CROSSOVER MORITA EQUIVALENCES FOR BLOCKS OF THE COVERING GROUPS OF THE SYMMETRIC AND ALTERNATING GROUPS

Radha Kessar, Dept. of Mathematics, Ohio State University, Columbus, Ohio

Mary Schaps, Dept. of Mathematics, Bar-Ilan University, Ramat-Gan, Israel

1. Introduction

In [S], Joanna Scopes discovered a method for generating Morita equivalences between blocks of symmetric groups and thus for showing that Donavan's conjecture, that there are only a finite number of Morita equivalence classes of blocks with a given defect group, holds for the blocks of the symmetric groups. This method has led in various different directions. It was generalized by Puig [P1] to demonstrate not only Morita equivalences but also the more restrictive Puig equivalences, thus establishing Puig's conjecture, that there are only a finite number of Puig equivalence classes for a given defect group, for blocks of the symmetric group. A variant was adapted by the first author to prove Donovan's conjecture for blocks of the Schur covers of the symmetric and alternating groups, [K]. A related technique was used in [Jo] for blocks of the general linear group, and an adaptation of the method was developed in [HK1], [HK2] to find Morita equivalences between blocks in various other algebraic groups. The method also lead Rickard to a way of demonstrating derived quivalences between blocks of symmetric groups, and this method was then taken up by Chuang and Rouquier [ChR] to show that for a given weight there is only one derived equivalence class of symmetric blocks which, along with [ChK], settled the Broue conjecture for symmetric blocks.

In this paper we intend to return to [K] and show that, in fact, the results therein reflected only half of the picture. The results in [K] demonstrated the existence of Morita equivalences between blocks of the covering groups $\tilde{S}_n$ of S_n or between blocks of the covering groups $\tilde{A}_n$ of A_n. We will now reconsider the situation and show that we can equally well get "crossovers" between blocks of $\tilde{A}_n$ and $\tilde{S}_n$. More specifically, the various characters are associated with strict partitions of n and the Morita eqivalences are obtained by an involution which is a variant of the Scopes involution used in Scopes' original work. The cases treated in [K] were those in which the involution is parity-preserving, and in this paper we will be interested in cases where it is parity-reversing.

Brundan and Kleshchev [BK] have studied various functors involving module categories of blocks of the covering groups of the symmetric group, which are then extended to the covering groups of the alternating groups. The existence of the crossovers we demonstrate in this paper would seem to have some relevance to these functors.

2. The projective representations of S_n and A_n, in characteristic 0.

The projective representations of the symmetric and alternating groups are currently studied as the linear representations of the covering groups $\tilde{A}_n$, $\tilde{S}_n^+$, and $\tilde{S}_n^-$, each of which has a central subgroup C_2 such that the quotient is A_n or S_n respectively. The differences between the two versions of $\tilde{S}_n$, which are said to be *isoclinic* to each other, are minor and barely affect the representation theory; they are similar to the differences between the quaternions and the dihedral group of order 8. We will generally write simply $\tilde{S}_n$, meaning one consistent choice.

Our group algebras will be considered over modular systems (K,R,k), where R is a complete discrete valuation ring, K is its quotient field, and k is the residue field of characteristic p. We assume that the characteristic is different from 2.

As usual, a partition $\lambda = (\lambda_1, \lambda_2, ...)$ of n is a nonincreasing sequence of non-negative integers summing to n. The partition will be called *strict* is it has no repeating parts.

Those characters of the covering groups which take the value -1 on the central element of order 2 (and hence are not simply inflations of the characters of the original groups) are determined by the strict partitions, but not in a one-to-one fashion. The mapping depends on the following concept.

Definition 2.1. *The parity $\epsilon(\lambda)$ of a strict partition λ is* 0 *or* 1 *respectively, depending on whether the sum of the parts with the number of parts is even or odd.*

This definition corresponds to the ordinary definition of an odd or even permutation if the integers in the partition are considered to represent the cycle lengths of a permutation, and in this sense we will speak of an odd or even partition.

An odd strict partition corresponds to one irreducible character of $\tilde{A}_n$ and to two irreducible character of $\tilde{S}_n$ with a common restriction to $\tilde{A}_n$. The partition gives the cycle structure of an element on which the two characters differ. An even strict partition corresponds to two conjugate irreducible characters of $\tilde{A}_n$ and to a single irrdeducible character of $\tilde{S}_n$.

Looking at this for a fixed group, in $\tilde{S}_n$ there are two ireducible characters associated with each strict partition of parity 1, and one with each strict partition of parity 0. In $\tilde{A}_n$ the parities are reversed, i.e., there are two irreducible characters associated with each strict partition of parity 0, and one for parity 1.

As with the representations of the symmetric group, it is possible to determine the degree of the irreducible character from the partition. To a strict partition we associate a diagram of n squares in rows corresponding to the distinct parts, with the parts staggered along a diagonal. The number of different ways to build up the partition step by step from the empty partition, so that each intermediate partition is strict, corresponds to the number of ways to fill in the numbers 1,...,n so that all rows and columns are increasing. This number is then multiplied by 2^r, where r is the greatest integer in half of $n-t$, where t is the number of free entries, i.e., the number of entries for which there is a choice of how to fill them in.(See [St].) While we will not need this formula explicitly, we will derive a related formula which involves counting the number of ways that the diagram of one partition can be built up from another by adding squares, and multiplying by an appropriate power of 2.

3. Combinatorics

We now consider the representation theory over the field k of characteristic p, where we have assumed that p is an odd prime. Because p is odd, the ordinary characters associated with a block are either all inflated or all faithful. We will consider only the latter case and refer to the block as faithful.

There is a procedure called removing p-bars, where each p-bar removed reduces the sum of the parts of the strict partition by p. When one removes the maximal number w of p-bars one arrives at a strict partition ν called the p-core. The integer w is called the weight of the block, and if it is greater than zero, then two characters belong to the same p-block if and only if one can remove the same number w of p-bars and arrive at the same p-core ν. In the special case $w = 0$, the character has defect zero, and if the parity is such that there are two characters corresponding to the core, then they give two distinct blocks.

We describe this procedure of removing p-bars, which will be very important in the sequel. The parts of the strict partition are represented as beads on an abacus with p-rods, labeled by the residues modulo p, $\{0, 1, ..., p-1\}$. Removal of a p-bar consists either of reducing a single part by p, which corresponds to lowering the position of one bead one place on its rod, or removing two parts which sum to p.

Fix a positive integer w and a p-core ν. Let $X = \{1, 2, \ldots, p-1\}$. For any $x \in X$, we define the Scopes involution $\widetilde{Sc_x}$ as in [K]. In particular, for $x \neq 1$, it corresponds in the abacus notation to exchanging the beads on runners x and $x-1$, and simultaneously, exchanging the beads on $p-x$ and $p-x+1$. In the particular case where $x = (p+1)/2$, we make only one exchange, since the two pairs of runners coincide. It was proven in [K], Lemma 4.7, that $\widetilde{Sc_x}$ preserves p-cores, so that $\mu = \widetilde{Sc_x}(\nu)$ will also be a p-core. We consider the case that $|\nu| > |\mu|$. Let $n = pw + |\nu|$.

Let J_n be the set of strict partitions of n with core ν, the union of the partitions with parity 0, denoted by J_n^+ and the partitions with parity 1, denoted by J_n^-. Set

$$m = pw + |\mu|.$$

Let J_m be the set of strict partitions of m with core μ. For any two strict partitions, λ and χ, let $\mathcal{M}(\lambda, \chi)$ be the number of sequences of strict partitions starting in λ and ending in χ such that each succesive term in the sequence is obtained from the previous one by the removal of a 1-bar. Set $\beta := \mathcal{M}(\nu, \mu)$

Definition 3.1. *Let ν and μ be as above. Then (ν, μ) form a w-compatible pair if the the following holds:*

(i)The map $\widetilde{Sc_x} : J_n \to J_m$ is one-to-one and onto.

(ii) For any $\lambda \in J_n$ and $\chi \in J_m$, $\mathcal{M}(\lambda, \chi) = 0$ if $\chi \neq \widetilde{Sc_x}(\lambda)$, and $\mathcal{M}(\lambda, \widetilde{Sc_x}(\lambda)) = \beta$.

(iii) For any $\lambda \in J_n$, $\epsilon(\lambda) + \epsilon(\widetilde{Sc_x}(\lambda) = \epsilon(\mu) + \epsilon(\nu)$.

Note that Proposition 4.9 of [K] gives a sufficient condition for ν and μ to form a w-compatible pair (however, there are examples of w-compatible pairs which do not satisfy the hypothesis of Proposition 4.9. of [K], see example below).

It was a shown in [K] that if ν and μ are w-compatible pairs having the same parity, then the corresponding blocks of $\tilde{S}_n$ and $\tilde{S}_m$ (or of $\tilde{A}_n$ and $\tilde{A}_m$) are Morita equivalent. Since we are interested in crossing over, we consider precisely Scopes involutions with reverse the parity, which are characterized by the following lemma. Since when $x = 1$, ν and μ have always the same parity, we will only discus the case $x \neq 1$.

Lemma 3.2. *With the notation above, for $x \in X$ satisfying $x \neq 1$, and core ν, the Scope involution $\widetilde{Sc_x}$ reverses the parity of ν and of all the elements in J_n, if the total number of parts congruent to a number in the set $C = \{x, x-1, p-x, p-x+1\}$ is odd, and preserves the parity if the total number of parts congruent to a number in C is even.*

Proof. The contribution of all the parts outside the set C is fixed, so we need consider only the contribution of the parts in C. The total number of parts is fixed under the Scopes involution, so the change occurs only in the size of each part. For each part there is a change of $+1$ or -1, which reverses the parity of that particular part. Thus if there is an odd number of parts, the total parity is reversed, and if there is an even number of parts, the total parity remains fixed. □

Example 3.3. *For $n = 13$ and $p = 5$, consider the block of $\tilde{S}_{13}$ of defect 2 and the block of $\tilde{A}_{12}$ of defect 2. In this case ν is (3), which is even, and $\mu = \tilde{Sc}_2(\nu)$ is (2), which is odd. These form a parity reversing 2-compatible pair. We now list the elements of J_{13} and J_{12} so that they correspond under* *the parity reversing Scopes involution for $x = 3$:*

$J_{13}^{+} = \{(13), (1,3,9), (1,4,8), (2,3,8), (3,4,6)\}$; $J_{13}^{-} = \{(3,10), (5,8), (1,3,4,5)\}$

$J_{12}^{-} = \{(12), (1,2,9), (1,4,7), (2,3,7), (2,4,6)\}$; $J_{12}^{+} = \{(2,10), (5,7), (1,2,4,5)\}$

In both cases, the total number of irreducible characters is 11*. Note that this example does not come under the purview of Proposition 4.9 of* [K].

4. The case $m = n - 1$

In order to show that the above example is not isolated, but that in fact there is an infinite family of examples of 2-compatible pairs, we discuss the case of $m = n - 1$ in detail. This is also intended to provide orientation for the more complicated general theory to follow.

Let b be a block of $\tilde{S}_n$ with core ν and c a block of $\tilde{A}_m$ with core μ, where we assume that $\mu = \widetilde{Sc_x}(\nu)$ for $x \neq 1$. We now consider the special case $m = n - 1$. By the rules for calculating the core, in the core ν, either there is no part congruent to x or there is no part congruent to $p - x$. If $x = (p+1)/2$, then ν consists of a single part equal to x. If $x \neq 1, (p+1)/2$, then in ν there is one more bead on runner x than on runner $x - 1$, or else one more bead on runner $p - x + 1$ than on $p - x$. In the latter case we will replace x by $p - x + 1$, which gives the same Scopes involution, and we may thus assume that there is one more bead on x than on $x - 1$, and none on the other two runners involved in the involution. Thus the total effect of the Scopes involution on ν is to reduce the highest part congruent to x by 1.

Lemma 4.1. *Suppose that ν and μ are as above, with $x = (p + 1)/2$, w=2, and $p > 2$. Then (ν,μ) form a 2-compatible pair.*

Proof. We must check three conditions. We first check (i). Since $x \neq 1$, the Scopes involution is one-to-one and onto for the total set of strict partitions of n. Thus to demonstrate (i), it suffices to show that the image of J_n is J_m. Let λ be a partition in J_n, and χ its image under $\widetilde{Sc}_x$. For the given x, the subset of X affected by the Scopes involution is $C = \{x, x - 1\}$. If both moves producing λ from ν were outside of C, then the same two moves produce χ as an element of J_m. If only one move is in C, it must be moving the top bead on runner x up one, and the corresponding moves produce χ as an element of J_n. If both moves are in C, then either the top bead on x is moved up 2, and the same move produces χ from μ, or there is a unique bead on x, and the two moves consist of moving that bead up one and adding a complementary pair. The corresponding moves produce χ in j_m.

To prove (ii), we note that in our case $\beta = 1$. The induction of characters is done by adding 1 to one of the parts of λ or adding a new part (1), in as many ways as this can be done while the partition remains strict. The analysis of cases in the proof of (i) is already sufficient to establish that

$\mathcal{M}(\lambda, \widetilde{Sc_x}(\lambda)) = 1$, since Scopes involution for $x \neq 1$ preseves the number of parts and the only one which can be changed is the highest part congruent to x.

If the 1 is added outside the set C effected by the Scopes involution, then we will not get a reduction to the correct core, since reduction to the core involves either moving a bead down on a single runner or removing a complementary pair. Similarly, adding 1 in the area effected by the Scope involution will can only produce the correct core if it moves a bead from the runner in μ which gained a bead from ν, i.e., from $x-1$ to x. The only way to add 1 to one of the parts of τ and get a strict partition corresponding to ν is to reverse the Scopes involution, $\mathcal{M}(\lambda, \tau) = 0$.

Condition (iii) on the parity follows immediately from the previous lemma, since parity is reversed both for the cores and for each of the elements of J_n. □

The case $w = 2$ is significant for the abelian defect group conjecture. The original germ of this paper came from an analysis of blocks with identical decomposition matrices in the second author's database of blocks of abelian defect group.

5. A character correspondence

All modules will be left modules unless otherwise stated. For a ring R, finite group G, and an RG-module V, the R-dual V^* of V is naturally a right RG-module, and we will use this fact without comment. Also, for groups G and $\hat{G}$, a $(RG, R\hat{G})$-module will be considered as an $R(G \times \hat{G}^{op})$-module and vice versa.

Definition 5.1. *Let F be a field, G and $\hat{G}$ finite groups and let b and c be central idempotents of FG and $F\hat{G}$ respectively such that FGb and $F\hat{G}c$ are split semi-simple algebras. We will denote by $Irr(G, b)$ the set of characters of simple FGb-modules and by $Irr(\hat{G}, c)$ the set of characters of simple $F\hat{G}c$-modules. For $\chi \in Irr(G, b)$, and $\tau \in Irr(G, c)$, and a finite dimensional $(FGb, F\hat{G}c)$-bimodule X, we will denote by $r(\chi, \tau, X)$ the multiplicity of the FGb-module V_χ as a summand of $X \otimes_{F\hat{G}} V_\tau$ where V_χ is a simple FG-module with character χ and V_τ is a simple $F\hat{G}$-module with character τ.*

Before proceeding we record the following fact.

Proposition 5.2. *Let F, G, $\hat{G}$, b, c and X be as in the above definition. For each $\chi \in Irr(G, b)$, let V_χ be a simple FG module with character χ and for each $\tau \in Irr(\hat{G}, c)$, let V_τ be a simple $F\hat{G}$ module with character τ. Then as $F(G \times \hat{G}^{op})$-modules, there is an isomorphism*

$$X \cong \sum_{\chi \in Irr(G,b), \tau \in Irr(\hat{G},c)} r(\chi, \tau, X) V_\chi \otimes_F V_\tau^*.$$

Let w be a positive integer. Let ν and $\mu := \tilde{Sc_x}(\nu)$ be two p-cores such that $x > 1$ and $|\nu| > |\mu|$. Let $n = pw + |\nu|$ and $m = pw + |\mu|$ and let b (respectively c) be the faithful blocks of $\tilde{S}_n$ (respectively $\tilde{S}_m$) with core ν (respectively μ). Note that b and c are also blocks of the double covers of the corresponding alternating groups as well. Let K be a field of characteristic 0 which is a splitting field for all subgroups of $\tilde{S}_n$.

Lemma 5.3. *Let $\alpha := n - m$ and let $\beta := \mathcal{M}(\nu, \mu)$. Suppose that ν and μ form a w-compatible pair and that ν and μ have opposite parities.*

(i) $|Irr(\tilde{S}_n, b)| = |Irr(\tilde{A}_m, c)|$. For each $\chi \in Irr(\tilde{S}_n, b)$,

$$\Sigma_{\tau\in Irr(\tilde{A}_m,c)} r(\chi, \tau, K\tilde{S}_n bc) = 2^{\frac{\alpha+1}{2}}\beta$$

and for each $\tau \in Irr(\tilde{A}_m, c)$,

$$\Sigma_{\chi\in Irr(\tilde{S}_n,b)} r(\chi, \tau, K\tilde{S}_n bc) = 2^{\frac{\alpha+1}{2}}\beta.$$

(ii) $|Irr(\tilde{A}_n, b)| = |Irr(\tilde{S}_m, c)|$ and for each $\phi \in Irr(\tilde{A}_n, b)$,

$$\Sigma_{\pi\in Irr(\tilde{S}_m,c)} r(\phi, \pi, K\tilde{S}_n bc) = 2^{\frac{\alpha+1}{2}}\beta$$

and for each $\pi \in Irr(\tilde{S}_m, c)$,

$$\Sigma_{\phi\in Irr(\tilde{A}_n,b)} r(\phi, \pi, K\tilde{S}_n bc) = 2^{\frac{\alpha+1}{2}}\beta.$$

Proof. (i) Note that the $(K\tilde{S}_n b, K\tilde{A}_m c)$-bimodule $K\tilde{S}_n bc$ represents induction from $\tilde{A}_m$ to $\tilde{S}_n$ followed by truncation at the block b. Let λ be a strict partition of n and γ a strict partition of m. Let θ be an irreducible character of $\tilde{S}_n$ corresponding to λ and η an irreducible character of $\tilde{S}_m$ corresponding to γ. It follows from the branching rules (see for example [HH]) that if θ is a constituent of $Ind_{\tilde{S}_m}^{\tilde{S}_n}(\eta)$, then $\mathcal{M}(\lambda, \gamma)$ is non-empty. Furthermore, if λ and γ have the same number of parts, then the multiplicity of θ as a constituent of $Ind_{\tilde{S}_m}^{\tilde{S}_n}(\eta)$ is $2^{\frac{\alpha-1}{2}}|\mathcal{M}(\lambda, \gamma)|$ if α is odd, is $2^{\frac{\alpha}{2}}|\mathcal{M}(\lambda, \gamma)|$ if α is even and $\epsilon(\gamma) = 0$ and is $2^{\frac{\alpha}{2}-1}|\mathcal{M}(\lambda, \gamma)|$ if α is even and $\epsilon(\gamma) = 1$. (Actually, this can be written as $2^{\frac{\alpha-\epsilon(\lambda)-\epsilon(\gamma)}{2}}|\mathcal{M}(\lambda, \gamma)|$).

Since in our situation, ν and μ have opposite parities and since $x > 1$, α is odd and for any strict partition λ, λ and $\widetilde{Sc_x}(\lambda)$ have the same number of parts. The fact that ν and μ are a w-compatible pair along with the above remarks yields that for any λ in J_n^+, $\widetilde{Sc_x}(\lambda)$ is in J_m^-, and if η is any one of the two characters of $\tilde{S}_m$ corresponding to $\widetilde{Sc_x}(\lambda)$, then the contribution of b to $Ind_{\tilde{S}_m}^{\tilde{S}_n}(\eta)$ is equal to $2^{\frac{\alpha-1}{2}}\beta$ copies of the unique irreducible character of $\tilde{S}_n$ corresponding to λ. Similarly, if λ in J_n^-, $\widetilde{Sc_x}(\lambda)$ is in J_n^+, and if η is the unique character of $\tilde{S}_m$ corresponding to $\widetilde{Sc_x}(\lambda)$, then the contribution of b to $Ind_{\tilde{S}_m}^{\tilde{S}_n}(\eta)$ is equal to the sum of $2^{\frac{\alpha-1}{2}}\beta$ copies of each of the two irreducible characters of $\tilde{S}_n$ corresponding to λ. The result now follows from the behaviour of irreducible characters of $\tilde{A}_m$ under induction to $\tilde{S}_m$.

(ii) The $(K\tilde{A}_nb, K\tilde{S}_mc)$-bimodule $K\tilde{S}_nbc$ represents induction from $\tilde{S}_m$ to $\tilde{S}_n$ followed by truncation at the block b, followed by restriction to $\tilde{A}_n$. The rest of the proof is analogous to (i).

□

6. Source algebra Equivalence

Let n, m, b and c be as in the previous section. Let (K, R, k) be a p-modular system such that K and k are splitting fields for all subgroups of $\tilde{S}_n$. In this section we will show that $R\tilde{S}_nb$ and $R\tilde{A}_mc$ are source algebra equivalent and that $R\tilde{A}_nb$ and $R\tilde{S}_mc$ are source algebra equivalent. The approach will be similar to that in [HK2]. However, for the equivalence between $R\tilde{A}_nb$ and $R\tilde{S}_mc$, we cannot apply [HK2] directly since $\tilde{S}_m$ is not a subgroup of $\tilde{A}_n$. In order to circumvent this problem, we switch from the pointed groups approach to an approach via p-permutation modules.

We fix some notation which will stay in effect for the rest of the paper. We let $H := \tilde{A}_n$ and $G := \tilde{S}_n$, $\hat{H} = \tilde{A}_m$ and $\hat{G} := \tilde{S}_m$. Set $E_1 = G \times \hat{G}^{op}$, $E_2 = G \times \hat{H}^{op}$, $E_3 = H \times \hat{G}^{op}$ and $E_4 = H \times \hat{H}^{op}$.

Let D be a defect group of the block c of $\tilde{A}_m$ (so D is a defect group of c as a block of $\tilde{S}_m$ and of b as a block of $\tilde{A}_n$ and of $\tilde{S}_n$). Let ΔD be the subgroup $\{(x, x^{-1}) \mid x \in D\}$ of $H \times \hat{H}^{op}$. Since $RGbc$ is a $H \times \hat{G}^{op}$ summand of the permutation module RG and D is a defect group of the block b of H and the block c of $\hat{G}$ all indecomposable $R(H \times \hat{G}^{op})$ summands of $RGbc$ have trivial source and a vertex which is conjugate in $H \times \hat{G}^{op}$ to a subgroup of ΔD.

Lemma 6.1. *Let $1 \leq i \leq 4$ and let*

$$RGbc = \oplus_{j\in J} W_j \oplus_{j'\in J'} Z_{j'} \tag{1}$$

be a direct sum decomposition of the RE_i -module $RGbc$, such that ΔD is a vertex of W_j for each $j \in J$ and ΔD is not a vertex of $Z_{j'}$ for any $j' \in J'$. Then $|J| = 2^{\frac{\alpha+1}{2}}\beta$ if $i = 2, 3, 4$, and $|J| = 2^{\frac{\alpha-1}{2}}\beta$ if $i = 1$.

Proof. Let E be a subgroup of $G \times \hat{G}^{op}$ containing $H \times \hat{H}^{op}$. The RE module $RGbc$ is a p-permutation module. Let V be an indecomposable p-permutation RE-module. By the relationship between the Brauer homomorphism and p-permutation modules given in Theorem 3.2 of [Br1], $V(\Delta D)$ is non-zero iff ΔD is contained in a vertex of V; ΔD is a vertex of V if and only if $V(\Delta D)$ is an indecomposable projective $kN_E(\Delta D)/\Delta D$ module; the correspondence $V \to V(\Delta D)$ induces a bijection between the isomorphism classes of indecomposable p-permutation RE-modules with vertex ΔD and the isomorphism classes of indecomposable projective $kN_E(\Delta D)/\Delta D$ modules. Furthermore, if T is a projective indecomposable $kN_E(\Delta D)/\Delta D$-module and $V(\Delta D, T)$ is the corresponding p-permutation RE-module with vertex ΔD, then the multiplicity of $V(\Delta D, T)$ as a summand of $RGbc$ is equal to the multiplicity of T as a summand of $RGbc(\Delta D)$.

Let Z be the subgroup of $G \times \hat{G}^{op}$ consisting of elements $(x, 1)$, where $x \in Z(D)$. Then Z is a normal subgroup of $N_E(\Delta D)$. Let $\bar{Z}$ be the image of $Z\Delta D$ under the canonical epimorphism onto $N_E(\Delta D)/\Delta D$. Then $\bar{Z} \cong Z(D)$ is a normal subgroup of $N_E(\Delta D)/\Delta D$.

Thus, the correspondence $T \to k \otimes_{k[\bar{Z}]} T$ induces a bijection between isomorphism classes of projective indecomposable $kN_E(\Delta D)/\Delta D$ modules and isomorphism classes of projective indecomposable $kN_E(\Delta D)/(Z\Delta D)$-modules.

Summarizing, if $\bar{T}$ is the projective indecomposable $k(N_E(\Delta D)/(Z\Delta D))$-module corresponding to the projective indecomposable $kN_E(\Delta D)/(\Delta D)$ module T, and if $V(\Delta D, T)$ is the corresponding p-permutation RE-module with vertex ΔD, then the multiplicity of $V(\Delta D, T)$ as a summand of $RGbc$ is equal to the multiplicity of $\bar{T}$ as a summand of $k \otimes_{k[\bar{Z}]} RGbc(\Delta D)$.

Since the ΔD-module structure of $RGbc$ is compatible with the conjugation action of D on the algebra $RGbc$, it follows that there is an isomorphism of $N_E(\Delta D)/\Delta D$-modules

$$RGbc(\Delta D) \cong kC_G(D)\mathrm{Br}_D(b)\mathrm{Br}_D(c), \tag{2}$$

where the $kN_E(\Delta D)/\Delta D$-module structure of $kC_G(D)\mathrm{Br}_D(b)\mathrm{Br}_D(c)$ is the natural one, that is, (x, y) acts by left multiplication by the element x of $N_G(D)$ and right multiplication by the element y of $N_{\hat{G}}(D)$. Since the pair (x, y) lies in $kN_E(\Delta D)/\Delta D$, the conjugation actions of x and y on D are the same.

By the local structure of faithful blocks of the double covers of the symmetric and alternating groups as described in [Ca] and [HH], $\tilde{S}_{m-|\mu|} = \tilde{S}_{n-|\nu|}$ and D can be chosen to be a Sylow p-subgroup of $\tilde{S}_{n-|\nu|}$.

Let σ be an element of $\tilde{S}_{|\mu|} - \tilde{A}_{|\mu|}$ and ρ be an element of $N_{\tilde{S}_{m-|\mu|}}(D) - \tilde{A}_{m-|\mu|}$. Let M be the subgroup of $G \times \hat{G}^{op}$ consisting of elements (x, x^{-1}), $x \in N_{A_{n-|\nu|}}(D) = N_{A_{m-|\mu|}}(D)$. Set

$$L_1 = \tilde{S}_{|\nu|} \times \tilde{S}^{op}_{|\mu|} \text{ and } R_1 =< (\rho, \rho^{-1}) >,$$

$$L_2 = \tilde{S}_{|\nu|} \times \tilde{A}^{op}_{|\mu|} \text{ and } R_2 =< (\rho, \sigma^{-1}\rho^{-1}) >,$$

$$L_3 = \tilde{A}_{|\nu|} \times \tilde{S}^{op}_{|\mu|} \text{ and } R_3 =< (\sigma\rho, \rho^{-1}) >,$$

$$L_4 = \tilde{A}_{|\nu|} \times \tilde{A}^{op}_{|\mu|} \text{ and } R_4 =< (\sigma\rho, \sigma^{-1}\rho^{-1}) > .$$

Then, for $1 \leq i \leq 4$,

$$N_{E_i}(\Delta D) = L_i Z M R_i.$$

The group L_i is isomorphic to its image under the canonical surjection onto $N_{E_i}(\Delta D)/\Delta D)$. Henceforth, we identify the groups L_i with these images.

Now $C_G(D) \cong \tilde{S}_{|\nu|} \times Z(D)$, and $\mathrm{Br}_D(b) = \bar{b}$ and $\mathrm{Br}_D(c) = \bar{c}$ where $\bar{b}$ and $\bar{c}$ are central idempotents of $\tilde{S}_{|\nu|}$ corresponding to the characters of $\tilde{S}_{|\nu|}$ and $\tilde{S}_{|\mu|}$ associated to the partitions ν and μ respectively. These characters have defect 0 because ν and μ are p-cores.(See [Ca]).

Thus, by (2) and the description of normalisers given above it follows that $N_{E_i}(\Delta D)/Z\Delta D \cong L_iMR_i/\Delta D$, and under this isomorphism,

$$k \otimes_{k\bar{Z}} RGbc(\Delta D) \cong k\tilde{S}_{|\nu|}\bar{b}\bar{c}.$$

Now, $\bar{b}$ and $\bar{c}$ are sums of defect 0 blocks of $\tilde{S}_{|\nu|}$ and $\tilde{S}_{|\mu|}$ respectively, hence $k\tilde{S}_{|\nu|}\bar{b}\bar{c}$ is semi-simple and projective as kL_i-module. On the other hand, L_i is of p'-index in $L_iMR_i/\Delta D$. Hence $k\tilde{S}_{|\nu|}\bar{b}\bar{c}$ is a semi-simple projective $kL_iMR_i/\Delta D$-module. Furthermore, $M/\Delta D$ is normal in $L_iMR_i/\Delta D$ and $M/\Delta D$ acts trivially on $k\tilde{S}_{|\nu|}\bar{b}\bar{c}$, thus the number of summands in a direct sum decomposition of $k\tilde{S}_{|\nu|}\bar{b}\bar{c}$ as $kL_iMR_i/\Delta D$-module is the same as the number of summands in a direct sum decomposition of $k\tilde{S}_{|\nu|}\bar{b}\bar{c}$ as $kL_iMR_i/M \cong kL_iR_i$-module.

Thus it remains to determine the structure of $k\tilde{S}_{|\nu|}\bar{b}\bar{c}$ as kL_iR_i-module.

Consider first the case that $\epsilon(\nu) = 1$ and $\epsilon(\mu) = 0$.

In this case $\bar{b}$ is the sum of the two blocks of defect zero of $\tilde{S}_{|\nu|}$ corresponding to the two simple projective modules V and V^a of $k\tilde{S}_{|\nu|}$ associated to the partition ν, and $\bar{c}$ is the block of defect zero of $k\tilde{S}_{|\mu|}$ corresponding to the unique simple projective module U of $k\tilde{S}_{|\mu|}$ associated to the partition ν. Let Y be the unique simple projective $k\tilde{A}_{|\nu|}$-module covered by V and V^a and let X and X^c be the two simple projective $k\tilde{A}_{|\nu|}$-modules covered by U.

It is standard Clifford theory that X and X^c are conjugate to each by the permutation σ defined above. Conjugation by ρ leaves them fixed. It is somewhat more surprising the two associated blocks V and V^a are conjugate under ρ; this is a result of the fact that V and V^a differ only on the conjugacy class of the odd element ν. From the conjugation rule in the covering group for preimages of odd permutations, conjugation by ρ multiplies the preimage of the conjugate by the central element, which takes character value -1 in the faithful blocks.

Now, any indecomposable kL_1 summand of $k\tilde{S}_{|\nu|}\bar{b}\bar{c}$ is isomorphic to either $V \otimes_k U^*$ or to $V^a \otimes_k U^*$. The multiplicity of $V \otimes_k U^*$ as a summand of $k\tilde{S}_{|\nu|}\bar{b}\bar{c}$ is equal to the multiplicity of V as a summand of $Ind_{\tilde{A}_{|\mu|}}^{\tilde{S}_{|\mu|}}(U)$ (see Proposition 5.2). Similarly, the multiplicity of $V^a \otimes_k U^*$ as a summand of $k\tilde{S}_{|\nu|}\bar{b}\bar{c}$ is equal to the multiplicity of V as a summand of $Ind_{\tilde{A}_{|\mu|}}^{\tilde{S}_{|\mu|}}(U^*)$. Thus, by the same arguments as given in Lemma 5.3, it follows that

$$k\tilde{S}_{|\nu|}\bar{b}\bar{c} \cong 2^{\frac{\alpha-1}{2}}\beta V \otimes_k U^* \oplus 2^{\frac{\alpha-1}{2}}\beta V^a \otimes_k U^*$$

as kL_1-module. Similarly,

$$k\tilde{S}_{|\nu|}\bar{b}\bar{c} \cong 2^{\frac{\alpha-1}{2}}\beta V\otimes_k X^* \oplus 2^{\frac{\alpha-1}{2}}\beta V^a\otimes_k X^* \oplus 2^{\frac{\alpha-1}{2}}\beta V\otimes_k X^{*c} \oplus 2^{\frac{\alpha-1}{2}}\beta V^a\otimes_k X^{*c}$$

as kL_2-module,

$$k\tilde{S}_{|\nu|}\bar{b}\bar{c} \cong 2^{\frac{\alpha+1}{2}}\beta Y \otimes_k U^*,$$

as kL_3-module, and

$$k\tilde{S}_{|\nu|}\bar{b}\bar{c} \cong 2^{\frac{\alpha+1}{2}}\beta Y \otimes_k X^* \oplus 2^{\frac{\alpha+1}{2}}\beta Y \otimes_k X^{*c}$$

as kL_4-module.

Now, as kL_1-module,

$${}^{(\rho,\rho^{-1})}(V \otimes_k U^*) \cong {}^{\rho}V \otimes_k {}^{\rho^{-1}}U^* \cong V^a \otimes_k U^*,$$

hence by Clifford theory, it follows that the kL_1R_1 module $k\tilde{S}_{|\nu|}\bar{b}\bar{c}$ is a direct sum of $2^{\frac{\alpha-1}{2}}\beta$ modules. Similarly, as kL_2-module,

$${}^{(\rho,\sigma^{-1}\rho^{-1})}(V \otimes_k X^*) \cong {}^{\rho}V \otimes_k {}^{\sigma^{-1}\rho^{-1}}X^* \cong V^a \otimes_k X^{*c},$$

and

$${}^{(\rho,\sigma^{-1}\rho^{-1})}(V \otimes_k X^{*c}) \cong {}^{\rho}V \otimes_k {}^{\sigma^{-1}\rho^{-1}}X^{*c} \cong V^a \otimes_k X^*,$$

hence the kL_2R_2-module $k\tilde{S}_{|\nu|}\bar{b}\bar{c}$ is a direct sum of $2^{\frac{\alpha+1}{2}}\beta$ modules.

As kL_3-module,

$${}^{(\sigma\rho,\rho^{-1})}(Y \otimes_k U^*) \cong {}^{\sigma\rho}Y \otimes_k {}^{\rho^{-1}}U^* \cong Y \otimes_k U^*,$$

hence the kL_3R_3-module $k\tilde{S}_{|\nu|}\bar{b}\bar{c}$ is a direct sum of $2^{\frac{\alpha+1}{2}}\beta$ modules.

Finally, as kL_4-module,

$${}^{(\sigma\rho,\sigma^{-1}\rho^{-1})}(Y \otimes_k X^*) \cong {}^{\sigma\rho}Y \otimes_k {}^{\sigma^{-1}\rho^{-1}}X^* \cong Y \otimes_k X^{*c},$$

hence the kL_4R_4-module $k\tilde{S}_{|\nu|}\bar{b}\bar{c}$ is a direct sum of $2^{\frac{\alpha+1}{2}}\beta$ modules.

The case $\epsilon(\nu) = 0$ is handled similarly. In this case, there is a single V but two associates U^* and U^{*a}. There are two conjugates Y and Y^c whereas there is now only one X^*. As before, ρ exchanges the associates, while σ and $\sigma\rho$ exchange the conjugates.

The restrictions to the L_i are now as follows:

$$k\tilde{S}_{|\nu|}\bar{b}\bar{c} \cong 2^{\frac{\alpha-1}{2}}\beta V \otimes_k U^* \oplus 2^{\frac{\alpha-1}{2}}\beta V \otimes_k U^{*a}$$

as kL_1-module. Similarly,

$$k\tilde{S}_{|\nu|}\bar{b}\bar{c} \cong 2^{\frac{\alpha+1}{2}}\beta V \otimes_k X^*,$$

as kL_2-module,

$$k\tilde{S}_{|\nu|}\bar{b}\bar{c} \cong 2^{\frac{\alpha-1}{2}}\beta Y\otimes_k U^* \oplus 2^{\frac{\alpha-1}{2}}\beta Y^c\otimes_k U^* \oplus 2^{\frac{\alpha-1}{2}}\beta Y\otimes_k U^{*a} \oplus 2^{\frac{\alpha-1}{2}}\beta Y^c\otimes_k U^{*a}$$

as kL_3-module, and

$$k\tilde{S}_{|\nu|}\bar{b}\bar{c} \cong 2^{\frac{\alpha+1}{2}}\beta Y \otimes_k X^* \oplus 2^{\frac{\alpha+1}{2}}\beta Y \otimes_k X^{*c}$$

as kL_4-module. With appropriate changes to reflect the action of the R_i, the remainder of the proof is virtually identical.

□

We have illustrated these ideas in an example at the end of the paper. It can be read at this point, but we do not insert it here in order not to interfere with the continuity of the proof of the theorem.

Lemma 6.2. *Let $1 \leq i \leq 4$. Denote by M_i be the image of the projection of E_i onto the first component, and $\hat{M}_i^{op}$ the image of the projection of E_i onto the second component. Let W be an indecomposable RE_i-module summand of $RGbc$ having vertex ΔD. Then $Res_{M_i}(W)$ is a progenerator for $mod(RM_ib)$ and $Res_{\hat{M}_i^{op}}(W)$ is a progenerator for $mod(R\hat{M}_i^{op}c)$*

Proof. First note that $Res_{M_i}RGbc$ and $Res_{\hat{M}_i^{op}}RGbc$ are projective. Thus, we need only show that if P is a projective indecomposable RM_ib-module then $Res_{M_i}(W)$ contains a summand isomorphic to P and that if Q is a projective indecomposable $R\hat{M}_i^{op}c$-module then $Res_{\hat{M}_i^{op}}(W)$ contains a summand isomorphic to P.

Let $i = 1$. The RE_1-module W is isomorphic to $RGbc\iota$ where ι is a primitive idempotent of

$$End_{RE_1}(RGbc) \cong (End_{RG}(RGbc))^{\hat{G}} \cong (cRGbc)^{\hat{G}}$$

when $cRGbc$ is considered as an $\hat{G}$ module via conjugation. The group ΔD being a vertex of W is equivalent to $\mathrm{Br}_D(\iota) \neq 0$. Thus, there is a primitive idempotent t of $(\iota RG\iota)^{\hat{G}}$ such that $\mathrm{Br}_D(t) \neq 0$. In other words, ι contains a source idempotent t of the block b of G. Since D is a defect group of the block b, this means that $b \in \mathrm{Tr}_D^G((RGb)^D\iota(RGb)^D)$, where Tr stands for relative trace and $(RGb)^D\iota(RGb)^D$ is the ideal of $(RGb)^D$ generated by ι (see Theorem 18.3 of [Th]). Consequently, $b \in RG\iota RG$ which means that ι does not belong to any maximal ideal of RGb. It follows that the RGb-module $W \cong RG\iota$ contains a summmand isomorphic to P for any projective indecomposable RM_ib-module P. Also, as is explained in the proof of Theorem 2.5 of [HK2], Lemma 3.8 of [Pu] implies that the $R(\hat{G} \times \hat{G}^{op})$ module $\iota RGbc\iota$ contains a summand isomorphic to $R\hat{G}c$. Consequently, $Res_{\hat{G}\times\hat{G}^{op}}(W)$ contains a summand isomorphic to $R\hat{G}c$, hence in particular, $Res_{\hat{G}^{op}}(W)$ is a progenerator for the category $mod(R\hat{G}^{op}c)$.

The proof for the case $i = 2$ is identical to that for the case $i = 1$.

Now let $i = 4$. Let σ be an element of $\tilde{S}_{|\mu|} - \tilde{A}_{|\mu|}$. As RE_4-modules, there is a decomposition,

$$RGbc = RHbc \oplus RHbc\sigma.$$

If W is isomorphic to a direct summand of $RHbc$, then the result follows exactly as for $i = 1$. Suppose W is isomorphic to a summand of $RHbc\sigma$. Since $RHbc \cong {}^{(1,\sigma^{-1})}RHbc\sigma$ as RE_4-modules, ${}^{(1,\sigma^{-1})}W$ is isomorphic to a direct summand, say V of $RHbc$ having ${}^{(1,\sigma^{-1})}\Delta D = \Delta D$ as vertex. In particular, $Res_H(W) = Res_H({}^{(1,\sigma^{-1})}W) \cong Res_HV$ and $Res_{\hat{H}^{op}}(W) \cong {}^{\sigma}Res_{\hat{H}^{op}}(V)$. As for the case $i = 1$, $Res_H(V)$ is a progenerator for $mod(RHb)$ and

$Res_{\hat{H}^{op}}(V)$ is a progenerator for $mod(R\hat{H}^{op}c)$. Since $Q \to {}^{\sigma}Q$ is a bijection on the isomorphism classes of projective indecomposable $R\hat{H}^{op}c$-modules, it follows that $Res_H(W)$ is a progenerator for $mod(RHb)$ and $Res_{\hat{H}^{op}}(W)$ is a progenerator for $mod(R\hat{H}^{op}c)$

It remains to do the case $i = 3$. The arguments for this case depend on the following two observations. First, we claim that $Res_{\hat{H}\times\hat{G}^{op}}R\hat{G}c$ is indecomposable. Indeed, the $K(\hat{H} \times \hat{H}^{op})$ module $K\hat{H}c$ is isomorphic to

$$\sum_{\phi\in Irr(\hat{H},c)} V_\phi \otimes_K V_\phi^*,$$

hence as $K(\hat{H} \times \hat{H}^{op})$ module $K\hat{H}c\sigma$ is isomorphic to

$$\sum_{\phi\in Irr(\hat{H},c)} V_\phi \otimes_K {}^{\sigma}V_\phi^*.$$

Since every block of positive defect contains both even and odd characters, it follows that $K\hat{H}c \ncong K\hat{H}c\sigma$ as $K(\hat{H} \times \hat{H}^{op})$ module and thus $R\hat{H}c \ncong R\hat{H}c\sigma$ as $R(\hat{H} \times \hat{H}^{op})$ module. On the other hand, $R\hat{H}c\sigma \cong {}^{(1,\sigma)}R\hat{H}c$ as $R(\hat{H} \times \hat{H}^{op})$ module, that is the $R(\hat{H}\times\hat{H}^{op})$ modules $R\hat{H}c\sigma$ and $R\hat{H}c$ are conjugate in $\hat{H}\times\hat{G}^{op}$. Since $R\hat{H}c$ and $R\hat{H}c\sigma$ are indecomposable $R(H\times\hat{H}^{op})$ modules, it follows that $R\hat{G}c = R\hat{H}c \oplus R\hat{H}c\sigma$ is indecomposable as $R(\hat{H} \times \hat{G}^{op})$-module.

Next, let U be an RE_1 indecomposable module summand of $RGbc$ and let W' be an indecomposable RE_3 summand of $Res_{E_3}(U)$. We claim that either $Res_{E_3}U \cong W'$ or $Res_{E_3}U \cong W' \oplus {}^{(1,\sigma)}W'$. Indeed, since the index of E_3 in E_1 is 2 and since p is odd, U is relatively E_3 projective, that is there is an indecomposable summand W'' of $Res_{E_3}U$ such that U is isomorphic to a direct summand of $Ind_{E_3}^{E_1}(W'')$. By the Mackey formula, it follows that $Res_{E_3}U$ is either indecomposable or a direct sum of W'' and ${}^{(1,\sigma)}W''$. Clearly, W' is isomorphic to one of W'' or ${}^{(1,\sigma)}W''$ proving the claim.

Now let $i = 3$ and let U be an RE_1 indecomposable module summand of $RGbc$ such that W is an indecomposable RE_3 summand of $Res_{E_3}(U)$. By the claim above either $Res_{E_3}U \cong W$ or $Res_{E_3}U \cong W \oplus {}^{(1,\sigma)}W$. Also, U has vertex ΔD. Hence, by the argument given for the case $i = 1$, $Res_{\hat{G}\times\hat{G}^{op}}(U)$ contains a summand isomorphic to $R\hat{G}c$. Since $Res_{\hat{H}\times\hat{G}^{op}}(R\hat{G}c)$ is indecomposable by the first claim above, it follows that either $Res_{\hat{H}\times\hat{G}^{op}}(W)$ or $Res_{\hat{H}\times\hat{G}^{op}}({}^{(1,\sigma)}W)$ has a $R(\hat{H} \times \hat{G}^{op})$-summand isomorphic to $R\hat{G}c$. Thus, either $Res_{\hat{G}^{op}}(W)$ or $Res_{\hat{G}^{op}}\,{}^{\sigma}(W)$ is a progenerator for the category $mod(R\hat{G}^{op}c)$. But $Res_{\hat{G}^{op}}\,{}^{\sigma}(W) = {}^{\sigma}(Res_{\hat{G}^{op}}W)$ and $Q \to {}^{\sigma}Q$ is a bijection on the isomorphism classes of projective indecomposable $R\hat{G}^{op}$-modules. Hence $Res_{\hat{G}^{op}}(W)$ is a progenerator for the category $mod(R\hat{G}^{op}c)$. Let V be an indecomposable summand of $Res_{E_4}W$ having vertex ΔD. Then by the arguments given for the case $i = 4$, $Res_H V$ is a progenerator for $mod(RHc)$, hence so is $Res_H W$.

$\square$

Theorem 6.3. *(i) Let W be an indecomposable summand of the $R(G \times \hat{H})$-module $RGbc$ having vertex ΔD. Then*

$$W \otimes_R - : (modR\hat{H}c) \to (modRGb)$$

is an equivalence. Consequently, RGb and $R\hat{H}c$ are source algebra equivalent.

(ii) Let W be an indecomposable summand of the $R(H \times \hat{G})$-module $RGbc$ having vertex ΔD. Then

$$W \otimes_R - : (modRHb) \to (modR\hat{G}c)$$

is an equivalence. Consequently, RHb and $R\hat{G}c$ are source algebra equivalent.

Proof. (i) Let

$$RGbc = \oplus_{j \in J} W_j \oplus_{j' \in J'} Z_{j'} \tag{3}$$

be a direct sum decomposition of the RE_2-module $RGbc$, such that ΔD is a vertex of W_j for each $j \in J$ and ΔD is not a vertex of $Z_{j'}$ for any $j' \in J'$.

Let $j \in J$. Demote by KW_j the KE_2-module $K \otimes_R W_j$. By the previous lemma, $Res_G(KW_j)$ is a progenerator for $mod(KGb)$ and $Res_{\hat{H}^{op}}(KW_j)$ is a progenerator for the category $mod(K\hat{H}c)$.

Thus writing

$$K \otimes_R W_j \cong \sum_{\chi,\tau} r(\chi, \tau, W_j)(V_\chi \otimes_K V_\tau^*),$$

where χ ranges over $Irr(G, b)$ and τ ranges over $Irr(\hat{H}, c)$, it follows that for each $\chi \in Irr(G, b)$,

$$\sum_{\tau \in Irr(\hat{H},c)} r(\chi, \tau, W_j) \geq 1. \tag{4}$$

and for each $\tau \in Irr(\hat{H}, c)$,

$$\sum_{\chi \in Irr(G,b)} r(\chi, \tau, W_j) \geq 1. \tag{5}$$

By Lemma 6.1, case $i = 2$, we know that $|J| = 2^{\alpha+1}\beta$. Now by Lemma 5.3

$$\sum_{\chi \in Irr(G,b)} r(\chi, \tau, K\tilde{G}bc) = 2^{\alpha+1}\beta,$$

and combining this with equation 5, we find that for each $j \in J$ and each $\chi \in Irr(G, b)$

$$\sum_{\tau \in Irr(\hat{H},c)} r(\chi, \tau, W_j) = 1. \tag{6}$$

Similarly combining Lemma 5.3 with equation 4 shows that for each $j \in J$ and each $\tau \in Irr(\hat{H}, c)$,

$$\sum_{\chi \in Irr(G,b)} r(\chi, \tau, W_j) = 1. \tag{7}$$

Finally, since the W_j account for all the characters, we deduce that $J' = \phi$.

Thus tensoring by KW_i induces a bijection between $Irr(H, c)$ and $Irr(G, b)$. Since W_i is R-free, and is projective as left RG-module and as right RH-module, by Theorem 2.4 of [Br2], it follows that W_i induces a Morita equivalence between $R\hat{H}c$ and RGb. Finally the equivalence of source algebras follows from a result of L.L.Scott (see [P2]) which says that the $R(G \times H^{op})$ module W_i has ΔD as vertex and trivial source.

(ii) This is identical to the proof in (i).

□

Example 6.4. *We return to the earlier example with* $n = 13$*,* $m = 12$*,* $p = 5$*. We assume that the* p*-core* ν *is* (3) *and that its image under the Scopes involution exchanging* 3 *and* 2 *is* (2)*. Thus in this example we have* $\epsilon(\nu) = 0$ *and* $\epsilon(\mu) = 1$*, which is the case which was not done explicitly in the proof of Lemma 6.1. We have already shown that this is a parity reversing* 2*-compatible pair. In this case* $\alpha = n - m = 1$*, and* $\beta = 1$*. We use the notation* $[a, b, c, ..]$ *for the preimage of the cycle* $(a, b, c, ...)$*. We let our defect group* D *be the elementary abelian subgroup generated by those preimages of* $(1, 2, 3, 4, 5)$ *and* $(6, 7, 8, 9, 10)$ *which are of order* 5 *rather than of order* 10*. By the conventions in the ATLAS* [Con]*, these are* $-[1, 2, 3, 4, 5]$ *and* $-[6, 7, 8, 9, 10]$*. For definiteness, we will choose that version of* $\tilde{S}_n$ *in which* $[1, 2]$ *is of order* 4*.*

The normalizers of D *in the various groups* E_i *defined in the proof of Lemma 6.1 depended on two permutations, which we can take to be*

$$\sigma = [11, 12] \in \tilde{S}_{|\mu|} - \tilde{A}_{|\mu|},$$

and

$$\rho = [1, 6][2, 7][3, 8][4, 9][5, 10] \in N_{\tilde{S}_{10}}(D) - N_{\tilde{A}_{10}}(D).$$

In order to define the block idempotents of the cores, we need one further permutation

$$\eta = [11, 12, 13] \in \tilde{S}_{|\nu|}.$$

The block idempotents of the defect zero blocks of the cores are then

$$\bar{c} = (1/2)(() - \sigma^2),$$

and

$$\bar{b} = \bar{c}((1/3)(2() - \eta - \eta^2).$$

The four groups are $\tilde{S}_{|\nu|} \tilde{\to} Q_6$*,* $\tilde{A}_{|\nu|} \tilde{\to} C_6$*,* $\tilde{S}_{|\mu|} \tilde{\to} C_4$*, and* $\tilde{A}_{|\mu|} \tilde{\to} C_2$*. The relevant irreducibles are given by* $\bar{b}$ *are one of degree 2 in* G *and two of degree one in* H*. The irreducibles cut out by* $\bar{c}$ *are two of degree 1 in* $\hat{G}$ *and one of degree 1 in* $\hat{H}$*.*

To count the irreducible modules as in Lemma 6.1, we must analyze $\tilde{S}_\nu \bar{b}\bar{c}$ *as an* RL_iR_i*-module for* $i = 1, \ldots, 4$. *The block algebra* $\tilde{S}_{|\nu|}\bar{b}\bar{c}$ *is a matrix block of dimension 4, and thus as an* L_1*-bimodule it is a sum of one copy each of the two distinct projective bimodules. In* L_2 *it restricts to two copies of the unique projective, in* L_3 *we get one copy each of all four projectives, and in* L_4 *there are two copies of each of the two projectives. In every case all projectives occur. Except for* L_2*, where there is is a unique projective, the effect of considering the* R_i*-action is to pair the projectives, creating indecomposable projective* RL_iR_i*-modules. Again, every indecomposable projective occurs. As predicted by Lemma 6.1, the total number of indecomposable projective* RL_iR_i*-modules is* 1 *for* $i = 1$ *and* 2 *for* $i = 2, 3, 4$.

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