MAXIMAL STRINGS IN THE CRYSTAL GRAPH OF SPIN REPRESENTATIONS OF THE SYMMETRIC AND ALTERNATING GROUPS

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Abstract. We define a block-reduced version of the crystal graph of spin representations of the symmetric and alternating groups, and separate it into layers, each obtained by translating the previous layer and, possibly, adding new defect zero blocks. We demonstrate that each layer has weight-preserving central symmetry, and study the sequence of weights occurring in the maximal strings.

The Broué conjecture, that a block with abelian defect group is derived equivalent to its Brauer correspondent, has been proven for blocks of cyclic defect group and verified for many other blocks. This paper is part of a study of the spin block case.

1. Introduction

Let G be a finite group and let k be a field of characteristic p, where p divides $|G|$. Assume that k is sufficiently large that it is a splitting field for all relevant finite groups. Let $kG = \oplus B_i$ be a decomposition of the group algebra into blocks, and let D_i be the defect group of the block B_i, of order p^{d_i}. By Brauer's Main Theorems, [A], there is a one-to-one correspondence between blocks of kG with defect group D_i and blocks of $kN_G(D_i)$ with defect group D_i. Let b_i be the block corresponding to B_i, called its Brauer correspondent.

Broué [B2] has conjectured that if D_i is abelian and B_i is a principal block, then B_i and b_i are derived equivalent, i.e., the bounded derived categories $D^b(B_i)$ and $D^b(b_i)$ are equivalent. In fact, it is generally believed by researchers in the field that the hypothesis t hat B_i be principal is unnecessary.

In the case of the symmetric groups, the attempt to prove Broué's conjecture led Rouquier and Chuang to a much stronger result. The p-blocks of the symmetric groups are determined by a partition ρ called the p-core and by a non-negative integer w called the weight. They showed that all blocks of a fixed weight are derived equivalent. Together with a result by Chuang and Kessar, showing that a certain block is Morita equivalent to the Brauer correspondent, this proved the Broué conjecture for the symmetric groups, and the result was extended to the alternating groups. Central to the Chuang-Rouquier method, called sl_2-categorization, was the use of Lie group methods, including reflection functors.

The symmetric and alternating groups have central extensions $\tilde{S}_n$ and $\tilde{A}_n$ with kernel C_2, the cyclic group of order 2, which we will loosely refer to as covering groups (although, for a few small values of n, the kernel of the extension is not contained in the commutator of the group). We assume henceforward that p is an odd prime, in which case all blocks can be divided into ordinary blocks, which are simply blocks of S_n or A_n or *spin blocks*, whose characters all take the value -1 on the non-trivial element of the center.

The covering groups of the alternating group are unique. The covering groups of the symmetric groups come in two versions, but the versions have isomorphic spin blocks so we will simply assume one consistent choice.

The irreducible representations of the symmetric group over a field of characteristic 0 correspond to partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$, with $\lambda_1 \geq \cdots \geq \lambda_r$, and we use the notation $|\lambda|$ to denote the sum of the parts. Over a field of prime characteristic p, the irreducible representations are grouped into blocks, each determined by a p-core ρ and a weight w. We will not review this standard material here.

The roles of $\tilde{S}_n$ and $\tilde{A}_n$ are much more symmetrical in the spin case then they are for the ordinary representations of the symmetric and alternating groups. Over a field of characteristic zero, the irreducible spin representations of both $\tilde{S}_n$ and $\tilde{A}_n$ correspond to partitions, but now they are strict partitions, containing no repeated parts, not even the part 1. Whereas in the symmetric case there was a one-to-one correspondence of partitions and irreducible representations, in the spin case the matter is determined by the parity of the partition. Let $n(\lambda)$ denote the number of part in the partition λ, and recall that $|\lambda|$ denotes the sum of the parts. Each strict partition λ of n has a parity

$$\varepsilon = \varepsilon(\lambda) \equiv |\lambda| - n(\lambda) \pmod 2.$$

The partition is called even if $\varepsilon = 0$ and odd if $\varepsilon = 1$. Over a field of characteristic 0, an even strict partition labels two conjugate irreducible representations of $\tilde{A}_n$ and one of $\tilde{S}_n$. An odd strict partition labels one irreducible representation of $\tilde{A}_n$ and two of $\tilde{S}_n$.

As with blocks of the symmetric groups, the spin blocks are determined by a non-negative integer, the weight w, and by a partition ρ called the p-bar core. However, the p-bar cores must be strict partitions, cannot contain any parts divisible by p, and cannot contain parts congruent to i and to $p-i$ for any i satisfying $1 \leq i \leq p-1$. Any pair (ρ, w) for $w > 0$ determines exactly one block of $\tilde{S}_n$ and one block of $\tilde{A}_n$. In the sequel, we will denote the pair (ρ, w) by ρ^w.

Returning temporarily to the symmetric group case, the irreducible modules for all the blocks of all the symmetric groups can be arranged into a labeled graph called the *crystal graph*, with the edges connecting irreducible modules of S_n to simple modules of S_{n+1} labeled by the

residues modulo p. A *maximal string* is a maximal connected sequence of simple modules joined by edges with the same label. The reflection functors of Chuang-Rouquier reflect the maximal strings around their midpoint, with the reflection preserving the weights of the blocks to which the simples belong. Furthermore, it was shown by Scopes [Sc] that the extremal points of the maximal strings are actually Morita equivalent. A Lie theoretic version of this result, due to Brundan and Kleshchev, can be found in Section 11 of [Kl].

The obstruction to making an immediate generalization of the Chuang-Rouquier result to the spin blocks of $\tilde{S}_n$ lies in the fact that blocks which, by the combinatorics of the reflection functors, should seemingly be derived equivalent, do not always have the same number of simple modules. Since the number of simples is invariant under derived equivalence, this meant that they could not in fact be in the same derived equivalence class.

Similarly, the generalization by Kessar of the Scopes result to the covering groups could not deal with all maximal strings, because if the cores at the two ends of a string had different parities, the number of simples in the corresponding blocks of $\tilde{S}_n$ differed by a factor of 2, and thus there can be no Morita equivalence.

The solution to this dilemma is the following, inspired by the results of [KS]:

Crossover Conjecture. (Kessar-Schaps) If p is an odd prime, then among all the spin blocks of $k\tilde{S}_n$ and $k\tilde{A}_n$, there are exactly two derived equivalence classes for each weight $w > 0$, and for each p-bar core there is exactly one block of weight w in each equivalence class. The extremal points of the maximal strings in the crystal graph correspond to Morita equivalent blocks, making the appropriate crossover from $k\tilde{S}_n$ to $k\tilde{A}_n$ if the parities differ.

In Section 2 we define the block-reduced crystal graph and give some of its properties. In Section 3 we prove that an edge of label i between two blocks in the block-reduced crystal graph implies the existence of an edge with that label in the actual crystal graph for every irreducible module in the in the block of smaller weight. A natural geometric realization of the graph is given in Section 4. A recursive construction for the maximal strings in given in Section 5.

2. The block-reduced crystal graph

We now define a version of the crystal graph of [LT] in which the vertices are not labels of irreducible modules but labels of blocks, each block being represented in the form ρ^w for some p-bar-core ρ. In the case $w = 0$, the corresponding label will correspond to two conjugate blocks of $\tilde{A}_n$ if ρ is of even parity, and two "associate" blocks of $\tilde{S}_n$ if ρ is of odd parity.

Recalling that p is odd, we set $t = (p-1)/2$.

Definition 2.1. The diagram of a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a set of r rows of boxes, with λ_i boxes in row i. Each row will be filled in, as far as possible, by repetitions of the sequence $0, 1, \ldots, t-1, t, t-1, \ldots, 1, 0$. The *content* $\gamma(\lambda)$ is the $(t+1)$-tuple $\gamma = (\gamma_0, \ldots, \gamma_t)$, where γ_i is the number of boxes containing the integer i.

Since we will be dealing largely with blocks, we need a representation for the p-bar cores which will be suited to our purposes. One standard way of representing strict partitions is on an abacus with p runners, numbered by the residues modulo p of the various parts. A part equal to $ap + i$, with $0 \leq i \leq p-1$, $a \geq 0$, will be represented by a bead in position a on runner i.

The process of reducing a partition λ to its p-bar-core consists of the removal of p-bars. This is done either by removing p from some part λ_j, provided that $\lambda_j - p$ is not a part of λ, or by removing two complementary parts equal to i and $p - i$ for some $0 \leq i \leq p-1$. In terms of the abacus notations, these possible moves correspond to moving one bead down into an empty space, removing the bottom bead on the 0-runner, or removing the bottom beads on two complementary runners. The effect any of these moves on the content $\gamma(\lambda)$ is always to subtract off a copy of the $t+1$-tuple $\delta = (2, 2, 2, ..., 2, 1)$. After w moves of these kinds have been made, the new content will be $\gamma(\lambda) - w\delta$.

A p-bar core is a strict partition from which no p-bars can be removed. In abacus notation it has no gaps on any runner, no beads on the 0-runner, and beads on only one of two complementary runners. Thus we can define a condensed numerical notation appropriate for representing p-bar cores:

Definition 2.2. Given a p-bar core $\rho = (\rho_1, \ldots, \rho_r)$, the *core t-tuple* is

$$c(\rho) = ((\ell_1, \varepsilon_1), \ldots, (\ell_t, \varepsilon_t)),$$

where the *length* ℓ_i is the number of parts of ρ congruent to i or $p - i$ modulo p, and ε_i is 0 if there is at least one part congruent to i, and 1 otherwise. We will also write $\rho = \rho(c)$ when $c = c(\rho)$.

Definition 2.3. We will extend the exponential notation ρ^w to the core t-tuples, so that if $c = c(\rho)$, then c^w will also be used to represent the block ρ^w. Furthermore, we set

$$|c^w| = |\rho| + wp,$$

this being the rank n of the symmetric (or alternating) group containing the block labeled by c^w. We further define the *layer number* of the block ρ^w or c^w to be

$$L(c^w) = \gamma_t(\rho(c)) + w.$$

We also define the *length composition* of the core c to be the composition given by

$$\alpha(c) = (\ell_1 + \varepsilon_1, \ldots, \ell_t + \varepsilon_t).$$

The *normalized* length composition $\beta(c)$ is the composition obtained by rearranging the elements of $\alpha(c)$ in ascending order.

Lemma 2.1. *Let c be the core t-tuple of a core ρ.*

(1) *The layer number of a p-core depends only on the orbit of its length composition under permutation by elements of S_t, and equals*

$$L(c) = \sum_{i=1}^{t} (\ell_i + \varepsilon_i - 1)(\ell_i + \varepsilon_i)/2.$$

(2) *For any block c^w, there are $2^t t!/|St_{S_t}(\alpha(c))|$ blocks with the same weight and normalized length composition, all with the same layer number.*

(3) *$|c^w| = (L(c) + w)p + ((-1)^{\varepsilon_1}\ell_1, \ldots, (-1)^{\varepsilon_t}\ell_t) \cdot (1, 2, \ldots, t)$, where the dot represents the scalar product.*

Proof.

(1) The layer number of a core t-tuple c depends only on the t coordinate of the content $\gamma(\rho)$), where $c = c(\rho)$. These numbers can be calculated separately for the parts on each runner and then summed together. There are two cases:

Case 1. $\varepsilon_i = 0$: Every part on the i runner is of the form $ap + i$. This contributes a copies of t, one for each copy of p. As there are ℓ_i parts of this form, corresponding to $a = 0, \ldots, \ell_i - 1$, we get the sum of these values of a, which is $(\ell_i - 1)(\ell_i)/2$. Adding the 0-valued ε_i gives the formula in the lemma.

Case 2. $\varepsilon_i = 1$: Every part on the $p - i$-runner is of the form $(a+1)p - i$. the number of copies of t in the row corresponding to the part $(a + 1)p - i$ is $(a + 1)$, since $i < t$ and thus the i boxes removed from the end of $(a+1)p$ do not contain copies of t. Thus $\gamma_t(\rho)$ contains the sum of the numbers $a + 1 = 1, \ldots, \ell_i$, which is $(\ell_i)(\ell_i + 1)/2 = (\ell_i + \varepsilon_i - 1)(\ell_i + \varepsilon_i)/2$.

(2) Since the formula for the layer number of a core deduced in the first part of the proof does not depend on the values of the ε_i, we get 2^t possibilities for every length composition α. Since the number of length compositions in the orbit of $\alpha(c)$ under the action of S_t is $|S_t|/|St(\alpha(c))|$, the required formula follows for the cores. However, raising each of these cores to the exponent w simply adds w to the layer number.

(3) The formula for the rank $|c|$ comes from representing the parts in the form $(ap + (-1)^{\varepsilon_i} i)$ and summing, with a running from ε_i to $\ell_i + \varepsilon_i - 1$.

□

Our aim in this section is to define the block-reduced crystal graph, but let us first review the definition of the crystal graph [LT] of irreducible modules for the covering groups of the symmetric and alternating groups. The vertices represent irreducible modules over the field k of characteristic p, which we have assumed large enough to be a splitting field for all the groups involved. Brundan and Kleshchev [BK] have shown that these irreducible modules can be labeled by partitions from the set of partitions which have distinct parts except for multiples of p

$$DP_p = \{\lambda = (\lambda_1, \ldots, \lambda_r) | \lambda_i = \lambda_{i+1} \to p | \lambda\}.$$

In order to get a single label for each irreducible module, we make a further set of restrictions:

(1) $\lambda_i - \lambda_{i+1} \leq p$
(2) $\lambda_i - \lambda_{i+1} = p$ implies that p does not divide λ_i.

Partitions in DP_p satisfying these additional conditions will be called p-restricted, and the set of such partitions will be denoted by RP_p.

Definition 2.4. Consider i in $I = \{0, \ldots, \mathrm{t}\}$. The *$i$-signature* of a partition λ is determined by the following rules, applied to the Young diagram of the partition with the boxes filled by residues:

(1) A "+" is added to the Young diagram wherever a box A labeled i could be added to produce a new Young diagram of an element λ^A of RP_p.
(2) A "-" is added to the Young diagram in any box B which could be removed to produce a new Young diagram of an element λ_B of RP_p.
(3) In the special case $i = 0$, where two boxes of the same residue can appear side by side, we can get a double plus or double minus provided both operations, one after the other, produce valid partitions.

The list of pluses and minuses, read from the left bottom to the top right is the signature of the partition. We then remove all instances of "+-", to get the reduced signature, which is of the form "$- - \cdots - + \cdots + +$", where either the list of pluses or of minuses might be empty. It has been shown that this is well-defined; the order in which the "+-" are removed does not affect the final result. The box corresponding to the leftmost "+", if any exist, is called *i-good* and the box corresponding to the rightmost "-", if any exist, is called *i-cogood*.

The following definition is essentially the definition given in [LT]:

Definition 2.5. The *crystal graph* for $\tilde{S}_n$ or $\tilde{A}_n$ has as vertices the elements of RP_p. A vertex labeled λ with $|\lambda| = n$ is connected by an edge labeled by i to a vertex μ with $|\mu| = n+1$ if there exists an i-good box A such that $\mu = \lambda^A$, or, equivalently, there is an i-cogood box B of μ such that $\lambda = \mu_B$.

We now modify this definition so that the vertices will be, not labels of irreducible modules, but labels of p-blocks.

Definition 2.6. The *block-reduced crystal graph* is a graph with vertices labeled by pairs ρ^w consisting of a p-bar core ρ and a non-negative integer weight w, i.e., labels of p-blocks. A block ρ^w will be connected to a block σ^v by an edge labeled i if some irreducible of the block ρ^w is connected to some irreducible of the block σ^v in the ordinary crystal graph by an edge labeled i, which can only happen if $|\rho| + wp$ differs from $|\sigma| + vp$ by 1.

Remark 2.1. If the partition λ is a core, then the signature will consist entirely of pluses or of minuses and is already reduced. It may be empty. If $c = c(\lambda)$ is the core t-tuple, then in each of three cases we define a number d:

(1) i=t: $d = (-1)^{\varepsilon_t}\ell_t$,
(2) $i = 0$: $d = ((-1)^{1-\varepsilon_1}2\ell_1 - (-1)^{1-\varepsilon_1}$,
(3) $0 < i < t$: $d = (-1)^{\varepsilon_i}\ell_i - (-1)^{\varepsilon_{i+1}}\ell_{i+1}$.

The reduced signature of the core consists of $|d|$ copies of $sgn(d)$. Since the length of the reduced signature will be seen soon to equal the length of a maximal string, these formulae will be justified in Theorem 5.2 below.

Example 1. Consider $p = 7$, $c = ((2,0),(2,0),(3,1))$. The corresponding partition λ equals $(18,11,9,8,4,2,1)$. We have the following signatures:

(1) $i = 0$ $- - -$,
(2) $i = 1$ $\emptyset$,
(3) $i = 2$ $+ + + + +$,
(4) $i = 3$ $- - -$.

Consider the following two values for i:

i=0:

0	1	2	3	2	1	0	0	1	2	3	2	1	0	0	1	2	3
0	1	2	3	2	1	0	0	1	2	3							
0	1	2	3	2	1	0	0	1									
0	1	2	3	2	1	–	–										
0	1	2	3														
0	1																
–																	

i=2:

0	1	2	3	2	1	0	0	1	2	3	2	1	0	0	1	2	3	+
0	1	2	3	2	1	0	0	1	2	3	+							
0	1	2	3	2	1	0	0	1										
0	1	2	3	2	1	0	0											
0	1	2	3	+														
0	1																	
0																		

3. The Scopes involutions

We now define the so-called Scopes involutions. These were defined by Scopes [Sc] in the symmetric group case, and generalized to the spin case by Kessar in [K]. In order to achieve compatibility with the labeling of the edges in the crystal graph, we will change the notations slightly. We let RP_p^+ represent the subset of RP_p consisting of element containing the part 1 and RP_p^- be the set not containing the part 1.

Definition 3.1. For $0 < i \leq t$, the *Scopes involution* $K_i : RP_p \to RP_p$ will interchange the parts congruent to i and $i+1$ and also interchange the parts congruent to the complements $p-i$ and $p-i-1$. For $i = 0$, we have an involution $K_0 : RP_p \to RP_p$ which interchanges Rp_p^+ and RP_p^-, sending the part $ap-1$ to $ap+1$, sending $ap+1$ to $ap-1$, and adding a part 1 is it was not previously present, or removing it, if it was present.

Remark 3.1. For $i > 0$, the mapping K_i is approximately the same as $\tilde{S}c_{i+1}$ of [K]. The definition of K_0 is different.

Remark 3.2. On cores, the Scopes involution K_i corresponds to adding (or removing) all the boxes in the Young diagram of the partition which correspond to the pluses (or minuses) of the signature. The Scopes involutions have simple descriptions in terms of the core t-tuple. For $i \neq 0, t$, we simply switch (ℓ_i, ε_i) and $(\ell_{i+1}, \varepsilon_{i+1})$, while for $i = t$ we reverse the value of ε_t, and for $i = 0$, we both reverse the value of ε_i and we replace ℓ_0 with $\ell_0 - (-1)^{\varepsilon_0}$. Thus for the example given at the end of the last section, with $\lambda = (18, 11, 9, 8.4, 2, 1)$ and $c = ((2,0),(2,0),(3,1))$ we have:

(1) $K_0(\lambda) = (18, 11, 9, 6, 4, 2)$, $c_0 = ((1,1),(2,0),(3,1))$,
(2) $K_1(\lambda) = \lambda$, $c_1 = c$,
(3) $K_2(\lambda) = (19, 12, 10, 8, 4, 3, 1)$, $c_2 = ((2,0),(3,1),(2,0))$,
(4) $K_3(\lambda) = (17, 10, 9, 8, 3, 2, 1)$, $c_3 = ((2,0),(2,0),(3,0))$.

4. Geometric realizations of the block-reduced crystal graph

We get a geometric realization of the block-reduced crystal graph in $\mathbf{R^{t+1}}$ by locating the vertex labeled by ρ^w at the point $\gamma(\rho^w)$ given by

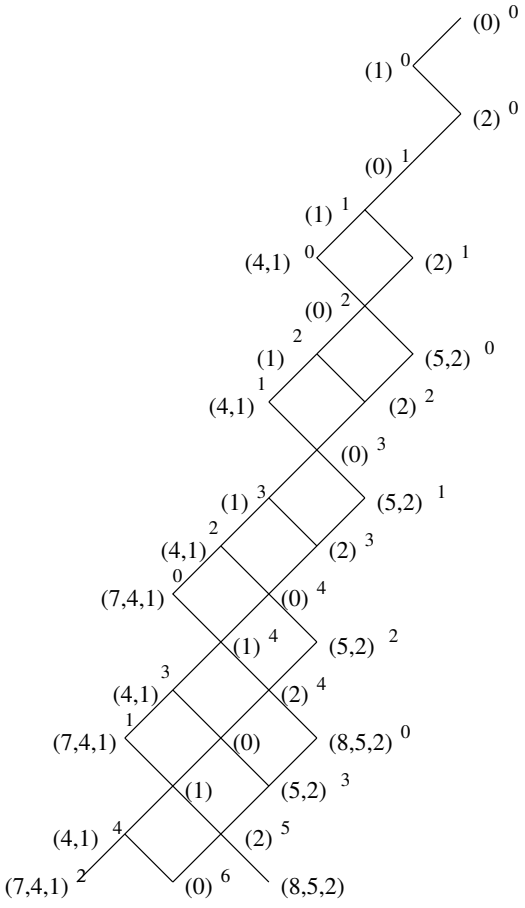

FIGURE 1

its content. Letting the variables be $x_0, ..., x_t$, we let an edge labeled by i be represented by a line of length one parallel to the x_i axis. The layer number of a block is then simply its t-coordinate. In Fig. 1, we give the geometric realization of the block-reduced crystal graph for $p = 3$. In Fig. 2, we give the geometric realizations for $p = 5$, in a three-dimensional representation. To make the three-dimensional representation easier to view, we have drawn the positive γ_t axis going down rather than up. Within each layer, the edges labeled 0 go from the upper right to the bottom left, and the edges labeled 1 from the upper left to the bottom right. All blocks in a horizontal line on the two-dimensional representation, in both figures, have the same rank, i.e., represent blocks in the same $\tilde{S}_n$ or the same $\tilde{A}_n$. Each layer has central symmetry determined by an involution called the flip involution, which will be defined shortly.

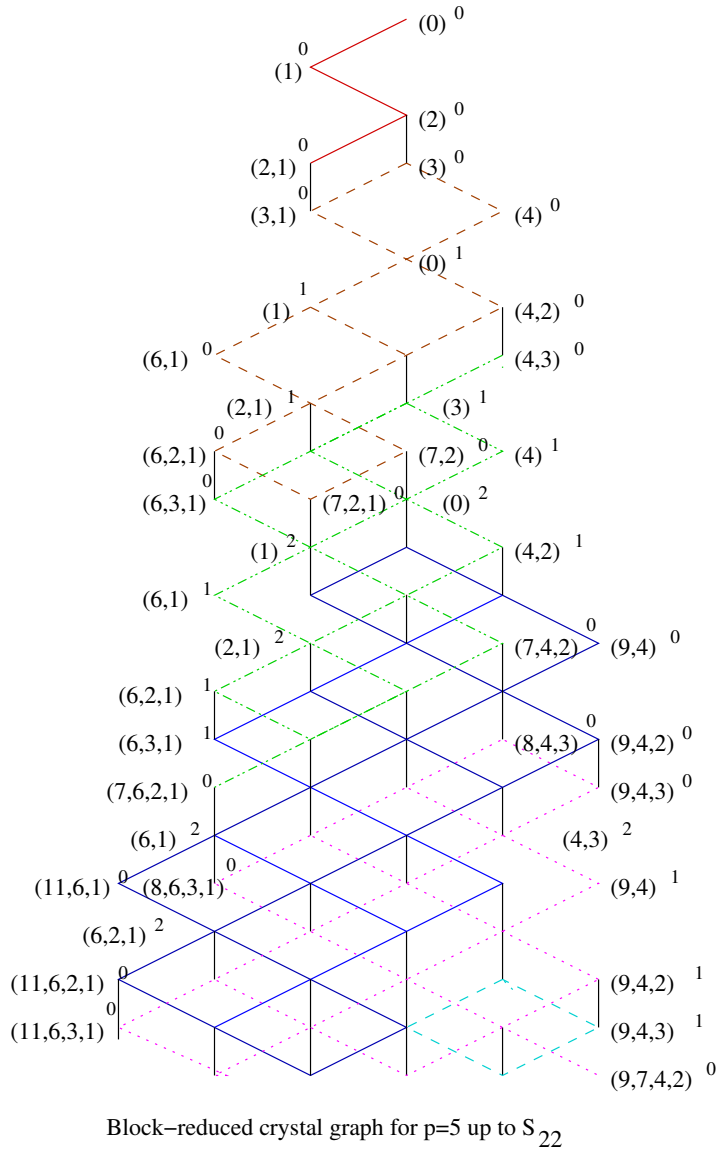

FIGURE 2

Note that each layer contains a translated copies of the previous layer. For the vertices this is automatic from the definitions. In the next section we will prove that this is true of the edges as well.

Definition 4.1. The *flip involution* of the block-reduced crystal graph is an involution τ which is most easily described in terms of the core t-tuples. For $c = ((\ell_1, \varepsilon_1), \ldots, (\ell_t, \varepsilon_t))$, we define $\tau(c) = ((\ell'_1, \varepsilon'_1), \ldots, (\ell'_t, \varepsilon'_t))$, where

$$\ell'_i = \ell_i - (-1)^{\varepsilon_i}; \varepsilon'_i = 1 - \varepsilon_i.$$

This is clearly an involution on cores, and can be extended to an involution of the entire block-reduced crystal graph by setting $\tau(c^w) = \tau(c)^w$.

Lemma 4.1. *For any normalized length composition $(m_1, \ldots, m_t)$, the corresponding cores with minimal and maximal rank are*

$$c^- = ((m_1 - 1, 1), \ldots, (m_t - 1, 1);$$

$$c^+ = \tau(c^-) = ((m_1, 0), \ldots, (m_t, 0)).$$

Any other core t-tuple c with the same normalized length composition can be obtained from c^+ (resp. c^-) by a sequence of Scopes involutions K_i which decrease (resp., increase) the rank.

Proof. We will describe the algorithm for getting from c^+ to c by rank decreasing involutions. The dual sequence of operations will go from c^- to $\tau(c)$ but will increase the rank at each K_i. As an aid to the reader, we will carry along an example for $p = 11$. Suppose that $c = ((3,0),(1,1),(4,1),(2,0),(1,0))$. The corresponding $c^+ = ((1,0),(2,0),(2,0),(3,0),(5,0))$ has rank 213.

The algorithm has four stages:

Step 1: Unshuffling. During this stage all ε are 0. The involution K_i for $1 \leq i \leq t$ is rank decreasing whenever $\ell_i + \varepsilon_i < \ell_{i+1} + \varepsilon_{i+1}$. Because the ℓ_i are non-decreasing as i increases, we can choose any minimal length product of the generators $s_i = (i, i+1)$ of S_t which will "unshuffle" $\alpha(c^+)$ so that all the $\ell_i + \varepsilon_i$ which will have $\varepsilon_i = 0$ in c will be in places $1, \ldots, s$, still in non-decreasing order, and the remaining elements, also in non-decreasing order, would be in places $s+1, \ldots, t$. In the example above, the result of this first step would be

$$((2,0),(5,0),(1,0),(2,0),(3,0)).$$

The new rank is 203.

Turning the corner. The pairs destined for $\varepsilon_i = 1$ are brought around the corner one after the other using: K_0, $K_0 \cdot K_1$, etc. At this point those with second term 0 are increasing and the others are decreasing. In the example, this would give

$$((4,1),(1,1),(1,0),(2,0),(3,0)).$$

The rank is now 185.

Reordering Each of the two sets should be separately reordered into the order they have in c. This can be done while decreasing the rank by representing each of the permutations by a minimal length product of the appropriate generators s_i.

$$((1,1),(4,1),(3,0),(2,0),(1,0)).$$

The rank is now 178.

Shuffling Intersperse the pairs with second term 1 among those of second term 0. All such permutation will be rank-decreasing: if $(\ell, 1)$ is in position i and $(\ell', 0)$ is in position $i+1$, then K_i will shift the ℓ'

beads on runner $i+1$ to the lower runner i, and the ℓ beads on runner $p-i$ to the lower runner $p-i-1$. In the example, we now get

$$c = ((3,0),(1,1),(4,1),(2,0),(1,0)).$$

The rank of c is 167. □

5. The combinatorics of strings

We are about to consider the relationship between the block-reduced crystal graph and the crystal graph. All the blocks of a given p-weight have the same number of labels of irreducible modules, the generating formula for the exact number being given by raising the generating function of the partitions to the power t, [Bo]. For the first three odd primes the sequences begin as follows, starting with $w = 0$:

(1) $p = 3$: 1, 1, 2, 3, 5, ...
(2) $p = 5$: 1, 2, 5, 10, ...,
(3) $p = 7$: 1, 3, 9, ...

The actual number of simples is either equal to the number of labels or is twice as large. In particular, for blocks of weight 1 it is either t or $2t = p-1$. Thus an edge in the block-reduced crystal graph, which connects two blocks of different weights, is also connecting two sets of different sizes. Our theorem will take this into account by considering the passage from the smaller weight to the larger.

Theorem 5.1. *Let ρ^w and σ^v be blocks such that $w \leq v$ and the ranks $|\rho^w|$ and $|\sigma^v|$ differ by 1. If $\gamma(\rho^w) - \gamma(\sigma^v)$ is either plus or minus the i-th basis vector e_i of $\mathbf{R^t}$, then in the crystal graph every label corresponding to the block ρ^w is connected by an i-labeled edge to an irreducible in σ^v.*

Proof. As shown in [Kl], the content of a spin block corresponds to a central character of a certain degenerate affine Hecke algebra, and thus there is a unique block for a given content. Thus, it suffices to show that any partition λ in RP_p corresponding to the block ρ^w has an i-good (or i-cogood) node, for the result of adding (or removing) the corresponding box will necessarily be in σ^v. We will give the proof for the case that $|\sigma^v| = |\rho^w| + 1$ and we need an i-good node, the other case being dual. Translating by the weight w, and letting e_i be the standard basis vector with 1 in position i and 0 elsewhere, we have that

$$\gamma(\rho) + e_i = \gamma(\sigma^{v-w}).$$

We now divide into the standard three cases:

Case 1. $i = t$: The t-runner of ρ is non-empty. Let ℓ_t be the number of beads on that runner. The lowest bead is the i-good bead, and the change from ρ to σ^{v-w} is obtained by moving that lowest bead to the $t+1$ runner. Now let λ be any partition in RP_p labeling irreducibles from the block ρ^w. The number of beads on the t-runner must be ℓ_t more than the number of beads on the $t+1$ runner, since moving beads

on a runner does not change the number, and adding p-bars does not change the difference. Beads on the t runner produce a "+" and beads on the $t+1$ runner produce a minus, except when a pair of adjacent beads cancel out one "+" and one "-". Thus, after canceling out pairs of "+-" from the signature to arrive at the reduced signature, there must be at least one "+" left, and therefore there must be an i-good node.

Case 2. $i = 0$: In this case, there are ℓ_1 beads on runner $p-1$, and σ is obtained from ρ by adding a bead to runner 1 and reducing if necessary. If λ is the chosen partition of ρ^w, there will be ℓ_1 more beads on the $p-1$-runner than on the 1 runner. If there are no adjacent beads on runners $p-1$, 0 and 1 (where, for example, $ap-1$ and ap are adjacent, and also ap and $ap+1$ are adjacent) then every bead on runner i would produce a "+" in the signature, every bead on runner 0 would produce a "+" and a "-" and every bead on runner 1 would produce a "-". Where there are adjacent beads, one "+" and one "-" do not appear in the signature. The total effect is that there are ℓ_1 more copies of "+" in the reduced signature than of "-", and thus an i-good node exists. In the special case of $\ell_1 = 0$, there is still one plus in the signature for adding a part equal to 1 at the bottom.

Case 3. $0 < i < t$: Here there are four runners involved, corresponding to the residues $i, i+1, p-i-1$, and $p-i$. The total number of beads on the "low" side is larger than the total number of beads on the "high" side, and thus the number of "+" signs in the reduced signature is positive, guaranteeing an i-good node. □

Corollary 5.1.1. *If ρ^0 is connected to σ^s by an edge labeled i, then every ρ^w is connected to σ^{s+w} by an edge i.*

Proof. By the proof of the theorem above, every λ in the block ρ^w has an i-good node, which necessarily sends it to a partition in σ^{s+w}, the unique block with content $\gamma(\rho^w) + e_i$. □

We now consider the internal points of the maximal strings. Let us begin with a core ρ and a Scopes involution K_i such that the resulting core $\tau = K_i(\rho)$ satisfies

$$|\rho| \neq |\tau|.$$

Although our goal is to give a complete combinatorial description of the entire i-string from ρ^w to τ^w for any non-negative integer w, the main result concerns the first step of the string.

Theorem 5.2. *Let ρ be a p-bar core, with $c(\rho) = ((\ell_1, \varepsilon_1), \ldots, (\ell_t, \varepsilon_t))$, and let K_i be a Scopes involution such that $|\rho| \neq |K_i(\rho)|$. Then the result of making the change corresponding to the i-good (or i-cogood) node is a partition μ in the block σ^s, with $c' = c(\sigma) = ((\ell'_1, \varepsilon'_1), \ldots, (\ell'_t, \varepsilon'_t))$, first step in a string whose length is the absolute value of d given as follow:*

(1) *$i = t$: $d = (-1)^{\varepsilon_t}\ell_t$; if $\ell_t = 1$, then $s = 0$, $\varepsilon'_t = 1 - \varepsilon_t$, $\ell'_t = \ell_t$; if $\ell_t > 1$, then $s = 2\ell_t - 2$, $\ell'_t = \ell_t - 2$;*
(2) *i=0: $d = ((-1)^{1-\varepsilon_1}2\ell_1 - (-1)^{\varepsilon_1}$; if $\ell_i > 0, s = \ell_1 + \varepsilon_1 - 1$, $\ell'_1 = \ell_1 - 1$; if $\ell_i = 0, s = 0, \ell'_1 = \ell_1 + 1$, $\varepsilon'_1 = 0$.*
(3) *$0 < i < t$: $d = (-1)^{\varepsilon_i}\ell_i - (-1)^{\varepsilon_i+1}\ell_{i+1}$; if $d = 1$, then $s = 0$; if $d > 1$, then $s = |d| - 1$; if $\varepsilon_i \neq \varepsilon_{i+1}, \ell'_i = \max(l_i - 1, 0), \ell'_{i+1} = \max(\ell_{i+1} - 1, 0)$; if $\varepsilon_i = \varepsilon_{i+1}$, then $\max(\ell_i, \ell_{i+1})$ is reduced by 2.*

Proof. We will do the proof for the direction of increasing rank, with the opposite direction being exactly dual.

(1) *$i = t$:* If K_t increases the rank, then $\varepsilon_t = 0$. There are ℓ_t "+" signs in the signature, corresponding to all the parts congruent to "t", and thus the string is of positive length $d = \ell_t$, that being the number of parts to be increased. The t-good node is at the end of the lowest row of length congruent to t.

    The simplest case is when $\ell_t = 1$. Then the i-good move, moving the unique bead on runner t to runner $t + 1$, is the entire Scopes involution. The new partition is also a core, so $s = 0$. Finally, $\ell'_t = \ell_t$ and ε_t is now 1 instead of 0.

    If $\ell_t > 1$, then moving the bottom bead to the $t + 1$-runner does not produce a core, but instead provokes a cascade of moves. All the $\ell_t - 1$ bead above drop down. Then the bead that was moved over cancels the bottommost bead, another move, and finally, the remaining beads on the t-runner drop again, for a total of $s = 2(\ell_t - 1)$ moves. In the core of the new partition, $\ell'_t = \ell_t - 2$ because of the double drop. The position ε_t remains equal to 0, unless $\ell'_t = 0$, in which case, by convention, we change ε_t to 1.
(2) *i=0:* If K_0 increases the rank, then $\varepsilon_1 = 1$, so the 1-runner is empty. In the 0-signature of ρ, there is one "+" for adding a part 1, which is 0-good, and every other row corresponding to a part congruent to $p - 1$ ends in 2 "+"-signs, for a total signature of length $2\ell_1 + 1$. (If we were dealing with the dual case, when $\varepsilon_1 = 0$,the length of the signature would be $2\ell_1 - 1$, so we combine these into the single formula $2\ell - (-1)^{\varepsilon_1}$).

    If $\ell_1 = 0$, then the unique move is the Scopes involution, so $s = 0$ and $\ell'_1 = 1$, $\varepsilon'_1 = 0$. In the non-trivial case that $\ell_1 > 0$, the result of the 0-good "+" is to create a bead on the 1-runner. This will cancel the bottom bead on the $p - 1$-runner, giving one move to reduce to the p-bar core, and then there will be an additional $\ell_1 - 1$ moves as the remaining beads move down, giving $s = \ell_1$. The new core will have $\ell'_i = \ell_i - 1$.
(3) *$0 < i < t$:* We consider two separate cases: short strings, in which $\varepsilon_i = \varepsilon_{i+1}$ and long strings, with $\varepsilon_i \neq \varepsilon_{i+1}$.

Case 1. $\varepsilon_i = \varepsilon_{i+1}$: Since we have assumed that K_i increases the rank, we must have either $\ell_i > \ell_{i+1}$, if $\varepsilon_i = 0$, or $\ell_i < \ell_{i+1}$, if $\varepsilon_i = 1$. We combine the two situations into one formula by writing $(-1)^{\varepsilon_i}(\ell_i - \ell_{i+1}) > 0$. We have $|\ell_i - \ell_{i+1}|$ i-normal nodes. The i good node moves the first possible bead from one runner to the adjacent runner, leaving $s = |\ell_i - \ell_{i+1}| - 1$ beads to drop down. The larger of ℓ_i and ℓ_{i+1} is reduced by 1 in the new core, and the smaller is increased by 1.

Case 2. $\varepsilon_i \neq \varepsilon_{i+1}$: In this case, there are $|\ell_i + \ell_{i+1}|$ copies of "+" in the signature, and this will be the length of the string. When the lowest of these moves over, it cancels a bead from the complementary runner if such exists. All the beads on both runners drop one place, giving a total $s = \ell_i + \ell_{i+1} - 1$ moves to reduce to the core. Each non-zero runner is reduced by one. Thus $\ell'_j = \max(\ell_j - 1, 0)$ for $j = i, i+1$.

□

Corollary 5.2.1. *The maximal strings in the block-reduced crystal graph are symmetrical, with the weights increasing toward the center and the successive differences decreasing.*

Proof. The duality in the definition of each step in the theorem shows that the strings are constructed symmetrically, with the ends of lower weight than the s-translated central part, each maximal i-string corresponding to an involution K_i. The difference between each weight and that of the block closer to the center is a linear function of ℓ_i and possibly ℓ_{i+1}, and thus the differences decrease as one approaches the center of the string. □

References

[A] J. Alperin, *Local Representation Theory*, Vol. 11, Cambridge Univ. Press, 1986.

[B1] M. Broué, *Isométries parfaites, types de blocs, catégories dérivées*, Astérisque **181-182** (1990), 61–92.

[B2] M. Broué, *Rickard equivalences and block theory*, Groups '93, Galway-Saint-Andrews Conference, vol. 1; London Math. Soc. Series, vol. 211, Cambridge Univ. Press, 1995, 58–79.

[Bo] Ch. Bessenrodt and J. Olsson, *On Cartan matrices and lower defect groups of the symmetric groups* Trans. Amer. Math. Soc. **358** (2006), 3623-3635.

[C1] J. Chuang, *The derived category of some blocks of symmetric groups and a conjecture of Broué*, J. Algebra **217** (1999), 114–155.

[CK] J. Chuang and R. Kessar, *Symmetric groups, wreath products, Morita equivalence, and Broué's abelian conjecture*, Bull. London Math. Soc., to appear.

[CR] J. Chuang and R. Rouquier, *Derived equivalences for symmetric groups and sl_2 categorifications*, preprint (2004).

[HK] G. Hiss and R. Kessar, *Scopes reduction and Morita equivalence classes of blocks in finite classical groups II*, preprint.

[K] R. Kessar, *Blocks and source algebras for the double covers of the symmetric and alternating groups*, J. Algebra **186** (1996), 872-933.

[Kl] A. Kleshchev, *Linear and Projective Representations of Symmetric Groups*, Cambridge Tracts in Mathematics, Cambridge Univ. Press, 2005.

[KS] R. Kessar and M. Schaps, *Crossover Morita equivalences for blocks of the covering groups of the symmetric and alternating groups*, preprint.

[L] R. Leabovich, *Morita equivalence classes for blocks of abelian defect group* $C_p \times C_p$, Master's thesis, Bar-Ilan University, 2004.

[LT] B. Leclerc and J.-Y. Thibon, *q-deformed Fock spaces and modular representations of spin symmetric groups*, J. Physics. A **30** (1997), 6163-6176.

[MS] J. Mueller and M. Schaps, *The Brauer conjecture for the faithful blocks of* $4.M_{22}$, preprint.

[O] T. Okuyama, *Some examples of derived equivalent blocks of finite groups*, preprint, Hokkaido, 1998.

[R1] J. Rickard, *Morita theory for derived equivalence*, J. London Math. Soc. **39** (1989), 436–456.

[Ro2] R. Rouquier *Block theory via stable and Rickard equivalences*, preprint, Paris, 2000.

[Sc] J. Scopes, *Cartan matrices and Morita equivalence for blocks of the symmetric group*, J. Algebra **142** (1991), 441–455.

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