make the origin of the closed-loop system an asymptotically stable equilibrium point. In this note, we are interested in feedback controllers that achieve boundedness of trajectories but not necessarily with convergence to an equilibrium point. Such situation arises in adaptive control [1], [8], where only the tracking error or both the tracking error and the parameter error converge to zero. Another example is the convergence to a zero-error manifold as in the servomechanism problem discussed in [7], [10], and [6]. Additional examples can be found in stabilization problems in the presence of disturbances as in [11] and [5], where only finite-time convergence to a set can be achieved. In most of these cases, it can be shown that the trajectories approach an attractive, positively invariant, compact set.

We consider a class of nonlinear systems similar to the one considered in [3] and characterize the performance of the state feedback controller as rendering a certain compact set positively invariant and asymptotically attractive. As in [3], we require the control law to be globally bounded and implement it using a high-gain observer. We show that, for sufficiently high observer gain, the output feedback controller recovers the performance of the state feedback controller. In particular, it renders a compact set of interest positively invariant and asymptotically attractive. Moreover, any compact subset of the region of attraction under state feedback can be included in the region of attraction under output feedback. Finally, trajectories under output feedback converge to those under state feedback as the observer gain approaches infinity.

II. Problem Formulation

We consider a nonlinear system represented by

$$\dot{x} = Ax + B\phi(x, z, d(t), u) \tag{1}$$

$$\dot{z} = \psi(x, z, d(t), u) \tag{2}$$

$$y = Cx \tag{3}$$

$$\zeta = q\Big(x, z, d(t)\Big) \tag{4}$$

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where

$u \in R^m$	control input;
$\zeta \in R^s$ and $y \in R^p$	measured outputs;
$x \in R^r$ and $z \in R^\ell$	state vector;
$d(t) \in R^d$	vector of signals that belongs to $\mathcal{M}_\mathcal{D}$, the set of all piecewise continuous functions from R to $\mathcal{D}$, where $\mathcal{D}$ is a compact subset of R^d.

The matrices A, B, and C represent p chains of integrators as in [3, Sec. II]. The state feedback control is assumed to be in the form

$$\dot{\vartheta} = \Gamma\Big(\vartheta, x, \zeta, d(t)\Big) \tag{5}$$

$$u = \gamma\Big(\vartheta, x, \zeta, d(t)\Big) \tag{6}$$

Assumption 1:

1) Γ and γ are locally Lipschitz functions in ϑ, x, and ζ, uniformly in d, over the domain of interest;
2) Γ and γ are globally bounded functions of x;
3) the closed-loop system is uniformly globally asymptotically stable with respect to the compact positively invariant set $\mathcal{A}$.[1]

Assumption 2: The functions q, ϕ and ψ are locally Lipschitz in x, z, and u uniformly in d over the domain of interest. Moreover, $\phi(x, z, d, \gamma(\vartheta, x, \zeta, d))$ is zero in $\mathcal{A}$ uniformly in d.

To implement the control (5) and (6), we use the state estimate $\hat{x}$ generated by the high-gain observer

$$\dot{\hat{x}} = A\hat{x} + B\phi_0(\hat{x}, \zeta, d(t), u) + H(y - C\hat{x}) \tag{7}$$

where the observer gain H is chosen as

$$H = \text{block diag}[H_1, \ldots, H_p] \quad H_i = \begin{bmatrix} \frac{\alpha_1^i}{\epsilon} \\ \frac{\alpha_2^i}{\epsilon^2} \\ \vdots \\ \frac{\alpha_{r_i-1}^i}{\epsilon^{r_i-1}} \\ \frac{\alpha_{r_i}^i}{\epsilon^{r_i}} \end{bmatrix}_{r_i \times 1} \tag{8}$$

ϵ is a positive constant to be specified, and the positive constants α_j^i are chosen such that the roots of

$$s^{r_i} + \alpha_1^i s^{r_i-1} + \cdots + \alpha_{r_i-1}^i s^1 + \alpha_{r_i}^i = 0$$

are in the open left-half plane, for all $i = 1, \ldots, p$. The function $\phi_0(x, \zeta, d(t), u)$ is a nominal model of $\phi(x, z, d(t), u)$ which is required to satisfy the following assumption.

Assumption 3: ϕ_0 is a locally Lipschitz function in x, ζ, and u, uniformly in d, over the domain of interest. Furthermore, it is globally bounded in x and zero in $\mathcal{A}$, uniformly in d.

Remark 1: The functions γ, Γ, and ϕ_0 are allowed to depend on d since some components of d may comprise reference signals that are available on line. They cannot, of course, depend on unknown disturbance signals. In the special case where the function ϕ is known and depends only on (x, ζ, u) and the known components of d, we can take $\phi_0 = \phi$. Taking $\phi_0 = 0$ yields a linear high-gain observer.

III. Performance Recovery

The objective of this section is to show that the output feedback controller recovers the performance of the state feedback controller for sufficiently small ϵ. The performance recovery manifests itself in three points. First, the compact set $\mathcal{A} \times \{x - \hat{x} = 0\}$ is a positively invariant set of the closed-loop system under output feedback and the closed-loop system is asymptotically stable with respect to $\mathcal{A} \times \{x - \hat{x} = 0\}$. Second, the output feedback controller achieves semiglobal stabilization; that is, for the initial states $(x_0, z_0, \vartheta_0) \in \mathcal{S}$, and $\hat{x}_0 \in \mathcal{Q}$, where $\mathcal{S}$ is any compact set containing $\mathcal{A}$ and $\mathcal{Q}$ is any compact subset of R^r, the set $\mathcal{S} \times \mathcal{Q}$ is included in the region of attraction under output feedback control. Third, the trajectory of (x, z, ϑ) under output feedback approaches the trajectory under state feedback as $\epsilon \to 0$.

For the purpose of analysis, we replace the observer dynamics by the equivalent dynamics of the scaled estimation error $D(\epsilon)\eta = x - \hat{x}$, where

$$D(\epsilon) = \text{block diag}[D_1, \ldots, D_p]$$
$$D_i = \text{diag}[\epsilon^{r_i-1}, \ldots, 1]_{r_i \times r_i}.$$

The closed-loop system can be represented by

$$\dot{x} = Ax + B\phi\Big(x, z, d(t), \gamma(\cdot)\Big) \tag{9}$$

$$\dot{z} = \psi\bigg(x, z, d(t), \gamma\Big(\vartheta, x - D(\epsilon)\eta, \zeta, d(t)\Big)\bigg) \tag{10}$$

$$\dot{\vartheta} = \Gamma\Big(\vartheta, x - D(\epsilon)\eta, \zeta, d(t)\Big) \tag{11}$$

$$\epsilon\dot{\eta} = A_0\eta + \epsilon Bg\Big(x, z, \vartheta, D(\epsilon)\eta, d(t)\Big) \tag{12}$$

where $g(\cdot) = \phi(x, z, d(t), \gamma(\cdot)) - \phi_0(\hat{x}, \zeta, d(t), \gamma(\cdot))$ and A_0 is a constant Hurwitz matrix. For simplicity, we write the system (9)–(11) as

$$\dot{\chi} = f_r(\chi, d(t), D(\epsilon)\eta) \tag{13}$$

where $\chi = [x^T, z^T, \vartheta^T]^T$ and $\chi(0) = [x_0^T, z_0^T, \vartheta_0^T]^T$. Then, the system under state feedback is given by

$$\dot{\chi} = f_r(\chi, d(t), 0). \tag{14}$$

The results of the analysis are given in the forthcoming theorems whose proofs can be found in [2]. The proofs are omitted for lack of space. They are very similar to the corresponding ones of [3], except that all bounds are calculated for $\mathbf{d} \in \mathcal{D}$. Most proofs follow a Lyapunov argument where the Lyapunov function is supplied by results from [9].

Theorem 1: Let Assumptions 1–3 hold. Then, there exists $\epsilon_1^\star > 0$ such that for every $0 < \epsilon \le \epsilon_1^\star$, the trajectories $(\chi(t, \epsilon), \eta(t, \epsilon))$ of the system (9)–(12) starting in $\mathcal{S} \times \mathcal{Q}$ are bounded for all $t \ge 0$ and all $d \in \mathcal{M}_\mathcal{D}$, and come arbitrarily close to $\mathcal{A} \times \{\eta = 0\}$ as time progresses.

Theorem 2: Under the conditions of Theorem 1, given any $\xi > 0$, there exist $\epsilon_2^\star = \epsilon_2^\star(\xi) > 0$ and $T_1 = T_1(\xi)$ such that, for every $0 < \epsilon \le \epsilon_2^\star$, we have

$$|\chi(t, \epsilon)|_\mathcal{A} + \|\eta(t, \epsilon)\| \le \xi \qquad \forall\, t \ge T_1 \qquad \forall\, d \in \mathcal{M}_\mathcal{D}. \tag{15}$$

Let $\chi_r(t)$ be the solution of (14) starting from $\chi(0)$. The following theorem shows that $\chi(t, \epsilon)$ converges to $\chi_r(t)$ as $\epsilon \to 0$, uniformly in t, for all $t \ge 0$.

Theorem 3: Under the conditions of Theorem 1, given any $\xi > 0$, there exists $\epsilon_3^\star > 0$ such that, for every $0 < \epsilon \le \epsilon_3^\star$ we have

$$\|\chi(t, \epsilon) - \chi_r(t)\| \le \xi \qquad \forall\, t \ge 0 \qquad \forall\, d \in \mathcal{M}_\mathcal{D}. \tag{16}$$

Next, we deal with local uniform asymptotic stability with respect to a compact, positively invariant set. We assume that the trajectory belongs to some ball around $\mathcal{A}$. First, we deal with the case where $\phi_0 = \phi$ and the system (14) is uniformly asymptotically stable with respect to $\mathcal{A}$.

Theorem 4: Let Assumptions 1–3 hold and assume that $\phi_0 = \phi$. Then, there exists $\epsilon_4^\star > 0$ such that, for every $0 < \epsilon \le \epsilon_4^\star$, the system (9)–(12) is uniformly asymptotically stable with respect to the compact positively invariant set $\mathcal{A} \times \{\eta = 0\}$.

Second, we deal with the case where the system (14) is uniformly exponentially stable, whether or not we know ϕ.

Theorem 5: Let Assumptions 1–3 hold and assume that the closed-loop system (14) is uniformly exponentially stable with respect to the set $\mathcal{A}$. Then, there exists $\epsilon_5^\star > 0$ such that, for every $0 < \epsilon \le \epsilon_5^\star$,

[1]We define uniform asymptotic stability with respect to a set in the spirit of [4, Def. 4.1, 4.12] and [13, Sec. 1.10, Def. 1]. The definition of a Lyapunov function with respect to a compact, positively invariant set is given in [9].

the system (9)–(12) is uniformly exponentially stable with respect to the set $\mathcal{A} \times \{\eta = 0\}$.

The Lyapunov function needed for this proof comes from an extension of the proof of [12, Th. 19.1, 19.2] to the case of exponential stability with respect to sets.

Third, we deal with the case where the system (14) is uniformly asymptotically stable with respect to $\mathcal{A}$, but $\phi_0 \neq \phi$. The following condition is imposed on the modeling error.

Assumption 4: There exist a C^1 function $V_3(t, \chi)$ defined on $[0, \infty) \times U$, where $U = \{\chi : |\chi|_{\mathcal{A}} \leq r_3, r_3 > 0\}$ is a neighborhood of $\mathcal{A}$ in Ω, and three functions ψ_1, ψ_2, and ψ_3, defined and continuous on U, which are positive–definite with respect to $\mathcal{A}$ (i.e., positive everywhere and zero only in $\mathcal{A}$) such that, for all $t \geq 0$, we have (17)–(20), shown at the bottom of the page for all $\chi \in U$ and all $\mathbf{d} \in \mathcal{D}$, for some positive constants $c_0 \geq 0, c_1 > 0$ and $a, b < 1$, such that $a + b = 1$.

Remark 2: Assumption 4 is similar to [3, Ass. 4] in the sense that it relates the modeling error magnitude and the rate of convergence of trajectories near the attractor (which is a set in the case at hand).

The recovery of asymptotic stability can now be stated as follows.

Theorem 6: Let Assumptions 1–4 hold. Then, there exists $\epsilon_6^\star > 0$ such that, for all $0 < \epsilon \leq \epsilon_6^\star$, the system (9)–(12) is uniformly asymptotically stable with respect to the compact positively invariant set $\mathcal{A} \times \{\eta = 0\}$.

In many cases, asymptotic stability with respect to a set, achieved under state feedback control, is not global and the region of attraction is a finite subset of the state space. This case is treated in [2] where it is shown that we can recover the same measures of performance as in the foregoing theorems. However, as in [3], we need a converse Lyapunov theorem that yields a Lyapunov function which goes to infinity at the boundary of an estimate of the region of attraction. This is done by extending the converse Lyapunov results of [9] to an estimate of the region of attraction. In the process of that extension $d(t)$ is further restricted, as in the following assumption which replaces Assumption 1.

Assumption 5: $d(t)$ belongs to the set of all continuously differentiable functions from R to $\mathcal{D}$ where the derivative $d'(t)$ of $d(t)$ belongs to a compact set. Items 1) and 2) of Assumption 1 hold. The closed-loop system under the state feedback control is uniformly asymptotically stable.

Theorem 7: Let Assumptions 2–5 hold. Let $\mathcal{R}$ be an open, connected subset of the region of attraction and $\mathcal{S}$ be any compact subset of $\mathcal{R}$ that contains $\mathcal{A}$. Then, the conclusions of Theorems 1, 2, 3, 4, 5, and 6 hold.

IV. Example

In order to illustrate the separation theory developed in the previous section, we apply it to the regulation problem of [6]. Several other examples can be found in [2]. The analysis of [6] applies to the closed-loop system under output feedback, including the high-gain observer. Because of the separation theory, we show that it is sufficient to consider the closed-loop system under state feedback. This simplifies the analysis because it eliminates the observer dynamics and the need to worry about the singularly perturbed nature of closed-loop system under output feedback. This part of the analysis has been already taken care of in the separation theory. Moreover, Theorem 3 shows that the trajectories under output feedback approach the trajectories under state feedback, a new result that is not shown in [6]. In presenting this example, we use the notation of [6] and design the state feedback controller to correspond to the output feedback controller of [6] when the observer is eliminated.

Consider the system

$$\begin{aligned} \dot{z} &= Z(\mu) z + p_0(x_1, \omega, \mu) \\ \dot{x} &= Fx + Gu + P(z, x, \omega, \mu) \\ e &= Hx \end{aligned} \tag{21}$$

where (F, G, H) represents a chain of n integrators and

$$P(z, x, \omega, \mu) = \begin{pmatrix} p_1(z, x_1, \omega, \mu) \\ p_2(z, x_1, x_2, \omega, \mu) \\ \cdots \\ p_{r-1}(z, x_1, x_2, \ldots, x_{r-1}, \omega, \mu) \\ p_r(z, x_1, x_2, \ldots, x_r, \omega, \mu) \end{pmatrix}$$

with state $x \in R^n$, control input $u \in R^m$, and regulated output $e \in R^m$. The system (21) is subject to an exogenous input $\omega \in R^d$ and $\mu \in \mathcal{P} \subset R^p$ is a vector of unknown parameters. Furthermore, $\mathcal{P}$ is a compact set, $p_0(\cdot)$ and $P(\cdot)$ are C^k functions of their arguments (for some large k), and $p_0(0, 0, \mu) = 0$, $P(0, 0, 0, \mu) = 0$. Without loss of generality we assume $0 \in \operatorname{int}(\mathcal{P})$. The exosystem $\dot{\omega} = S\omega$ is neutrally stable (the matrix S has distinct eigenvalues on the imaginary axis).

Assumption A: The eigenvalues of $Z(\mu)$ have negative real part, for all $\mu \in \mathcal{P}$. Moreover, the equation

$$\frac{\partial \zeta(\omega, \mu)}{\partial \omega} S\omega = Z(\mu)\zeta(\omega, \mu) + p_0(0, \omega, \mu) \tag{22}$$

has a solution $\zeta(\omega, \mu)$ defined for all ω, μ.

Given Assumption A and the structure of F, G, H and $P(\cdot)$, a routine calculation shows that the system (23), shown at the bottom of the page, has a unique and globally defined solution $\pi^a(\omega, \mu),\ c^a(\omega, \mu)$

$$\begin{aligned} \psi_1(\chi) \leq & V_3(t, \chi) \leq \psi_2(\chi) && (17) \\ \frac{\partial V_3}{\partial t} + \frac{\partial V_3}{\partial \chi} f_r(\chi, \mathbf{d}, 0) \leq & -\psi_3(\chi) && (18) \\ \|\phi(x, z, \mathbf{d}, \gamma(\vartheta, x, \zeta, \mathbf{d})) - \phi_0(x, \zeta, \mathbf{d}, \gamma(\vartheta, x, \zeta, \mathbf{d}))\| & \\ \leq & c_0 \psi_3^a(\chi) && (19) \\ \left\| \frac{\partial V_3}{\partial \chi}(t, \chi) \right\| \leq & c_1 \psi_3^b(\chi) && (20) \end{aligned}$$

$$\begin{aligned} \frac{\partial (\zeta, \pi^a)(\omega, \mu)}{\partial \omega} S\omega &= \begin{pmatrix} Z(\mu)\zeta(\omega, \mu) + p_0(H\pi^a(\omega, \mu), \omega, \mu) \\ F\pi^a(\omega, \mu) + Gc^a(\omega, \mu) + P(\zeta(\omega, \mu), \pi^a(\omega, \mu), \omega, \mu) \end{pmatrix} \\ 0 &= H\pi^a(\omega, \mu) \end{aligned} \tag{23}$$

such that $\pi^a(0,\mu)=0$ and $c^a(0,\mu)=0$ for all μ. Hereafter, it is assumed that the function $c^a(\omega,\mu)$ satisfies the following assumption.

Assumption B: For some set of real numbers $a_0, a_1, \ldots, a_{q-1}$, the identity

$$\begin{aligned} L_s^q c^a(\omega,\mu) =& a_0 c^a(\omega,\mu) + a_1 L_s c^a(\omega,\mu) \\ &+ \cdots + a_{q-1} L_s^{q-1} c^a(\omega,\mu) \end{aligned} \tag{24}$$

holds for all ω, μ, where $L_s = (\partial/\partial\omega) S\omega$. Moreover, the polynomial equation

$$s^q - a_{q-1}s^{q-1} - \cdots - a_1 s - a_0 = 0$$

has distinct roots on the imaginary axis.

Simple routine calculations show that, under Assumption B, there exist a $q \times q$ matrix Φ, a $1 \times q$ row vector Γ, and a globally defined mapping $\tau^a(\omega,\mu)$ such that

$$\begin{aligned} \frac{\partial \tau^a(\omega,\mu)}{\partial\omega} =& \Phi\tau^a(\omega,\mu) \\ c^a(\omega,\mu) =& \Gamma\tau^a(\omega,\mu). \end{aligned} \tag{25}$$

In fact, this happens for

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{q-1} \end{pmatrix}$$

$$\tau^a(\omega,\mu) = \begin{pmatrix} c^a(\omega,\mu) \\ L_s c^a(\omega,\mu) \\ \cdots \\ L_s^{q-2} c^a(\omega,\mu) \\ L_s^{q-1} c^a(\omega,\mu) \end{pmatrix}$$

$$\Gamma = (1 \quad 0 \quad 0 \quad \cdots \quad 0).$$

Hereafter, we propose a feedback law for which we prove the existence of an attractive zero-error invariant manifold. Furthermore, this manifold can be made semiglobally attractive. Set $\tilde{z} = z - \zeta(\omega,\mu)$, $\tilde{x} = x - \pi^a(\omega,\mu)$, and

$$\begin{aligned} \eta = \begin{pmatrix} e \\ e^{(1)} \\ \vdots \\ e^{(r-1)} \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_r \end{pmatrix} \\ = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 + \tilde{p}_1(\tilde{z}, \tilde{x}_1, \omega, \mu) \\ \vdots \\ \tilde{x}_r + \tilde{p}_{r-1}(\tilde{z}, \tilde{x}_1, \ldots, \tilde{x}_{r-1}, \omega, \mu) \end{pmatrix}. \end{aligned} \tag{26}$$

Consider now a feedback law of the form

$$\begin{aligned} \dot{\xi}_1 =& \Phi\xi_1 + Ne \\ u =& M\eta + T\xi_1. \end{aligned} \tag{27}$$

Similar to the proof of [6, Prop. 1], it is straightforward to prove the following property.

Proposition 1: Suppose Assumptions A and B hold and (27) stabilizes the linear approximation of (21) at the equilibrium point $(\xi_1, z, x) = (0,0,0)$, $(\omega,\mu) = (0,0)$. Then there exists a $q \times q$ matrix $\prod$ satisfying

$$\Phi \prod = \prod \Phi \quad T\prod = \Gamma \tag{28}$$

where Φ and Γ are defined as in (25). Consequently, the closed-loop system

$$\begin{aligned} \dot{\xi}_1 =& \Phi\xi_1 + NHx \\ \dot{z} =& Z(\mu)z + p_0(x_1,\omega,\mu) \\ \dot{x} =& Fx + G(M\eta + T\xi_1) + P(z,x,\omega,\mu) \\ \dot{\omega} =& S\omega \end{aligned} \tag{29}$$

has a globally defined center manifold

$$\begin{aligned} \mathcal{M}_c = \{(\xi_1, z, x, \omega) : \xi_1 = \prod \tau^a(\omega,\mu), \\ z = \zeta(\omega,\mu), x = \pi^a(\omega,\mu)\} \end{aligned} \tag{30}$$

at $(\xi_1, z, x, \omega) = (0,0,0,0)$.

Now, let us design the state feedback controller that makes $\mathcal{M}_c$ semiglobally attractive. The issue here is to choose N, T and M such that this goal is achieved. Let $\tilde{\xi}_1 = \xi_1 - \prod\tau^a(\omega,\mu)$. Then, in the coordinates $(\tilde{z}, \tilde{x}, \tilde{\xi}_1)$, the closed-loop system becomes

$$\begin{aligned} \dot{\tilde{\xi}}_1 =& \Phi\tilde{\xi}_1 + NH\tilde{x} \\ \dot{\tilde{z}} =& Z(\mu)\tilde{z} + \tilde{p}_0(H\tilde{x}, \exp(St)\omega^0, \mu) \\ \dot{\tilde{x}} =& F\tilde{x} + G(M\eta + T\tilde{\xi}_1) + \tilde{P}(z, x, \exp(St)\omega^0, \mu) \end{aligned} \tag{31}$$

where ω^0 represents the value at time $t=0$ of the state of the exosystem. System (31) is an uncertain system because the actual values of μ and ω^0 are unknown. We assume that the initial value ω^0 belongs to an *a priori* known compact set $\mathcal{W} \in R^d$. The invariant manifold reduces to the origin $(\tilde{\xi}_1, \tilde{z}, \tilde{x}) = (0,0,0)$ where the regulation error $e = \tilde{x}_1$ is zero. Thus, output regulation is achieved if the origin is attractive.

In order to be able to use the separation results of the previous section, Assumption C of [6] is modified as follows.

Assumption C: There exists a positive–definite smooth function $V(\tilde{z})$ satisfying

$$\alpha_1\|\tilde{z}\|^2 \le V(\tilde{z}) \le \alpha_2\|\tilde{z}\|^2 \tag{32}$$

$$\frac{\partial V}{\partial \tilde{z}}(Z(\mu)\tilde{z} + \tilde{p}_0(H\tilde{x}, \exp(St)\omega^0, \mu)) \le -\alpha_3\|\tilde{z}\|^2 + c|H\tilde{x}|^2 \tag{33}$$

for all $\tilde{z}, \tilde{x}, t$ and all $(\omega^0, \mu) \in \mathcal{W} \times \mathcal{P}$, where $\alpha_i > 0$ and $c \ge 0$.

For N choose any matrix such that the pair (Φ, N) is controllable. Then, given any compact set $\mathcal{S}$ of initial conditions $(\tilde{\xi}_1(0), \tilde{z}(0), \tilde{x}(0)) \in R^q \times R^{n-r} \times R^r$, find (via backstepping methods and high-gain feedback, for example) a pair of matrices M and T such that the origin is locally exponentially stable with a basin of attraction that includes the set $\mathcal{S}$.

In order to apply our separation results, we consider the system

$$\begin{aligned} \dot{z} =& Z(\mu)z + p_0(x_1, \omega, \mu) \\ \dot{\eta} =& A\eta + G\Big(u + \tilde{p}_r(z, \eta, \omega, \mu)\Big) \\ \dot{\omega} =& S\omega \\ e =& H\eta. \end{aligned}$$

This system fits the model (1)–(4) with μ being the vector of bounded disturbances (constant in this case).

We consider the state feedback controller

$$\begin{aligned} \dot{\xi}_1 =& \Phi\xi_1 + Ne \\ u =& M\eta + T\xi_1. \end{aligned}$$

This controller achieves semiglobal tracking uniformly in ω and μ. Global boundedness of the control law with respect to η is achieved by saturation outside a region of interest.

We showed, by construction, that the closed-loop system under state feedback is exponentially stable with respect to the compact positively invariant zero-error manifold $\mathcal{M}_c$ with $\mathcal{S}$ being an estimate of the region of attraction.

To implement the controller, we use a linear high-gain observer. Boundedness, ultimate boundedness, and convergence of trajectories under the output feedback controller (starting in $\mathcal{S} \times \mathcal{Q}$, where $\mathcal{Q}$ is a compact subset of R^n) are guaranteed by Theorem 7. Moreover, Theorem 7 guarantees exponential stability with respect to the compact positively invariant set $\mathcal{M}_c \times \{\eta - \xi_0 = 0\}$, where ξ_0 is the estimate of η.

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Observers for Discrete-Time Systems with Multiple Delays

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***Abstract*—In this note, a useful and systematic approach to design observers for discrete-time systems with multiple delays, under general conditions, is presented. The main feature of the proposed technique is that necessary and sufficient conditions for asymptotic stability are derived while the observer's order is independent from the number of delays and is equal to the state dimension. To illustrate efficiency of the proposed technique, two numerical examples are provided.**

***Index Terms*—Asymptotic stability, discrete-time systems, Kalman-type observer, multiple delays.**

I. INTRODUCTION

If the state estimation of linear systems with extensions to the descriptor case in presence or not of unknown inputs is well understood by now, designing an observer for time-delay systems remains, however, a challenge as can be shown through the works developed in this field [1]–[11]. On the other hand, we notice that most of observers design methods were developed for continuous time-delay systems, we refer the reader to [1], [4], [6], [8], [11], and the references therein, with very few extensions to discrete-time models particularly with multiple delays.

Trinh *et al.* [9] have proposed a memoryless reduced-order state observer which is an extension of the recent works by Leyva–Ramos *et al.* [4] and themselves [8]. The main result performed there is to reduce the observer's order to only the number of unstable and/or poorly damped eigenvalues of the system. In the presence of unknown inputs, which act on the outputs, necessary and sufficient conditions for the existence of an observer were established in [7]. However, the observer's order is proportional to the number of delays that leads to high computational requirements and technical difficulties when the time-delay increases and when the observer implementation is considered.

Motivated by a recent result [2] where only sufficient conditions for asymptotic stability are derived, we propose here a simple and systematic approach to design observers for discrete-time systems with multiple delays. Thanks to the Lyapunov approach and a useful design of some arbitrary matrices and parameters, necessary and sufficient conditions for asymptotic convergence are established.

The idea consists, in fact, of showing that the proposed algorithm is equivalent to a modified global Kalman observer when the arbitrary matrix, namely $\overline{Q}_k$ in this note, is appropriately chosen.

Numerical examples are provided.

II. PROBLEM FORMULATION

Consider a discrete-time system with multiple delays of the form

$$x_{k+1} = \sum_{i=0}^{r} A_i x_{k-i} + \sum_{j=0}^{s} B_j u_{k-j} \tag{1}$$

$$y_k = C x_k + D u_k \tag{2}$$

where $x_k \in R^n$, $u_k \in R^m$ and $y_k \in R^p$ denote the state, input, and output vectors, respectively, at time instant k. A_i, B_j, C and D are constant matrices of appropriate dimensions. The integers $r \geq 0$ and $s \geq 0$ are the number of time-delays assumed to be known.

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