GENERALIZED IGUSA FUNCTIONS AND IDEAL GROWTH IN NILPOTENT LIE RINGS

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Abstract. We introduce a new class of combinatorially defined rational functions and apply them to deduce explicit formulae for local ideal zeta functions associated to the members of a large class of nilpotent Lie rings which contains the free class-2-nilpotent Lie rings and is stable under direct products. Our results unify and generalize a substantial number of previous computations. We show that the new rational functions, and thus also the local zeta functions under consideration, enjoy a self-reciprocity property, expressed in terms of a functional equation upon inversion of variables. We establish a conjecture of Grunewald, Segal, and Smith on the uniformity of normal zeta functions of finitely generated free class-2-nilpotent groups.

1. Introduction

The objective of this paper is twofold. The first aim is to introduce a new class of combinatorially defined multivariate rational functions and to prove that they satisfy a self-reciprocity property, expressed in terms of a functional equation upon inversion of variables. The second is to apply these rational functions to describe explicitly the local ideal zeta functions associated to a class of combinatorially defined Lie rings. We start with a discussion of the latter application before formulating and explaining the new class of rational functions.

1.1. **Finite uniformity for ideal zeta functions of nilpotent Lie rings.** Given an additively finitely generated ring $\mathcal{L}$, i.e. a finitely generated $\mathbb{Z}$-module with some bi-additive, not necessarily associative multiplication, the ideal zeta function of $\mathcal{L}$ is the Dirichlet generating series

$$\zeta^{\triangleleft}_{\mathcal{L}}(s) = \sum_{I \triangleleft \mathcal{L}} |\mathcal{L} : I|^{-s}, \tag{1.1}$$

where I runs over the (two-sided) ideals of $\mathcal{L}$ of finite additive index in $\mathcal{L}$ and s is a complex variable. Prominent examples of ideal zeta functions include the Dedekind zeta functions, enumerating ideals of rings of integers of algebraic number fields and, in particular, Riemann's zeta function $\zeta(s)$.

It is not hard to verify that, for a general ring $\mathcal{L}$, the ideal zeta function $\zeta^{\triangleleft}_{\mathcal{L}}(s)$ satisfies an Euler product whose factors are indexed by the rational primes:

$$\zeta^{\triangleleft}_{\mathcal{L}}(s) = \prod_{p \text{ prime}} \zeta^{\triangleleft}_{\mathcal{L}(\mathbb{Z}_p)}(s),$$

where, for a prime p,

$$\zeta^{\triangleleft}_{\mathcal{L}(\mathbb{Z}_p)}(s) = \sum_{I \triangleleft \mathcal{L}(\mathbb{Z}_p)} |\mathcal{L}(\mathbb{Z}_p) : I|^{-s}$$

enumerates the ideals of finite index in the completion $\mathcal{L}(\mathbb{Z}_p) := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ or, equivalently, the ideals of finite p-power index in $\mathcal{L}$. Here $\mathbb{Z}_p$ denotes the ring of p-adic integers; note that ideals of $\mathcal{L}(\mathbb{Z}_p)$ are, in particular, $\mathbb{Z}_p$-submodules of $\mathcal{L}(\mathbb{Z}_p)$. It is, in contrast, a deep result that the Euler factors $\zeta^{\triangleleft}_{\mathcal{L}(\mathbb{Z}_p)}(s)$ are rational functions in the parameter p^{-s}; cf. [11, Theorem 3.5].

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Computing these rational functions explicitly for a given ring $\mathcal{L}$ is, in general, a very hard problem. Solving it is usually rewarded by additional insights into combinatorial, arithmetic, or asymptotic aspects of ideal growth. It was shown by du Sautoy and Grunewald [8] that the problem, in general, involves the determination of the numbers of $\mathbb{F}_p$-rational points of finitely many algebraic varieties defined over $\mathbb{Q}$. Only under additional assumptions on $\mathcal{L}$ may one hope that these numbers are given by finitely many polynomial functions in p. We say that the ideal zeta function of $\mathcal{L}$ is *finitely uniform* if there are finitely many rational functions $W_1^{\triangleleft}(X,Y),\dots,W_N^{\triangleleft}(X,Y) \in \mathbb{Q}(X,Y)$ such that for any prime p there exists $i \in \{1,\dots,N\}$ such that

$$\zeta^{\triangleleft}_{\mathcal{L}(\mathbb{Z}_p)}(s) = W_i^{\triangleleft}(p,p^{-s}).$$

If a single rational function suffices (i.e. $N = 1$), we say that the ideal zeta function of $\mathcal{L}$ is *uniform*. While finite uniformity dominates among low-rank examples, including most of those included in the book [9] and those computed by Rossmann's computer algebra package Zeta [22, 23], it is not ubiquitous: for a non-uniform example in rank 9, see [7] and [29]. In general, the ideal zeta function of a direct product of rings is not given by a simple function of the ideal zeta functions of the factors. It is not even clear whether (finite) uniformity of the latter implies (finite) uniformity of the former.

1.1.1. *Main results.* In this paper we give constructive proofs of (finite) uniformity of ideal zeta functions associated to the members of a large class of nilpotent Lie rings of nilpotency class at most 2.

Definition 1.1. Let $\mathfrak{L}$ denote the class of nilpotent Lie rings of nilpotency class at most two which is closed under direct products and contains the following Lie rings:

(1) the free class-2-nilpotent Lie rings $\mathfrak{f}_{2,d}$ on d generators, for $d \geqslant 2$; cf. Section 5.2.
(2) the free class-2-nilpotent products $\mathfrak{g}_{d,d'} = \mathbb{Z}^d * \mathbb{Z}^{d'}$, for $d, d' \geqslant 0$; cf. Section 5.3.
(3) the higher Heisenberg Lie rings $\mathfrak{h}_d$ for $d \geqslant 1$; cf. Section 5.4.

Note that $\mathfrak{L}$ contains the free abelian Lie rings $\mathbb{Z}^d = \mathfrak{g}_{d,0} = \mathfrak{g}_{0,d}$.

Our main "global" result produces explicit formulae for almost all Euler factors of the ideal zeta functions associated to Lie rings obtained from the members of $\mathfrak{L}$ by base extension with general rings of integers of number fields. In particular, we show that these zeta functions are finitely uniform and, more precisely, that the variation of the Euler factors is uniform among unramified primes with the same decomposition behaviour in the relevant number field.

Theorem 1.2. *Let $\mathcal{L}$ be an element of $\mathfrak{L}$, $g \in \mathbb{N}$, and $\mathbf{f} = (f_1,\dots,f_g) \in \mathbb{N}^g$. There exists an explicitly described rational function $W^{\triangleleft}_{\mathcal{L},\mathbf{f}} \in \mathbb{Q}(X,Y)$ such that the following holds:*

Let $\mathcal{O}$ be the ring of integers of a number field and set $\mathcal{L}(\mathcal{O}) = \mathcal{L} \otimes \mathcal{O}$. If a rational prime p factorizes in $\mathcal{O}$ as $p\mathcal{O} = \mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_g$, for pairwise distinct prime ideals $\mathfrak{p}_i$ in $\mathcal{O}$ of inertia degrees $(f_1,\dots,f_g)$, then

$$\zeta^{\triangleleft}_{\mathcal{L}(\mathcal{O}),p}(s) = W^{\triangleleft}_{\mathcal{L},\mathbf{f}}(p,p^{-s}).$$

In particular, $\zeta^{\triangleleft}_{\mathcal{L}(\mathcal{O})}(s)$ is finitely uniform and $\zeta^{\triangleleft}_{\mathcal{L}}(s) = \zeta^{\triangleleft}_{\mathcal{L}(\mathbb{Z})}(s)$ is uniform.

A special case of Theorem 1.2 establishes part of a conjecture of Grunewald, Segal, and Smith on the normal subgroup growth of free nilpotent groups under extension of scalars. In [11], they introduced the concept of the *normal zeta function*

$$\zeta^{\triangleleft}_G(s) = \sum_{H \triangleleft G} |G : H|^{-s}$$

of a torsion-free finitely generated nilpotent group G, enumerating the normal subgroups of G of finite index in G. As G is nilpotent, it also satisfies an Euler product decomposition

$$\zeta^{\triangleleft}_G(s) = \prod_{p \text{ prime}} \zeta^{\triangleleft}_{G,p}(s),$$

whose factors enumerate the normal subgroups of G of p-power index. If G has nilpotency class two, then its normal zeta function coincides with the ideal zeta function of the associated Lie ring $\mathcal{L}_G := G/Z(G) \oplus Z(G)$; see [11, Remark on p. 206]. Thus, $\zeta^{\triangleleft}_G(s) = \zeta^{\triangleleft}_{\mathcal{L}_G}(s)$. Moreover, every class-2-nilpotent Lie ring $\mathcal{L}$ arises in this way and gives rise to a torsion-free finitely generated nilpotent group $G(\mathcal{L})$; see [34, Section 1.2] for details. Theorem 1.2 thus has a direct corollary pertaining to the normal zeta functions of the finitely generated class-2-nilpotent groups corresponding to the Lie rings in $\mathfrak{L}$. Since the groups associated to the free class-2-nilpotent Lie rings $\mathfrak{f}_{2,d}$ are the finitely generated free class-2-nilpotent groups $F_{2,d} = G(\mathfrak{f}_{2,d})$, Theorem 1.2 implies the Conjecture on p. 188 of [11] for the case $* = \triangleleft$ and $c = 2$. The conjecture for normal zeta functions had previously been established only for $d = 2$ ([11, Theorem 3]; see also Section 1.1.2). We are not aware of any other case for which the conjecture has been proven or refuted.

Theorem 1.2 is a consequence of the following uniform "local" result. Throughout the paper, $\mathfrak{o}$ will denote a compact discrete valuation ring of arbitrary characteristic and residue field of characteristic p and cardinality q. Thus, $\mathfrak{o}$ may, for instance, be a finite extension of the ring $\mathbb{Z}_p$ of p-adic integers (of characteristic zero) or a ring of formal power series of the form $\mathbb{F}_q[\![T]\!]$ (of positive characteristic). The *$\mathfrak{o}$-ideal zeta function*

$$\zeta^{\triangleleft \mathfrak{o}}_L(s) = \sum_{I \triangleleft L} |L : I|^{-s}$$

of an $\mathfrak{o}$-algebra L of finite $\mathfrak{o}$-rank is defined as in (1.1), with I ranging over the $\mathfrak{o}$-ideals of L, viz. $(\operatorname{ad} L)$-invariant $\mathfrak{o}$-submodules of L. Note that every element $\mathcal{L}$ of $\mathfrak{L}$ may, after tensoring over $\mathbb{Z}$ with $\mathfrak{o}$, be considered a free and finitely generated $\mathfrak{o}$-Lie algebra. Given an $\mathfrak{o}$-module R, we write $L(R) = L \otimes_{\mathfrak{o}} R$.

Theorem 1.3. *Let $\boldsymbol{\mathcal{L}} = (\mathcal{L}_1, \dots, \mathcal{L}_g)$ be a family of elements of $\mathfrak{L}$ and $\boldsymbol{f} = (f_1, \dots, f_g) \in \mathbb{N}^g$. There exists an explicit rational function $W^{\triangleleft}_{\boldsymbol{\mathcal{L}},\boldsymbol{f}} \in \mathbb{Q}(X, Y)$ such that the following holds:*

Let $\mathfrak{o}$ be a compact discrete valuation ring and $(\mathfrak{O}_1, \dots, \mathfrak{O}_g)$ be a family of finite unramified extensions of $\mathfrak{o}$ with inertia degrees $(f_1, \dots, f_g)$. Consider the $\mathfrak{o}$-Lie algebra

$$L = \mathcal{L}_1(\mathfrak{O}_1) \times \cdots \times \mathcal{L}_g(\mathfrak{O}_g).$$

For every finite extension $\mathfrak{O}$ of $\mathfrak{o}$, of inertia degree f over $\mathfrak{o}$, say, the $\mathfrak{O}$-ideal zeta function of $L(\mathfrak{O})$ satisfies

$$\zeta^{\triangleleft \mathfrak{O}}_{L(\mathfrak{O})}(s) = W^{\triangleleft}_{\boldsymbol{\mathcal{L}},\boldsymbol{f}}(q^f, q^{-fs}).$$

The rational function $W^{\triangleleft}_{\boldsymbol{\mathcal{L}},\boldsymbol{f}}$ satisfies the functional equation

$$W^{\triangleleft}_{\boldsymbol{\mathcal{L}},\boldsymbol{f}}(X^{-1}, Y^{-1}) = (-1)^{N_0} X^{\binom{N_0}{2}} Y^{N_0+N_1} W^{\triangleleft}_{\boldsymbol{\mathcal{L}},\boldsymbol{f}}(X, Y), \tag{1.2}$$

where

$$N_0 = \operatorname{rk}_{\mathfrak{o}} L = \sum_{i=1}^{g} f_i \operatorname{rk}_{\mathbb{Z}}(\mathcal{L}_i) \quad \text{and} \quad N_1 = \operatorname{rk}_{\mathfrak{o}}(L/Z(L)) = \sum_{i=1}^{g} f_i \operatorname{rk}_{\mathbb{Z}}(\mathcal{L}_i/Z(\mathcal{L}_i)).$$

Theorem 1.2 is readily deduced from Theorem 1.3. Indeed, let $\mathcal{L}$ be a nilpotent Lie ring as in the statement of Theorem 1.2, and let $\mathcal{O}$ be the ring of integers of a number field. Suppose that the rational prime p is unramified in $\mathcal{O}$ and decomposes as $p\mathcal{O} = \mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_g$, where the $\mathfrak{p}_i$ are distinct prime ideals of $\mathcal{O}$ of inertia degrees f_i. Then $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \mathfrak{O}_1 \times \cdots \times \mathfrak{O}_g$, where each $\mathfrak{O}_i/\mathbb{Z}_p$ is an unramified extension of inertia degree f_i. Therefore,

$$\mathcal{L}(\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p) \simeq \mathcal{L}(\mathfrak{O}_1) \times \cdots \times \mathcal{L}(\mathfrak{O}_g).$$

Hence, by Theorem 1.3 we have

$$\zeta^{\triangleleft}_{\mathcal{L}(\mathcal{O}),p}(s) = \zeta^{\triangleleft \mathbb{Z}_p}_{\mathcal{L}(\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{Z}_p)}(s) = W^{\triangleleft}_{(\mathcal{L},\dots,\mathcal{L}),(f_1,\dots,f_g)}(p, p^{-s})$$

for an explicit rational function $W^{\triangleleft}_{(\mathcal{L},\dots,\mathcal{L}),(f_1,\dots,f_g)} \in \mathbb{Q}(X, Y)$. Setting $W^{\triangleleft}_{\mathcal{L},\boldsymbol{f}} = W^{\triangleleft}_{(\mathcal{L},\dots,\mathcal{L}),(f_1,\dots,f_g)}$, we obtain Theorem 1.2.

Remark 1.4. Our description of the rational function $W^{\triangleleft}_{\mathcal{L},\boldsymbol{f}}$ is so explicit that one may, in principle, read off the (local) *abscissa of convergence* $\alpha^{\triangleleft\mathfrak{O}}_{L(\mathfrak{O})}$ of $\zeta^{\triangleleft\mathfrak{O}}_{L(\mathfrak{O})}(s)$, viz.

$$\alpha^{\triangleleft\mathfrak{O}}_{L(\mathfrak{O})} := \inf\left\{\alpha \in \mathbb{R}_{>0} \mid \zeta^{\triangleleft\mathfrak{O}}_{L(\mathfrak{O})}(s) \text{ converges on } \{s \in \mathbb{C} \mid \Re(s) > \alpha\}\right\} \in \mathbb{Q}_{>0};$$

cf. Remark 4.23.

Remark 1.5. We emphasize that Theorem 1.3 makes no restriction on the residue characteristic of $\mathfrak{o}$. In this regard it strengthens, for the class of Lie rings under consideration, the result [34, Theorem 1.2], which establishes the functional equation (1.2) for all $\mathfrak{o}$ whose residue characteristic avoids finitely many prime numbers; cf. [34, Corollary 1.3]. In the global contexts of ideal zeta functions of rings of the form $\mathcal{L}(\mathcal{O})$ for number rings $\mathcal{O}$, Theorem 1.3 shows that the finitely many Euler factors for which the functional equation (1.2) fails must be among those indexed by primes that ramify in $\mathcal{O}$.

In [24, Conjecture 1.4] it was suggested that a functional equation should hold for *all* local factors $\zeta^{\triangleleft}_{\mathfrak{f}_{2,2}(\mathcal{O}),p}(s)$, where $\mathfrak{f}_{2,2}$ is the Heisenberg Lie ring and $\mathcal{O}$ is a number ring; if p ramifies in $\mathcal{O}$, then the symmetry factor must be modified from that of (1.2). Some cases of the conjecture were proved in [25, Corollary 3.13]. There is computational evidence, due to T. Bauer, that other Lie rings in the class $\mathfrak{L}$ also exhibit the remarkable property of the local factors $\zeta^{\triangleleft}_{\mathcal{L}(\mathcal{O}),p}(s)$ at ramified primes p being described by rational functions satisfying functional equations. Those computations, together with the results of this paper, suggest the following natural question: how do the local factors $\zeta^{\triangleleft}_{\mathcal{L}(\mathcal{O}),p}(s)$ behave at ramified primes, and how does the structure of $\mathcal{L}$ govern their behaviour?

1.1.2. *Previous and related work.* Theorems 1.2 and 1.3 generalize and unify several previously known results.

(1) The most classical may be the formula for the $\mathfrak{o}$-ideal zeta function

$$\zeta_{\mathfrak{o}^n}(s) := \zeta^{\triangleleft\mathfrak{o}}_{\mathfrak{o}^n}(s) = \prod_{i=1}^{n} \frac{1}{1-q^{-s+i-1}} \tag{1.3}$$

of the (abelian Lie) ring $\mathfrak{o}^n = \mathfrak{g}_{0,n}(\mathfrak{o}) = \mathfrak{g}_{n,0}(\mathfrak{o})$; cf. [11, Proposition 1.1].

(2) The ideal zeta functions of the so-called *Grenham Lie rings* $\mathfrak{g}_{1,d}$ were given in [30, Theorem 5].

(3) Formulae for the ideal zeta functions of the free class-2-nilpotent Lie rings $\mathfrak{f}_{2,d}$ on d generators are the main result of [31].

(4) The paper [24] contains formulae for all local factors of the ideal zeta functions of the Lie rings $\mathfrak{f}_{2,2}(\mathcal{O}) = \mathfrak{g}_{1,1}(\mathcal{O}) = \mathfrak{h}_1(\mathcal{O})$, i.e. the *Heisenberg Lie ring* over an arbitrary number ring $\mathcal{O}$, which are indexed by primes unramified in $\mathcal{O}$. The uniform nature of these functions had already been established in [11, Theorem 3]. Formulae for factors indexed by non-split primes are given in [25].

(5) The ideal zeta functions of the Lie rings $\mathfrak{h}_d \times \mathfrak{o}^r$ were computed in [11, Proposition 8.4], whereas for the direct products $\mathfrak{h}_d \times \cdots \times \mathfrak{h}_d$ they were computed in [1].

(6) The ideal zeta function of the Lie ring $\mathfrak{g}_{2,2}$ was computed in [21, Theorem 11.1].

Some of the members of the family of Lie rings $\mathfrak{L}$ have previously been studied with regards to related counting problems, each leading to a different class of zeta functions. We mention specifically four such classes: first, the *subring zeta function* of a (class-2-nilpotent Lie) ring $\mathcal{L}$, enumerating the finite index subrings of $\mathcal{L}$; second, the *proisomorphic zeta function* of $G(\mathcal{L})$, the finitely generated nilpotent group associated to $\mathcal{L}$ via the Mal'cev correspondence, enumerating the subgroups of finite index of $G(\mathcal{L})$ whose profinite completions are isomorphic to the one of $G(\mathcal{L})$; third, the *representation zeta function* of $G(\mathcal{L})$, enumerating the twist-isoclasses of complex irreducible representations of $G(\mathcal{L})$; fourth, the *class number zeta function* of $G(\mathcal{L})$, enumerating the class numbers (i.e. numbers of conjugacy classes) of congruence quotients of this group (see [17]).

The subring zeta functions of the Grenham Lie rings $\mathfrak{g}_{1,d}$ were computed in [32]. Those of the free class-2-nilpotent Lie rings $\mathfrak{f}_{2,d}$ are largely unknown, apart from $d = 2$ ([11]) and $d = 3$ ([9, Theorem 2.16], due to G. Taylor). The proisomorphic zeta functions of the members of a combinatorially defined class of groups that includes the Grenham groups $G(\mathfrak{g}_{1,d})$ were computed in [3], their normal zeta functions in [35]. The representation zeta functions of the free class-2-nilpotent groups $F_{2,d} = G(\mathfrak{f}_{2,d})$ were computed in [28, Theorem B], the ones of the groups $G(\mathfrak{g}_{d,d'})$ in [36, Theorem A]. The class number zeta functions of the groups $F_{2,d}$ and $G(\mathfrak{g}_{d,d})$ may be found in [18, Corollary 1.5].

1.1.3. *Methodology.* Our approach to computing the explicit rational functions mentioned in Theorems 1.3 and 1.2 hinges on the following considerations. Fix a prime p and a class-2-nilpotent Lie ring $\mathcal{L}$ and consider, for simplicity, the pro-p completion $L = \mathcal{L}(\mathbb{Z}_p)$ of $\mathcal{L}$. Given a $\mathbb{Z}_p$-sublattice $\Lambda \leqslant L$, set $\overline{\Lambda} := (\Lambda + L')/L'$ and $\Lambda' := \Lambda \cap L'$. Here we write $L' = [L, L]$ for the commutator subring of L. Clearly, Λ is a $\mathbb{Z}_p$-ideal of L if and only if $[\overline{\Lambda}, L] \subseteq \Lambda'$. This allows us, for fixed $\overline{\Lambda}$, to reduce the problem of enumerating such Λ' to the problem of enumerating subgroups of the finite abelian p-group $L'/[\overline{\Lambda}, L]$. The isomorphism type of the latter is given by the ($\mathbb{Z}_p$)-*elementary divisor type* of $[\overline{\Lambda}, L]$ in L', viz. the partition $\lambda(\Lambda) = (\lambda_1, \dots, \lambda_c)$ with the property that

$$L'/[\overline{\Lambda}, L] \simeq \mathbb{Z}_p/(p^{\lambda_1}) \times \cdots \times \mathbb{Z}_p/(p^{\lambda_c}).$$

For general Lie rings $\mathcal{L}$, controlling this type for varying Λ is a hard problem that may be dealt with by studying suitably defined p-adic integrals with sophisticated tools from algebraic geometry, including Hironaka's resolution of singularities in characteristic zero.

If, however, $\mathcal{L}$ is an element of the class $\mathfrak{L}$, then the elementary divisor type of $[\overline{\Lambda}, L]$ is determined, in a complicated but *combinatorial* manner, by so-called "projection data"; cf. Definition 4.1. These are the respective elementary divisor types of the projections of $\overline{\Lambda}$ onto various direct summands of L/L'. The technical tool we use to keep track of the resulting infinitude of finite enumerations are the *generalized Igusa functions* introduced in Section 3. An intrinsic advantage of this combinatorial point of view over the general (and typically immensely more powerful) algebro-geometric approach is that, structurally, $\mathbb{Z}_p$ only enters as a compact discrete valuation ring. The effect of passage to various other such local rings, including those of positive characteristic, is therefore easy to control.

For an informal overview of the combinatorial aspects of our approach to counting $\mathfrak{o}$-ideals, see Section 4.1.

1.2. **Counting ideals with generalized Igusa functions.** Our key to Theorem 1.3 is the systematic deployment of a new class of combinatorially defined multivariate rational functions, which we call *generalized Igusa functions*. Expecting that they will be of interest independently of questions pertaining to ideal growth in rings, we explain them here separately.

Generalized Igusa functions interpolate between two well-used classes of rational functions:

(1) A function we refer to as the *Igusa zeta function of degree n* plays a key role in numerous previous computations (for instance [30, 32, 31, 21, 28, 24, 25, 5, 35]):

$$I_n(Y; X_1, \dots, X_n) = \sum_{I \subseteq \{1,\dots,n\}} \binom{n}{I}_Y \prod_{i \in I} \frac{X_i}{1 - X_i} \in \mathbb{Q}(Y, X_1, \dots, X_n).$$

Here, $\binom{n}{I}_Y$ denotes the Gaussian multinomial; see (2.2). For instance,

$$\zeta_{\mathfrak{o}^n}(s) = I_n(q^{-1}; ((q^{n-i-s})^i)_{i=1}^n) \tag{1.4}$$

cf. (1.3) and [33, Example 2.20].

(2) In [24], the *weak order zeta function*

$$I_n^{\mathrm{wo}}((X_I)_{I \in \mathcal{P}([n]) \setminus \{\varnothing\}}) = \sum_{I_1 \subsetneq \cdots \subsetneq I_l \subseteq [n]} \prod_{j=1}^{l} \frac{X_{I_j}}{1 - X_{I_j}} \in \mathbb{Q}((X_I)_{I \in \mathcal{P}([n]) \setminus \{\varnothing\}}) \tag{1.5}$$

played a decisive role; cf. [24, Definition 2.9].

The main protagonist of Section 3 is the *generalized Igusa function* $I^{\mathrm{wo}}_{\underline{n}}(Y_1, \dots, Y_m; \mathbf{X})$, a rational function associated to a composition $\underline{n} = (n_1, \dots, n_m)$, with variables $\mathbf{X}$ indexed by the subwords of the word $a_1^{n_1} \dots a_m^{n_m}$ in "letters" $a_1, \dots, a_m$; cf. Definition 3.5 for details. It interpolates between the two classes of rational functions just mentioned: the Igusa function of degree n for the trivial composition (n) and the weak order zeta function for the all-one composition $(1, \dots, 1)$ of n; see Example 3.6.

Remark 1.6. Igusa functions are not to be confused with, but are related to, a class of p-adic integrals known as Igusa's local zeta function; cf. [6]. For a detailed explanation of the connection between I_n and work of Igusa, as well as further generalizations and applications, see [13].

1.3. **Organization and notation.**

1.3.1. In Section 2 we recall a number of preliminary notions and results used to enumerate lattices and finite abelian p-groups. In Section 3 we define the generalized Igusa functions and prove that they satisfy functional equations. In Section 4, these new functions are put to use to compute a general formula (cf. Theorem 4.21) for local ideal zeta functions of Lie rings satisfying the general combinatorial Hypothesis 4.5. In Section 5 we verify that the members of the class $\mathfrak{L}$ (cf. Definition 1.1) satisfy Hypothesis 4.5, complete the proof of Theorem 1.3, and attend to a number of special cases.

1.3.2. We write $\mathbb{N} = \{1, 2, \dots\}$ and, for a subset $X \subseteq \mathbb{N}$, set $X_0 = X \cup \{0\}$. For $m, n \in \mathbb{N}_0$ we denote $[n] = \{1, \dots, n\}$, $[n, m] = \{n, n+1, \dots, m\}$, and $(n, m) = \{n+1, \dots, m-1\}$. Given a finite subset $J \subseteq \mathbb{N}_0$, we write $J = \{j_1, \dots, j_r\}_<$ to imply that $j_1 < \dots < j_r$. We write $J - n$ for the set $\{j - n \mid j \in \mathbb{N}\}$. The power set of a set S is denoted $\mathcal{P}(S)$.

A *composition of* n *with* r *parts* is a family $(\lambda_1, \dots, \lambda_r) \in \mathbb{N}_0^r$ such that $\sum_{i=1}^r \lambda_i = n$. A *partition of* n *with* r *parts* is a composition of n with r parts such that $\lambda_1 \geqslant \dots \geqslant \lambda_r$. We occasionally obtain partitions from multisets by arranging their elements in non-ascending order. Our notation for the dual partition of a partition λ is λ'. Given partitions $\mu = (\mu_1, \dots, \mu_c)$ and $\lambda = (\lambda_1, \dots, \lambda_c)$ we write $\mu \leqslant \lambda$ if $\mu_i \leqslant \lambda_i$ for all $i \in [c]$, i.e. if the Young diagram of μ is included in the Young diagram of λ.

We write $t = q^{-s}$, where s denotes a complex variable.

2. Preliminaries

In this preliminary section, we collect some fundamental notions.

2.1. **Gaussian binomials and classical Igusa functions.** For a variable Y and integers $a, b \in \mathbb{N}_0$ with $a \geqslant b$, the associated *Gaussian binomial* is

$$\binom{a}{b}_Y = \frac{\prod_{i=a-b+1}^{a}(1 - Y^i)}{\prod_{i=1}^{b}(1 - Y^i)} \in \mathbb{Z}[Y].$$

A simple computation shows that

$$\binom{a}{b}_{Y^{-1}} = Y^{b(b-a)} \binom{a}{b}_Y. \tag{2.1}$$

Given $n \in \mathbb{N}$ and a subset $J = \{j_1, \dots, j_r\}_< \subseteq [n-1]$, the associated *Gaussian multinomial* is defined as

$$\binom{n}{J}_Y = \binom{n}{j_r}_Y \binom{j_r}{j_{r-1}}_Y \cdots \binom{j_2}{j_1}_Y \in \mathbb{Z}[Y]. \tag{2.2}$$

We omit the proof of the following simple lemma, which is similar to [24, Lemma 2.14].

Lemma 2.1. *Let* $n \in \mathbb{N}$ *and* $P = \{p_1, \dots, p_r\}_< \subseteq J \subseteq [n-1]$. *Then*

$$\binom{n}{J}_Y = \binom{n}{P}_Y \prod_{j=1}^{r} \binom{p_j - p_{j-1}}{J \cap (p_{j-1}, p_j) - p_{j-1}}_Y.$$

Definition 2.2. ([24, Definition 2.5]) Let $n \in \mathbb{N}$. Given variables Y and $\mathbf{X} = (X_1, \dots, X_n)$, we define the *Igusa functions of degree n*

$$I_n(Y;\mathbf{X}) = \frac{1}{1-X_n} \sum_{I \subseteq [n-1]} \binom{n}{I}_Y \prod_{i \in I} \frac{X_i}{1-X_i} = \sum_{I \subseteq [n]} \binom{n}{I}_Y \prod_{i \in I} \frac{X_i}{1-X_i} \in \mathbb{Q}(Y, X_1, \dots, X_n),$$

$$I_n^{\circ}(Y;\mathbf{X}) = \frac{X_n}{1-X_n} \sum_{I \subseteq [n-1]} \binom{n}{I}_Y \prod_{i \in I} \frac{X_i}{1-X_i} \in \mathbb{Q}(Y, X_1, \dots, X_n).$$

An important feature of these functions is that they satisfy a functional equation upon inversion of the variables; it is immediate from [30, Theorem 4] that, for all $n \in \mathbb{N}$,

$$I_n(Y^{-1};\mathbf{X}^{-1}) = (-1)^n X_n Y^{-\binom{n}{2}} I_n(Y;\mathbf{X}), \tag{2.3}$$

$$I_n^{\circ}(Y^{-1};\mathbf{X}^{-1}) = (-1)^n X_n^{-1} Y^{-\binom{n}{2}} I_n^{\circ}(Y;\mathbf{X}). \tag{2.4}$$

2.2. **Subgroups of finite abelian groups, Birkhoff's formula, and Dyck words.** It is well-known that, given a pair of partitions $\mu \leqslant \lambda$ and a prime p, the number $a(\lambda, \mu; p)$ of finite abelian p-groups of isomorphism type μ contained in a fixed finite abelian p-group of isomorphism type λ is given by a polynomial in p. More precisely, set

$$\alpha(\lambda, \mu; Y) = \prod_{k \geqslant 1} Y^{\mu'_k(\lambda'_k - \mu'_k)} \binom{\lambda'_k - \mu'_{k+1}}{\lambda'_k - \mu'_k}_{Y^{-1}} \in \mathbb{Q}[Y]. \tag{2.5}$$

Then, by a result attributed to Birkhoff in [4], $a(\lambda, \mu; p) = \alpha(\lambda, \mu; p)$.

In practical applications invoking infinitely many instances of this formula, as in [24, 15], it proved advantageous to sort pairs of partitions by their "overlap types" indexed by Dyck words, as we now recall.

Let $c \in \mathbb{N}$. A *Dyck word of length* $2c$ is a word

$$w = \mathbf{0}^{L_1} \mathbf{1}^{M_1} \mathbf{0}^{L_2 - L_1} \mathbf{1}^{M_2 - M_1} \cdots \mathbf{0}^{L_r - L_{r-1}} \mathbf{1}^{M_r - M_{r-1}}$$

in letters $\mathbf{1}$ and $\mathbf{0}$, both occurring c times each (hence $L_r = M_r = c$), and, crucially, no initial segment of w contains more ones than zeroes (or, equivalently, $M_i \leqslant L_i$ for all $i \in [r]$). Here, both the L_i and M_i are assumed to be positive. Below, we will use the notational conventions $M_0 = L_0 = 0$ and $L_{r+1} = L_r = c$, $M_{r+1} = M_r = c$. We write $\mathcal{D}_{2c}$ for the set of all Dyck words of length $2c$. See [24, Section 2.4] or [26, Example 6.6.6] for further details on Dyck words.

We say that two partitions λ and μ, both with c parts and satisfying $\mu \leqslant \lambda$, have *overlap type* $w \in \mathcal{D}_{2c}$, written $w(\lambda, \mu) = w$, if

$$\begin{aligned} \lambda_1 \geqslant \cdots \geqslant \lambda_{L_1} \geqslant \mu_1 \geqslant \cdots \geqslant \mu_{M_1} > \lambda_{L_1+1} \geqslant \cdots \geqslant \lambda_{L_2} \geqslant \mu_{M_1+1} \geqslant \cdots \geqslant \mu_{M_2} > \cdots \\ > \lambda_{L_{r-1}+1} \geqslant \cdots \geqslant \lambda_c \geqslant \mu_{M_{r-1}+1} \geqslant \cdots \geqslant \mu_c. \end{aligned} \tag{2.6}$$

In Definition 4.11 we slightly modify this definition to suit the specific needs of this paper.

2.3. **Gaussian multinomials and symmetric groups.** In Section 3, the following Coxeter group theoretic interpretation of the Gaussian multinomials will be useful. Recall that the symmetric group $W = S_n$ of degree n is a Coxeter group, with Coxeter generating system $S = (s_1, \dots, s_{n-1})$, where $s_i = (i\, i{+}1)$ denotes the standard transposition. The *Coxeter length* $\ell(w)$ of an element $w \in S_n$ is the length of a shortest word for w with elements from S. We define the (*right*) *descent set* $\mathrm{Des}(w) = \{i \in [n-1] \mid \ell(ws_i) < \ell(w)\}$. It is well-known ([27, Proposition 1.7.1]) that the Gaussian multinomials (2.2) satisfy

$$\binom{n}{J}_Y = \sum_{w \in S_n,\, \mathrm{Des}(w) \subseteq J} Y^{\ell(w)}. \tag{2.7}$$

Let w_0 denote the unique ℓ-longest element in S_n, of length $\ell(w_0) = \binom{n}{2}$. Then, for all $w \in S_n$,

$$\ell(ww_0) = \ell(w_0) - \ell(w), \qquad \mathrm{Des}(ww_0) = [n-1] \setminus \mathrm{Des}(w); \tag{2.8}$$

cf. [12, Section 1.8].

2.4. **A note on ramification.** Let $\mathfrak{o}$ be a compact discrete valuation ring of arbitrary characteristic. Let $\mathfrak{m}$ denote the maximal ideal of $\mathfrak{o}$ and let $\pi \in \mathfrak{o}$ be a uniformizer, i.e. any element such that $\mathfrak{m} = \pi\mathfrak{o}$. Let $\mathfrak{O}$ be a finite extension of $\mathfrak{o}$, with maximal ideal $\mathfrak{M}$ and uniformizer Π. Let $f = [\mathfrak{O}/\mathfrak{M} : \mathfrak{o}/\mathfrak{m}]$ be the inertia degree of the extension $\mathfrak{O}/\mathfrak{o}$, and let e be its ramification index; this means that $\pi\mathfrak{O} = \mathfrak{M}^e$. We will need the following standard fact.

Lemma 2.3. *Let $\mathfrak{O}$ be a finite extension of $\mathfrak{o}$ with ramification index e and inertia degree f. Let $\tau \in \mathbb{N}_0$. Suppose that $\tau = ge + h$, where $g \in \mathbb{N}_0$ and $h \in [e-1]_0$. Then the following isomorphism of $\mathfrak{o}$-modules holds:*

$$\mathfrak{O}/\mathfrak{M}^\tau \simeq \left(\mathfrak{o}/\mathfrak{m}^{g+1}\right)^{hf} \times (\mathfrak{o}/\mathfrak{m}^g)^{(e-h)f}.$$

In particular, if $\mathfrak{O}/\mathfrak{o}$ is unramified (i.e. $e = 1$), then $\mathfrak{O}/\mathfrak{M}^\tau \simeq (\mathfrak{o}/\mathfrak{m}^\tau)^f$ as $\mathfrak{o}$-modules.

Proof. Let $\beta_1, \dots, \beta_f \in \mathfrak{O}$ be a collection of elements whose reductions modulo $\mathfrak{M}$ constitute an $\mathfrak{o}/\mathfrak{m}$-basis of the residue field $\mathfrak{O}/\mathfrak{M}$. The set $\{\beta_i \Pi^j \mid i \in [f], j \in [e-1]_0\}$ provides a basis for $\mathfrak{O}$ as an $\mathfrak{o}$-module; see, for instance, the proof of [19, Proposition II.6.8]. Now it is clear that $\mathfrak{M}^\tau = \Pi^\nu \mathfrak{O}$ is the $\mathfrak{o}$-linear span of the set

$$\{\pi^{g+1}\beta_i\Pi^j \mid i \in [f], j \in [0, h-1]\} \cup \{\pi^g \beta_i \Pi^j \mid i \in [f], j \in [h, e-1]\}. \qquad \square$$

Definition 2.4. For $\tau \in \mathbb{N}_0$ and $e, f \in \mathbb{N}$, let $\{\tau\}_{e,f} = \{(g+1)^{(hf)}, g^{((e-h)f)}\}$ be the ef-element multiset consisting of the element $g+1$ with multiplicity hf and the element g with multiplicity $(e-h)f$, where $\tau = ge + h$ and $h \in [e-1]_0$, as in Lemma 2.3.

3. Generalized Igusa functions

In Section 3.1 we introduce generalized Igusa functions and prove that they satisfy functional equations. In Section 3.2 we record an identity involving weak order zeta functions, motivated by our applications of Igusa functions in ideal growth in Section 5.

3.1. **Generalized Igusa functions and their functional equations.** Let $\underline{n} = (n_1, \dots, n_m)$ be a composition of $N = \sum_{i=1}^m n_i$ with m parts. Consider the poset $C_{\underline{n}}$ of subwords of the word $v_{\underline{n}} := a_1^{n_1} a_2^{n_2} \dots a_m^{n_m}$ in "letters" $a_1, a_2, \dots, a_m$, each occurring with respective multiplicity n_i. This poset is naturally isomorphic to the lattice

$$C_{n_1} \times \dots \times C_{n_m},$$

the product of the chains of lengths n_i with the product order, which we denote by "$\leqslant$". We write $\hat{1} = v_{\underline{n}}$ and $\hat{0}$ for the empty word.

We denote by $\mathrm{WO}_{\underline{n}}$ the chain (or order) complex of $C_{\underline{n}}$. An element $V \in \mathrm{WO}_{\underline{n}}$ is a (possibly empty) chain, or flag, of non-empty subwords of $v_{\underline{n}}$, of the form $V = \{v_1 < \dots < v_t\}$. On $\mathrm{WO}_{\underline{n}}$ we consider the partial order defined by refinement of flags, also denoted by "$\leqslant$". Consider the natural map

$$\underline{\pi} : C_{\underline{n}} \to [n_1]_0 \times \dots \times [n_m]_0,$$
$$v = a_1^{\alpha_1} \dots a_m^{\alpha_m} \mapsto (\alpha_1, \dots, \alpha_m) =: (\pi_1(v), \dots, \pi_m(v)).$$

Definition 3.1. We consider the induced morphism of posets

$$\underline{\varphi} : \mathrm{WO}_{\underline{n}} \to \prod_{i=1}^m \mathcal{P}([n_i - 1]),$$
$$V = \{v_1 < \dots < v_t\} \mapsto (\{\pi_i(v_j) \mid j \in [t]\} \cap [n_i - 1])_{i=1}^m =: (\varphi_i(V))_{i=1}^m.$$

We say that $V \in \mathrm{WO}_{\underline{n}}$ has *full projections* if

$$\underline{\varphi}(V) = ([n_1 - 1], \dots, [n_m - 1]) =: K.$$

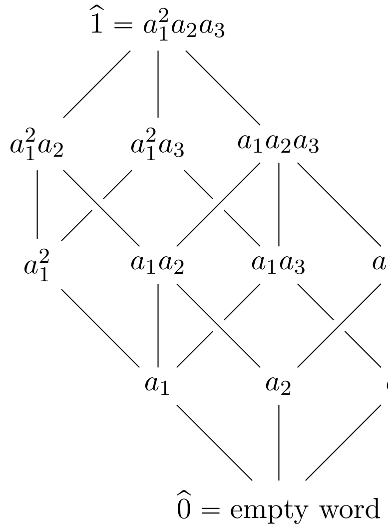

FIGURE 1. The poset $C_{\underline{n}}$ for $\underline{n} = (2,1,1)$.

Remark 3.2. We observe that the flag $V = \{v_1 < \cdots < v_t\} \in \mathrm{WO}_{\underline{n}}$ has full projections if, and only if, for all $j \in [t]_0$, the word v_{j+1}/v_j is squarefree, i.e. contains at most one copy of each letter $a_1, \dots, a_m$.

Definition 3.3. Let $V = \{v_1 < \cdots < v_t\} \in \mathrm{WO}_{\underline{n}}$. We define

$$W_V(\mathbf{X}) = \prod_{j=1}^{t} \frac{X_{v_j}}{1 - X_{v_j}} \in \mathbb{Q}(X_{v_1}, \dots, X_{v_t})$$

and

$$\binom{\underline{n}}{V}_{\mathbf{Y}} = \prod_{i=1}^{m} \binom{n_i}{\varphi_i(V)}_{Y_i} \in \mathbb{Q}(Y_1, \dots, Y_m),$$

where $\underline{\varphi}(V) = (\varphi_1(V), \dots, \varphi_m(V))$.

Example 3.4. Let $\underline{n} = (3,2,2)$. The flag $V = \{a_2a_3 < a_1a_2^2a_3\} \in \mathrm{WO}_{(3,2,2)}$ does not have full projections, as $\underline{\varphi}(V) = (\{1\},\{1\},\{1\})$. We note that

$$W_V(\mathbf{X}) = \frac{X_{a_2a_3}X_{a_1a_2^2a_3}}{(1 - X_{a_2a_3})(1 - X_{a_1a_2^2a_3})}$$

and

$$\binom{\underline{n}}{V}_{\mathbf{Y}} = \binom{3}{1}_{Y_1}\binom{2}{1}_{Y_2}\binom{2}{1}_{Y_3} = (1 + Y_1 + Y_1^2)(1 + Y_2)(1 + Y_3).$$

The following is the key combinatorial tool of this paper.

Definition 3.5. The *generalized Igusa function associated with the composition* $\underline{n}$ is

$$I^{\mathrm{wo}}_{\underline{n}}(\mathbf{Y}; \mathbf{X}) := \sum_{V \in \mathrm{WO}_{\underline{n}}} \binom{\underline{n}}{V}_{\mathbf{Y}} W_V(\mathbf{X}) \in \mathbb{Q}(Y_1, \dots, Y_m, (X_r)_{r \leqslant v_{\underline{n}}}),$$

Example 3.6.

(1) For $\underline{n} = (N)$, the trivial composition of N, we recover $I^{\mathrm{wo}}_{(N)}(\mathbf{Y}; \mathbf{X}) = I_N(Y; \mathbf{X})$, the classical Igusa zeta function recalled in Definition 2.2.
(2) For $\underline{n} = (1, \dots, 1)$, the all-one composition of N, we recover $I^{\mathrm{wo}}_{(1,\dots,1)}(\mathbf{Y}; \mathbf{X}) = I^{\mathrm{wo}}_N(\mathbf{X})$, the weak order zeta function recalled in (1.5). We note that the variables $\mathbf{Y}$ do not appear in this case, as all the polynomials $\binom{\underline{n}}{V}_{\mathbf{Y}}$ are equal to the constant 1.

(3) For $\underline{n} = (2,1)$ we obtain

$$I^{\mathrm{wo}}_{(2,1)}(\mathbf{Y};\mathbf{X}) = \frac{1}{1-X_{a_1^2a_2}}\left(1+\frac{X_{a_2}}{1-X_{a_2}}+\frac{X_{a_1^2}}{1-X_{a_1^2}}+\right.$$
$$(1+Y_1)\left(\frac{X_{a_1}}{1-X_{a_1}}+\frac{X_{a_1a_2}}{1-X_{a_1a_2}}+\frac{X_{a_1}}{1-X_{a_1}}\frac{X_{a_1a_2}}{1-X_{a_1a_2}}+\right.$$
$$\left.\left.\frac{X_{a_1}}{1-X_{a_1}}\frac{X_{a_1^2}}{1-X_{a_1^2}}+\frac{X_{a_2}}{1-X_{a_2}}\frac{X_{a_1a_2}}{1-X_{a_1a_2}}\right)\right).$$

Remark 3.7. Generalized Igusa functions associated with the all-one compositions also coincide with certain instances of generating functions associated with chain partitions in [2, Section 4.9].

The following "combinatorial reciprocity theorem" is the main result of this section.

Theorem 3.8. *The generalized Igusa function associated with the composition $\underline{n}$ of $N = \sum_{i=1}^m n_i$ satisfies the following functional equation:*

$$I^{\mathrm{wo}}_{\underline{n}}(\mathbf{Y}^{-1};\mathbf{X}^{-1}) = (-1)^N X_{v_{\underline{n}}}\left(\prod_{i=1}^m Y_i^{-\binom{n_i}{2}}\right) I^{\mathrm{wo}}_{\underline{n}}(\mathbf{Y};\mathbf{X}).$$

For the proof of Theorem 3.8 we require a number of preliminary results. The first records simple but crucial "inversion properties" of the rational functions $W_V(\mathbf{X})$.

Lemma 3.9. *For all $V \in \mathrm{WO}_{\underline{n}}$,*

$$W_V(\mathbf{X}^{-1}) = (-1)^{|V|}\sum_{Q\leqslant V} W_Q(\mathbf{X}).$$

Proof. This is a trivial consequence of the observation that

$$\frac{X^{-1}}{1-X^{-1}} = -\left(1+\frac{X}{1-X}\right). \quad \square$$

We fix some notation used in the rest of this section. We let $\mathrm{WO}^{\times}_{\underline{n}}$ denote the subcomplex of $\mathrm{WO}_{\underline{n}}$ of flags of *proper* subwords of $v_{\underline{n}}$. When dealing with tuples of sets, we will abuse notation and use set theoretical operations for componentwise operations. For instance, for $I = (I_1,\dots,I_m) \in \prod_{i=1}^m \mathcal{P}([n_i-1])$ we write $I^c := K\backslash I$ for $([n_1-1]\backslash I_1,\dots,[n_m-1]\backslash I_m)$.

The following analogue of [32, Lemma 7] is critical for our analysis.

Proposition 3.10. *For all $I \in \prod_{i=1}^m \mathcal{P}([n_i-1])$,*

$$\sum_{\substack{V\in\mathrm{WO}^{\times}_{\underline{n}}\\ \underline{\varphi}(V)\supseteq I}} W_V(\mathbf{X}^{-1}) = (-1)^{N-1}\sum_{\substack{V\in\mathrm{WO}^{\times}_{\underline{n}}\\ \underline{\varphi}(V)\supseteq I^c}} W_V(\mathbf{X}). \tag{3.1}$$

Proof. Let $I \in \prod_{i=1}^m \mathcal{P}([n_i-1])$. The inversion properties established in Lemma 3.9 yield that

$$\sum_{\substack{V\in\mathrm{WO}^{\times}_{\underline{n}}\\ \underline{\varphi}(V)\supseteq I}} W_V(\mathbf{X}^{-1}) = \sum_{\substack{V\in\mathrm{WO}^{\times}_{\underline{n}}\\ \underline{\varphi}(V)\supseteq I}} (-1)^{|V|}\sum_{Q\leqslant V} W_Q(\mathbf{X}) = \sum_{V\in\mathrm{WO}^{\times}_{\underline{n}}} W_V(\mathbf{X})\sum_{\substack{S\geqslant V\\ \underline{\varphi}(S)\supseteq I}}(-1)^{|S|}.$$

We are left with proving that, for all $V \in \mathrm{WO}^{\times}_{\underline{n}}$,

$$\sum_{\substack{S\geqslant V\\ \underline{\varphi}(S)\supseteq I}}(-1)^{|S|} = \begin{cases}(-1)^{N-1}, & \text{if } \underline{\varphi}(V)\supseteq I^c,\\ 0, & \text{otherwise.}\end{cases} \tag{3.2}$$

Write $V = \{v_1 < \cdots < v_t\}$ and set $v_0 := \widehat{0}$ and $v_{t+1} := \widehat{1}$. Set

$$I_V := I \cup \underline{\varphi}(V) \in \prod_{i=1}^{m} \mathcal{P}([n_i - 1]).$$

The sum in (3.2) runs over refinements S of the flag V, subject to additional constraints on the projection of S given by I: we say that a refinement S of V is *admissible* if $\underline{\varphi}(S) \supseteq I_V$. As $\underline{\varphi}$ is a poset morphism, the sum in (3.2) runs exactly over the admissible refinements of V.

We will construct such refinements of V "locally". More precisely, let $j \in [t]_0$. We say that S is a *refinement of V between v_j and v_{j+1}* if $S \geqslant V$ and S and V coincide outside the interval $[v_j, v_{j+1}]$. We further say that $S \geqslant V$ has *full projections between v_j and v_{j+1}* if $\underline{\varphi}(S \cap [v_j, v_{j+1}])$ is an m-tuple of intervals.

We set

$$I_V^{(j)} := (I_{V,i} \cap [\pi_i(v_j), \pi_i(v_{j+1})])_{i=1}^m \in \prod_{i=1}^{m} \mathcal{P}([n_i - 1]).$$

Informally, $I_V^{(j)}$ dictates the constraints on a refinement S of V between v_j and v_{j+1}. More precisely, we say that a refinement S of V between v_j and v_{j+1} is *j-admissible* if $\underline{\varphi}(S) \supseteq I_V^{(j)}$. We further define

$$F_j(V, I) := \sum_{\substack{S \geqslant V \\ j\text{-admissible}}} (-1)^{|S \setminus V|} = \sum_{\substack{S \geqslant V \\ j\text{-admissible}}} (-1)^{|(S \setminus V) \cup \{v_j, v_{j+1}\}|}.$$

Clearly, given j-admissible refinements V_j of V for all $j \in [t]_0$, the flag $S := \bigcup_{j=0}^{t} V_j$ is an admissible refinement of V and any ("global") admissible refinement of V can be constructed in this way. The sum in (3.2) may thus be rewritten as follows:

$$\sum_{\substack{S \geqslant V \\ \underline{\varphi}(S) \supseteq I}} (-1)^{|S|} = \sum_{\substack{S \geqslant V \\ \underline{\varphi}(S) \supseteq I}} (-1)^{|V| + |S \setminus V|} = (-1)^t \sum_{\substack{S \geqslant V \\ \underline{\varphi}(S) \supseteq I}} (-1)^{|S \setminus V|} = (-1)^t \prod_{j=0}^{t} F_j(V, I). \tag{3.3}$$

We prove (3.2) distinguishing the two cases

(I) $I_V = \underline{\varphi}(V)$ (equivalently, $I \subseteq \underline{\varphi}(V)$) and
(II) $I_V \neq \underline{\varphi}(V)$ (equivalently, $I \backslash \underline{\varphi}(V) \neq \varnothing$).

Case (I): Assume first that $I \subseteq \underline{\varphi}(V)$. In this case, the condition $\underline{\varphi}(S) \supseteq I$ is trivially satisfied for any flag $S \geqslant V$, as $\underline{\varphi}$ is a poset morphism, and thus any refinement of V is admissible. Moreover, in this case, $\underline{\varphi}(V) \supseteq I^c$ if and only if V has full projections. In other words, (3.2) may be rewritten as follows:

$$\sum_{S \geqslant V} (-1)^{|S|} = \begin{cases} (-1)^{N-1}, & \text{if } V \text{ has full projections,} \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Let $j \in [t]_0$. As in the case under consideration all local refinements are j-admissible, $F_j(V, I)$ is given in terms of the Möbius function of the interval $[v_j, v_{j+1}]$ in the lattice $C_{\underline{n}}$. Indeed, by Philip Hall's theorem (see, for instance, [27, Proposition 3.8.5]),

$$F_j(V, I) = -\mu(v_j, v_{j+1}) = \begin{cases} (-1)^{|v_{j+1}| - |v_j| + 1}, & \text{if } [v_j, v_{j+1}] \text{ is a Boolean algebra,} \\ 0, & \text{otherwise;} \end{cases}$$

cf. [27, Example 3.8.4]. Using (3.3) we may therefore rewrite the LHS of (3.2) as

$$(-1)^t \prod_{j=0}^{t} F_j(V, I) = (-1)^t \prod_{j=0}^{t} (-\mu(v_j, v_{j+1}))\,.$$

It is nonzero if and only if all of its factors are nonzero. The interval $[v_j, v_{j+1}]$ is a Boolean algebra if and only if the word v_{j+1}/v_j is squarefree. By Remark 3.2, this happens for all $j \in [t]_0$ if and only if V has full projections. In this case we obtain

$$\sum_{S \geqslant V} (-1)^{|S|} = (-1)^t \sum_{S \geqslant V} (-1)^{|S \setminus V|} = (-1)^t \prod_{j=0}^{t} F_j(V, I) =$$

$$(-1)^t \prod_{j=0}^{t} (-\mu(v_j, v_{j+1})) = (-1)^{2t+1} (-1)^{\sum_{j=0}^{t}(|v_{j+1}| - |v_j|)} = (-1)^{N-1},$$

proving (3.4) and therefore (3.2) in the case $I \subseteq \underline{\varphi}(V)$.

Case (II): Assume now that $I \setminus \underline{\varphi}(V) \neq \varnothing$. Note that $\underline{\varphi}(V) \supseteq I^c$, the condition invoked in (3.2), holds if and only if $I_V = K$, i.e. if and only if $I_V^{(j)}$ is a tuple of intervals for all $j \in [t]_0$.

We claim that, in the case under consideration, the following holds for all $j \in [t]_0$:

$$F_j(V, I) = \begin{cases} (-1)^{|v_{j+1}| - |v_j| + 1}, & \text{if } I_V^{(j)} \text{ is a tuple of intervals,} \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

We now prove this claim by induction on the degree of the word v_{j+1}/v_j.

If v_{j+1} covers v_j, then $F_j(V, I) = 1$ trivially. So assume that (3.5) holds for $|v_{j+1}/v_j| \leqslant \ell$, for some $1 < \ell \in \mathbb{N}$, and suppose that $|v_{j+1}/v_j| = \ell + 1$. Let ρ_j denote the number of different letters in v_{j+1}/v_j.

Assume first that $I_V^{(j)}$ is a tuple of intervals, viz.

$$I_V^{(j)} = \left([\pi_i(v_j), \pi_i(v_{j+1})] \cap [n_i - 1]\right)_{i=1}^{m}.$$

Informally, this means that a j-admissible refinement S of V needs to have full projections between v_j and v_{j+1}. This condition forces the first element of $S \setminus V$ to lie on the ρ_j-dimensional hypercube above v_j: it is obtained by multiplying v_j with at most one copy of each of the ρ_j relevant letters. We may therefore write $F_j(V, I)$ as a sum of $2^{\rho_j} - 1$ summands, indexed by the words $v^{(1)}, \dots, v^{(2^{\rho_j}-1)}$ covering v_j in $C_{\underline{n}}$:

$$F_j(V, I) = -\sum_{k=1}^{2^{\rho_j}-1} \sum_{\substack{S \geqslant V \ j\text{-adm.,} \\ \min(S \setminus V) = v^{(k)}}} (-1)^{|S \setminus V|},$$

where, for each $k \in [2^{\rho_j} - 1]$, the inner sum is taken over j-admissible refinements S of V having $v^{(k)}$ as smallest element greater than v_j. Each of these sums is known by induction from (3.5). Indeed, since the flags S also have full projections between $v^{(k)}$ and v_{j+1}, we obtain

$$F_j(V, I) = -\sum_{k=1}^{2^{\rho_j}-1} (-1)^{|v_{j+1}| - |v^{(k)}| + 1} = (-1)^{|v_{j+1}| - |v_j| + 1},$$

establishing (3.5) in the first case.

Suppose now that $I_V^{(j)}$ is not a tuple of intervals. Informally, this means that a j-admissible refinement S of V is not required to have full projections between v_j and v_{j+1}. Without loss of generality we can assume that the first "requirement gap" in $I_V^{(j)}$ is directly above v_j, that is if $\underline{\alpha} = (\alpha_1, \dots, \alpha_m)$ is the m-tuple of (componentwise) minima of $I_V^{(j)} \setminus \underline{\pi}(v_j)$, there is at least one $i \in [m]$ with $\alpha_i > \pi_i(v_j) + 1$. Given a j-admissible refinement S of V, the word $\min(S \setminus V)$, the smallest word in S greater than v_j, clearly belongs to the interval $(v_j, v_{\underline{\alpha}}]$ of subwords of $v_{\underline{\alpha}} := a_1^{\alpha_1} \dots a_m^{\alpha_m}$ which v_j strictly divides. Consider the subset

$$Y := \{v \in (v_j, v_{\underline{\alpha}}] \mid [v, v_{\underline{\alpha}}] \text{ is a Boolean algebra}\}.$$

We rewrite the sum defining $F_j(V,I)$ according to whether or not $\min(S\backslash V) \in Y$:

$$F_j(V,I) = \sum_{\substack{S \geqslant V \text{ } j\text{-adm.},\\ \min(S\backslash V)\notin Y}} (-1)^{|S\backslash V|} + \sum_{\substack{S \geqslant V \text{ } j\text{-adm.},\\ \min(S\backslash V)\in Y}} (-1)^{|S\backslash V|}. \tag{3.6}$$

Clearly, the first summand in (3.6) is zero. Indeed, we may further subdivide it by fixing the minimal element $\min(S\backslash V)$. Each of the resulting summands is zero by applying (3.5) inductively to the refined flag $V \cup \{v\}$, replacing v_j by v.

The second summand in (3.6) is zero, too. Indeed, without loss of generality we may assume that

$$I_V^{(j)} = ((\{\pi_i(v_j)\} \cup [\alpha_i, \pi_i(v_{j+1})]) \cap [n_i - 1])_{i=1}^m .$$

(Otherwise, an argument similar to the one for the first summand in (3.6) proves the claim.) Under this assumption, the induction hypothesis yields

$$\sum_{\substack{S \geqslant V \text{ } j\text{-adm.},\\ \min(S\backslash V)\in Y}} (-1)^{|S\backslash V|} = -\sum_{[v,v_{\underline{\alpha}}] \text{ Boolean}} (-1)^{|v_{j+1}|-|v|} = (-1)^{|v_{j+1}|-|v_{\underline{\alpha}}|+1} \sum_{Z\subseteq\{0,1\}^{\rho_j}} (-1)^{|Z|} = 0.$$

This proves (3.5) in the second case.

Suppose now $I_V = K$. Since $I_V^{(j)}$ is a tuple of intervals for all $j \in [t]_0$, we get, by (3.5),

$$\sum_{\substack{S \geqslant V\\ \underline{\varphi}(S)\supseteq I_V}} (-1)^{|S\backslash V|} = (-1)^t \prod_{j=0}^{t} F_j(V,I) = (-1)^{2t+1}(-1)^{\sum_{j=0}^t |v_{j+1}|-|v_j|} = (-1)^{N-1}$$

as desired.

Suppose now $I_V \neq K$. This means there exists $j \in [t]_0$ such that $I_V^{(j)}$ is not a tuple of intervals. By (3.5) we have $F_j(V,I) = 0$, thus the product in (3.3) is also zero, proving (3.2) in the last case. □

Proof of Theorem 3.8. The sum defining the generalized Igusa function can be rewritten as

$$I_{\underline{n}}^{\mathrm{wo}}(\mathbf{Y};\mathbf{X}) = \sum_{V\in \mathrm{WO}_{\underline{n}}} \binom{n}{V}_{\mathbf{Y}} W_V(\mathbf{X}) = \frac{1}{1-X_{v_{\underline{n}}}} \sum_{V\in \mathrm{WO}_{\underline{n}}^{\times}} \binom{n}{V}_{\mathbf{Y}} W_V(\mathbf{X}). \tag{3.7}$$

Inverting the variable in the factor $\frac{1}{1-X_{v_{\underline{n}}}}$ on the RHS of (3.7) simply gives a factor $-X_{v_{\underline{n}}}$. Thus Theorem 3.8 is equivalent to the identity

$$\sum_{V\in \mathrm{WO}_{\underline{n}}^{\times}} \binom{n}{V}_{\mathbf{Y}^{-1}} W_V(\mathbf{X}^{-1}) = (-1)^{N-1} \left(\prod_{i=1}^m Y_i^{-\binom{n_i}{2}}\right) \sum_{V\in \mathrm{WO}_{\underline{n}}^{\times}} \binom{n}{V}_{\mathbf{Y}} W_V(\mathbf{X}). \tag{3.8}$$

Writing $\underline{S_n} = S_{n_1} \times \cdots \times S_{n_m}$, $\underline{w} = (w_1, \ldots, w_m)$, $\mathrm{Des}(\underline{w}) = \mathrm{Des}(w_1) \times \cdots \times \mathrm{Des}(w_m)$, and using the identity (2.7), the LHS of (3.8) becomes

$$\begin{aligned}\sum_{V\in \mathrm{WO}_{\underline{n}}^{\times}} \binom{n}{V}_{\mathbf{Y}^{-1}} W_V(\mathbf{X}^{-1}) &= \sum_{V\in \mathrm{WO}_{\underline{n}}^{\times}} \left(\sum_{\substack{\underline{w}\in \underline{S_n}\\ \mathrm{Des}(\underline{w})\subseteq \underline{\varphi}(V)}} \prod_{i=1}^m Y_i^{-\ell(w_i)} \right) W_V(\mathbf{X}^{-1}) \\ &= \sum_{\underline{w}\in \underline{S_n}} \left(\prod_{i=1}^m Y_i^{-\ell(w_i)}\right) \sum_{\substack{V\in \mathrm{WO}_{\underline{n}}^{\times}\\ \underline{\varphi}(V)\supseteq \mathrm{Des}(\underline{w})}} W_V(\mathbf{X}^{-1}).\end{aligned}$$

For $i \in [m]$ we denote by $w_0^{(i)}$ the longest element in S_{n_i}, of length $\ell(w_0^{(i)}) = \binom{n_i}{2}$. By Proposition 3.10 and the identities (2.8) we can rewrite

$$\begin{aligned}
&\sum_{\underline{w} \in \underline{S}_{\underline{n}}} \left(\prod_{i=1}^{m} Y_i^{-\ell(w_i)} \right) \sum_{\substack{V \in \mathrm{WO}_{\underline{n}}^{\times} \\ \underline{\varphi}(V) \supseteq \mathrm{Des}(\underline{w})}} W_V(\mathbf{X}^{-1}) \\
&= (-1)^{N-1} \sum_{\underline{w} \in \underline{S}_{\underline{n}}} \left(\prod_{i=1}^{m} Y_i^{-\ell(w_i)} \right) \sum_{\substack{V \in \mathrm{WO}_{\underline{n}}^{\times} \\ \underline{\varphi}(V) \supseteq \mathrm{Des}(\underline{w})^c}} W_V(\mathbf{X}) \\
&= (-1)^{N-1} \left(\prod_{i=1}^{m} Y_i^{-\binom{n_i}{2}} \right) \sum_{\underline{w} \in \underline{S}_{\underline{n}}} \left(\prod_{i=1}^{m} Y_i^{\ell(w_i w_0^i)} \right) \sum_{\substack{V \in \mathrm{WO}_{\underline{n}}^{\times} \\ \underline{\varphi}(V) \supseteq \mathrm{Des}(\underline{w}\underline{w}_0)}} W_V(\mathbf{X}) \\
&= (-1)^{N-1} \left(\prod_{i=1}^{m} Y_i^{-\binom{n_i}{2}} \right) \sum_{V \in \mathrm{WO}_{\underline{n}}^{\times}} \binom{\underline{n}}{V}_{\mathbf{Y}} W_V(\mathbf{X}),
\end{aligned}$$

proving (3.8) and thus Theorem 3.8. □

3.2. **Weak order zeta functions and generalized Igusa functions.**

We record an identity between instances of weak order zeta functions which will be useful in Section 5.3.3 and may be of independent interest. The identity compares instances of weak order zeta functions associated with the all-one-compositions $\underline{g}$ and $\underline{2g}$, with g and $2g$ parts, respectively, and holds when substituting for the variables monomials satisfying certain relations.

In the current section, we call a subword of the word $\widehat{1} = v_{\underline{2g}} := a_1 \cdots a_{2g}$ *radical* if it is of the form $w = \prod_{i \in \mathcal{J}} a_i a_{i+g}$ for some $\mathcal{J} \subseteq [g]$; see also Definition 4.13. We observe that any subword $r \leqslant v_{\underline{2g}}$ may be written uniquely in the form $r = \sqrt{r} \cdot r' r''$, where $\sqrt{r} = \prod_{i \in \mathcal{I}} a_i a_{i+g}$ is a radical word, whereas $r' = \prod_{i \in \mathcal{I}'} a_i$ and $r'' = \prod_{i \in \mathcal{I}''} a_{i+g}$, and the subsets $\mathcal{I}, \mathcal{I}', \mathcal{I}'' \subseteq [g]$ are disjoint. Likewise, we define the *radical* $\sqrt{S}$ of a flag $S \in \mathrm{WO}_{\underline{2g}}$ to be the flag of radicals of the words of S.

In the following result, we omit the non-occurring variable Y from the generalized Igusa functions $I_{\underline{g}}^{\mathrm{wo}}$ and $I_{\underline{2g}}^{\mathrm{wo}}$; cf. our remark in Example 3.6 (2).

Proposition 3.11. *Let $g \in \mathbb{N}$. Suppose that the numerical data $\mathbf{y}$ satisfy $y_r = y_{\sqrt{r}} \cdot \prod_{i \in \mathcal{I}' \cup \mathcal{I}''} y_{a_i}$. Then*

$$I_{\underline{2g}}^{\mathrm{wo}}(\mathbf{y}) = \left(\prod_{i=1}^{g} \frac{1 + y_{a_i}}{1 - y_{a_i}} \right) I_{\underline{g}}^{\mathrm{wo}}(\mathbf{z}), \tag{3.9}$$

where $z_{\prod_{i \in \mathcal{I}} a_i} = y_{\prod_{i \in \mathcal{I}} a_i a_{i+g}}$ for all $\mathcal{I} \subseteq [g]$.

Proof. By sorting the flags in $\mathrm{WO}_{\underline{2g}}$ by their radicals, we may partition the domain of summation of the LHS of (3.9) as follows:

$$\mathrm{WO}_{\underline{2g}} = \bigcup_{R \in \mathrm{WO}_{\underline{g}}} \{ S \in \mathrm{WO}_{\underline{2g}} \mid \sqrt{S} = R \}.$$

The claim is equivalent to showing that, for all $R \in \mathrm{WO}_{\underline{g}}$,

$$\sum_{\substack{S \in \mathrm{WO}_{\underline{2g}}: \\ \sqrt{S} = R}} W_S(\mathbf{y}) = \left(\prod_{i=1}^{g} \frac{1 + y_{a_i}}{1 - y_{a_i}} \right) W_R(\mathbf{z}) = \prod_{i=1}^{g} \left(1 + 2 \frac{y_{a_i}}{1 - y_{a_i}} \right) W_R(\mathbf{z}). \tag{3.10}$$

Let $S = \{ s_1 < \cdots < s_t \} = \{ \sqrt{s_1} \cdot s_1' s_1'' < \cdots < \sqrt{s_t} \cdot s_t' s_t'' \} \in \mathrm{WO}_{\underline{2g}}$, where, as above, for $k \in [t]$, $s_k' = \prod_{i \in \mathcal{I}'} a_i$, $s_k'' = \prod_{i \in \mathcal{I}''} a_{i+g}$ and $\sqrt{s_k} = \prod_{i \in \mathcal{I}_k} a_i a_{i+g}$ is radical. Denote $J(S) = \{ y_{s_1}, \ldots, y_{s_t} \}$, the monomials appearing in

$$W_S(\mathbf{y}) = \prod_{l=1}^{t} \frac{y_{s_l}}{1-y_{s_l}}.$$

For $j \in [g]$, set $y_{a_j} J(S) := \{y_{a_j} y \mid y \in J(S)\}$. As before we set $s_0 = \widehat{0}$ and $s_{t+1} = \widehat{1} = v_{\underline{2g}}$.

We claim that, for all $j \in [g]$ and all $S \in \mathrm{WO}_{\underline{2g}}$ with $\sqrt{S} = R$ and the property that, for all $s \in S$ if $a_j|s$ or $a_{g+j}|s$ then $a_j a_{g+j}|s$, the following identity holds:

$$\sum_{\substack{\overline{S} \in \mathrm{WO}_{\underline{2g}}:\, \sqrt{\overline{S}}=R,\\ J(\overline{S}) \subset J(S) \cup y_{a_j} J(S)}} W_{\overline{S}}(\mathbf{y}) = \left(1 + 2\frac{y_{a_j}}{1-y_{a_j}}\right) W_S(\mathbf{y}). \tag{3.11}$$

It is easy to see that (3.10) follows by repeated application of (3.11) for $j \in [g]$.

We prove (3.11) by induction on t, the induction base ($t=0$) being trivial; we observe that our assumption on the numerical data implies that $y_{a_j} = y_{a_{g+j}}$. The RHS may therefore be written as

$$\left(\prod_{l=1}^{i} \frac{y_{s_l}}{1-y_{s_l}}\right)\left(1 + \frac{y_{a_j}}{1-y_{a_j}} + \frac{y_{a_{g+j}}}{1-y_{a_{g+j}}}\right)\left(\prod_{l=i+1}^{t} \frac{y_{s_l}}{1-y_{s_l}}\right).$$

The summand 1 in the central factor arises from the flag $\overline{S} = S$, with $W_S(\mathbf{y}) = \prod_{i=1}^{t} \frac{y_{s_l}}{1-y_{s_l}}$. The other two summands account for flags $\overline{S}$ with $J(\overline{S}) = y_{a_j} J(S)$, i.e. for flags whose words differ from those of S by at most an extra factor a_j or a_{g+j} (but not both, as they share with S the radical R), and which do feature at least one such a "augmented" word. We will call such flags *a_j-augmentations (of S)*. It remains to show that

$$\sum_{\substack{\overline{S} \in \mathrm{WO}_{\underline{2g}}:\\ a_j\text{-augmentation of } S}} W_{\overline{S}}(\mathbf{y}) = \left(\prod_{l=1}^{t} \frac{y_{s_l}}{1-y_{s_l}}\right) \frac{y_{a_j}}{1-y_{a_j}}; \tag{3.12}$$

the argument for a_{g+j} is identical.

We note that there exists a unique $i \in [t]$ such that $a_j|s_{i+1}$ but $a_j \nmid s_i$. For all a_j-augmentations $\overline{S}$ of S, the last $t-i$ words coincide with $s_{i+1}, \dots, s_t$. Therefore $\prod_{l=i+1}^{t} \frac{y_{s_l}}{1-y_{s_l}}$ divides all relevant $W_{\overline{S}}(\mathbf{y})$. Without loss of generality we may thus assume that $i = t$, i.e. that *no word of S is divisible by a_j*.

The claimed identity in (3.12) will become clear by interpreting the trivial identity

$$\left(\prod_{l=1}^{t} \frac{y_{s_l}}{1-y_{s_l}}\right) \frac{y_{a_j}}{1-y_{a_j}} = \left(\prod_{l=1}^{t-1} \frac{y_{s_l}}{1-y_{s_l}}\right)\left(\frac{y_{a_j} y_{s_t}}{1-y_{a_j} y_{s_t}} + \frac{y_{s_t}}{1-y_{s_t}} \frac{y_{a_j} y_{s_t}}{1-y_{a_j} y_{s_t}} + \frac{y_{a_j}}{1-y_{a_j}} \frac{y_{a_j} y_{s_t}}{1-y_{a_j} y_{s_t}}\right). \tag{3.13}$$

Informally, the RHS of (3.13) reflects the three alternatives for the first occurrence of a_j in an a_j-augmentation of $\overline{S}$.

1. The first summand arises from the a_j-augmentation $\overline{S} = \{\dots < s_{t-2} < s_{t-1} < a_j s_t\}$.
2. The second summand arises from the a_j-augmentation $\overline{S} = \{\dots < s_{t-1} < s_t < a_j s_t\}$.
3. The third summand arises from all a_j-augmentations of S whose last *two* words are divisible by a_j, the last one being $a_j s_t$, viz. a_j-augmentations of $S \backslash \{s_t\}$. All the relevant $W_{\overline{S}}(\mathbf{y})$ are therefore divisible by $\frac{y_{a_j} y_{s_t}}{1-y_{a_j} y_{s_t}}$. By induction hypothesis, (3.12) yields

$$\left(\prod_{l=1}^{t-1} \frac{y_{s_l}}{1-y_{s_l}}\right) \frac{y_{a_j}}{1-y_{a_j}} = \sum_{\substack{\overline{S} \in \mathrm{WO}_{\underline{2g}}:\\ a_j\text{-augmentation of } S \backslash \{s_t\}}} W_{\overline{S}}(\mathbf{y}).$$

This proves the claim, and hence the proposition. □

4. Counting $\mathfrak{o}$-ideals in combinatorially defined $\mathfrak{o}$-Lie algebras

In this section we compute the $\mathfrak{o}$-ideal zeta functions of $\mathfrak{o}$-Lie algebras satisfying a certain combinatorial condition (Hypothesis 4.5) in terms of the generalized Igusa functions introduced in Section 3. This prepares the proof of Theorem 1.3, given in Section 5.

4.1. **Informal overview.** We start by summarizing the principal ideas behind our approach, which greatly generalize those of [24]. Let L be an $\mathfrak{o}$-Lie algebra with derived subalgebra $L' = [L, L]$. As noted in Section 1.1.3, if L is class-2-nilpotent, then an $\mathfrak{o}$-sublattice $\Lambda \leqslant L$ is an $\mathfrak{o}$-ideal if $[\overline{\Lambda}, L] \leqslant \Lambda \cap L'$, where $\overline{\Lambda} = (\Lambda + L')/L'$. For simplicity of exposition we will assume, in this overview, that $L' = Z(L)$, i.e. that L has no abelian direct summands. By an argument going back to [11, Lemma 6.1], the computation of $\zeta_L^{\triangleleft \mathfrak{o}}(s)$ is reduced to a summation over pairs $(\overline{\Lambda}, \mathrm{M})$, where $\overline{\Lambda} \leqslant L/L'$ and $\mathrm{M} \leqslant L'$ are $\mathfrak{o}$-sublattices such that $[\overline{\Lambda}, L] \leqslant \mathrm{M}$. Recall that the $\mathfrak{O}$-elementary divisor type of a finite-index $\mathfrak{O}$-sublattice $\Lambda \leqslant \mathfrak{O}^n$, where $\mathfrak{O}$ is a compact discrete valuation ring with maximal ideal $\mathfrak{M}$, is the partition $(\lambda_1, \dots, \lambda_n)$ such that

$$\mathfrak{O}^n/\Lambda \simeq \mathfrak{O}/\mathfrak{M}^{\lambda_1} \times \cdots \times \mathfrak{O}/\mathfrak{M}^{\lambda_n}.$$

Given the $\mathfrak{o}$-elementary divisor type $\lambda(\overline{\Lambda})$ of $[\overline{\Lambda}, L]$, the lattices M satisfying this condition are enumerated by Birkhoff's formula (2.5).

An essential ingredient of our method, therefore, is an effective description of the $\mathfrak{o}$-elementary divisor type $\lambda(\overline{\Lambda})$ in terms of the structure of $\overline{\Lambda}$. For the $\mathfrak{o}$-Lie algebras considered in this paper, this is accomplished as follows. The quotient L/L' decomposes, as an $\mathfrak{o}$-module, into a direct sum of m components, which are viewed as free modules over finite extensions $\mathfrak{O}_1, \dots, \mathfrak{O}_m$ of $\mathfrak{o}$. For each component, we consider the $\mathfrak{O}_i$-elementary divisor type $\nu^{(i)}$ of the $\mathfrak{O}_i$-lattice generated by the projection of $\overline{\Lambda}$ onto that component. These are the projection data of Definition 4.1 below. The crucial Hypothesis 4.5 requires that the parts of the partition $\lambda(\overline{\Lambda})$ be given by the minima of term-by-term comparisons among the elementary divisor types appearing in the projection data. Assuming Hypothesis 4.5, we deduce a purely combinatorial expression for $\zeta_L^{\triangleleft \mathfrak{o}}(s)$ in Proposition 4.10.

Analogously to the argument of [24], we break up the sum in Proposition 4.10 into finitely many pieces on which the Gaussian multinomial coefficients—arising via the factors $\beta(\nu^{(i)}; q_i)$ and $\alpha(\lambda(\boldsymbol{\nu}), \mu; q)$, in the notation used there—and the dual partitions occurring in the definition (2.5) of $\alpha(\lambda(\boldsymbol{\nu}), \mu; q)$ are constant. The sum over each piece yields a product of Gaussian multinomials and geometric progressions; these, in turn, are assembled into generalized Igusa functions introduced in Section 3. As in [24], Dyck words of fixed length turn out to be suitable indexing objects for the finitely many pieces.

The technical complexity of the current paper, in comparison to [24], reflects the fact the translation between projection data and the elementary divisor type $\lambda(\overline{\Lambda})$ is considerably more involved. While the data determining $\lambda(\overline{\Lambda})$ in [24] were just a collection of integers, here they are a collection of partitions (the $\nu^{(i)}$ defined above). A more sophisticated combinatorial machinery, viz. the weak orders of Section 3.1, is required to keep track of the relative sizes of the parts of these different partitions; this is necessary in order to specify domains of summation over which the dual partition $\lambda(\overline{\Lambda})'$ is constant.

In Section 4.2 we define the concept of projection data and enumerate lattices $\overline{\Lambda} \leqslant L/L'$ with fixed projection data. In Section 4.3 we introduce and explain the combinatorial structure behind Hypothesis 4.5 and deduce Proposition 4.10, giving a general formula for $\mathfrak{o}$-ideal zeta functions of $\mathfrak{o}$-Lie algebras satisfying Hypothesis 4.5. In Section 4.4 we state the section's main result, viz. Theorem 4.21, and prove it modulo an auxiliary claim, viz. Proposition 4.20, whose rather technical proof is given in Section 4.5.

Throughout, let $\mathfrak{o}$ be a complete discrete valuation ring with finite residue field of cardinality q, and let $\mathfrak{O}_1, \dots, \mathfrak{O}_h$ be finite extensions of $\mathfrak{o}$. Let $\pi \in \mathfrak{o}$ be a uniformizer. For each $i \in [h]$, let e_i be the ramification index and f_i be the inertia degree of $\mathfrak{O}_i$ over $\mathfrak{o}$. Let $q_i = q^{f_i}$

be the cardinality of the residue field of $\mathfrak{O}_i$. For each $i \in [h]$, the local ring $\mathfrak{O}_i$ is a free $\mathfrak{o}$-module of rank $e_i f_i$. Let $(n_1, \ldots, n_h) \in \mathbb{N}_0^h$ and set $n = \sum_{i=1}^h e_i f_i n_i$. Consider a family $\widetilde{\boldsymbol{\nu}} = (\nu^{(1)}, \ldots, \nu^{(h)})$ of partitions $\nu^{(i)}$, each with n_i parts.

4.2. **Counting lattices with fixed projections.** Consider the $\mathfrak{o}$-module

$$\Omega = \mathfrak{O}_1^{n_1} \times \cdots \times \mathfrak{O}_h^{n_h}$$

and, for each $i \in [h]$, let $\pi_i : \Omega \to \mathfrak{O}_i^{n_i}$ be the projection onto the i-th component. Choosing an $\mathfrak{O}_i$-basis $(e_1^{(i)}, \ldots, e_{n_i}^{(i)})$ of $\mathfrak{O}_i^{n_i}$ and an $\mathfrak{o}$-basis $(\alpha_1^{(i)}, \ldots, \alpha_{e_i f_i}^{(i)})$ of each $\mathfrak{O}_i$, it is clear that the collection $\left\{\alpha_j^{(i)} e_k^{(i)}\right\}_{ijk}$ constitutes an $\mathfrak{o}$-basis of Ω that allows us to identify Ω with $\mathfrak{o}^n$.

Definition 4.1. For an $\mathfrak{o}$-sublattice $\Lambda \leqslant \mathfrak{o}^n$, we write $\nu^{(i)} = \nu(\pi_i(\Lambda))$ for the elementary divisor type of the $\mathfrak{O}_i$-sublattice of $\mathfrak{O}_i^{n_i}$ generated by $\pi_i(\Lambda)$. Note that $\nu^{(i)}$ is a partition with n_i parts. The family

$$\boldsymbol{\nu}(\Lambda) = (\nu^{(1)}, \ldots, \nu^{(h)})$$

of partitions is called the *projection data* of Λ with respect to Ω.

For any partition $\nu = (\nu_1, \ldots, \nu_N)$ with N parts, set $J_\nu = \{d \in [N-1] \mid \nu_d > \nu_{d+1}\}$. For a variable Y, we define

$$\beta(\nu; Y) = \binom{N}{J_\nu}_{Y^{-1}} Y^{\sum_{d=1}^{N-1} d(N-d)(\nu_d - \nu_{d+1})} \in \mathbb{Q}[Y]. \tag{4.1}$$

We observe that $\beta(\nu; Y) = \alpha(\lambda, \nu; Y)$, the "Birkhoff polynomial" (2.5), where λ is any partition whose parts are all at least ν_1. It follows that $\beta(\nu; q)$ enumerates the $\mathfrak{o}$-sublattices of $\mathfrak{o}^N$ of elementary divisor type ν. Proposition 4.2, which generalizes this formula, is an analogue of [24, Lemma 2.4] and key to our method. Recall the formula (1.3) for the zeta function of an abelian (Lie) algebra of finite rank over a compact discrete valuation ring.

Proposition 4.2. *Let $\widetilde{\boldsymbol{\nu}} = (\nu^{(1)}, \ldots, \nu^{(h)})$ be as above. Then*

$$\sum_{\substack{\Lambda \leqslant \mathfrak{o}^n \\ \boldsymbol{\nu}(\Lambda) = \widetilde{\boldsymbol{\nu}}}} |\mathfrak{o}^n : \Lambda|^{-s} = \frac{\zeta_{\mathfrak{o}^n}(s)}{\prod_{i=1}^h \zeta_{\mathfrak{O}_i^{n_i}}(s)} \left(\prod_{i=1}^h \beta(\nu^{(i)}; q_i)\right) t^{\sum_{i=1}^h \left(\sum_{j=1}^{n_i} \nu_j^{(i)}\right) f_i}.$$

Proof. Recall that for every $i \in [h]$ there is a natural embedding of rings $\iota_i : \mathfrak{O}_i \hookrightarrow \mathrm{Mat}_{e_i f_i}(\mathfrak{o})$ that sends an element $y \in \mathfrak{O}_i$ to the matrix representing the $\mathfrak{o}$-linear operator $x \mapsto xy$ on $\mathfrak{O}_i$ with respect to the chosen $\mathfrak{o}$-basis $\{\alpha_j^{(i)}\}_{j=1}^{e_i f_i}$. Moreover, $\det \iota_i(y) = \mathrm{N}_{\mathfrak{O}_i/\mathfrak{o}}(y)$ for all $y \in \mathfrak{O}_i$. This map extends naturally to an embedding of matrix rings $\mathrm{Mat}_{n_i}(\mathfrak{O}_i) \hookrightarrow \mathrm{Mat}_{e_i f_i n_i}(\mathfrak{o})$ that we continue to denote by ι_i.

Consider the set $\mathcal{H} = \{(H_1, \ldots, H_h) \mid \forall i \in [h] : H_i \leqslant \mathfrak{O}_i^{n_i}\}$. For $H \in \mathcal{H}$, denote

$$\Sigma_H = \sum_{\substack{\Lambda \leqslant \mathfrak{o}^n \\ \pi_i(\Lambda) = H_i}} |\mathfrak{o}^n : \Lambda|^{-s}.$$

Thus

$$\sum_{\substack{\Lambda \leqslant \mathfrak{o}^n \\ \boldsymbol{\nu}(\Lambda) = \widetilde{\boldsymbol{\nu}}}} |\mathfrak{o}^n : \Lambda|^{-s} = \sum_{\substack{H \in \mathcal{H} \\ \nu(H_i) = \nu^{(i)}}} \Sigma_H. \tag{4.2}$$

For every $i \in [h]$, let $B_i \in \mathrm{Mat}_{n_i}(\mathfrak{O}_i)$ be a matrix whose rows comprise an $\mathfrak{O}_i$-basis of H_i. Let $B \in \mathrm{Mat}_n(\mathfrak{o})$ be the block-diagonal matrix with blocks $\iota_i(B_i)$. We observe that the map $\mathrm{Mat}_n(\mathfrak{o}) \to \mathrm{Mat}_n(\mathfrak{o})$, $B' \mapsto B'B$ induces a bijection between the set of $\mathfrak{o}$-lattices $\Lambda \leqslant \mathfrak{o}^n$ such that $\pi_i(\Lambda) = \mathfrak{O}_i^{n_i}$ for all $i \in [h]$ and the set of lattices $\Lambda \leqslant \mathfrak{o}^n$ such that $\pi_i(\Lambda) = H_i$ for all $i \in [h]$. Furthermore, $\det B = \prod_{i=1}^h \mathrm{N}_{\mathfrak{O}_i/\mathfrak{o}}(\det B_i)$; cf., for instance, [14, Theorem 1]. The norms preserve normalized valuation, and hence $|\det B|_{\mathfrak{o}} = \prod_{i=1}^{h} q_i^{-\sum_{j=1}^{n_i} \nu_j^{(i)}}$. We conclude that

$$\Sigma_H = t^{\sum_{i,j} \nu_j^{(i)} f_i} \Sigma_{\mathbf{0}} = \prod_{i=1}^{h} |\mathfrak{O}_i^{n_i} : H_i|^{-s} \Sigma_{\mathbf{0}}, \tag{4.3}$$

where $\mathbf{0} = (\mathfrak{O}_1^{n_1}, \dots, \mathfrak{O}_h^{n_h}) \in \mathcal{H}$. Thus

$$\zeta_{\mathfrak{o}^n}(s) = \sum_{H \in \mathcal{H}} \Sigma_H = \Sigma_{\mathbf{0}} \sum_{H \in \mathcal{H}} \prod_{i=1}^{h} |\mathfrak{O}_i^{n_i} : H_i|^{-s} = \Sigma_{\mathbf{0}} \prod_{i=1}^{h} \zeta_{\mathfrak{O}_i^{n_i}}(s). \tag{4.4}$$

It follows immediately from (4.3) and (4.4) that

$$\Sigma_H = \frac{\zeta_{\mathfrak{o}^n}(s)}{\prod_{i=1}^{h} \zeta_{\mathfrak{O}_i^{n_i}}(s)} t^{\sum_{i,j} \nu_j^{(i)} f_i},$$

and substitution of this expression into (4.2) implies our claim. □

4.3. **Rewriting the $\mathfrak{o}$-ideal zeta functions of suitable $\mathfrak{o}$-Lie algebras.** Now let L be a class-2-nilpotent $\mathfrak{o}$-Lie algebra. We assume that its derived subalgebra L' is isolated, viz. L/L' is torsion-free. Let further $L' \subseteq A \subseteq Z(L)$ be a central, isolated subalgebra. Suppose that

$$L/A \simeq \mathfrak{O}_1^{n_1} \times \cdots \times \mathfrak{O}_h^{n_h}. \tag{4.5}$$

Fixing such an isomorphism, we obtain projections $\pi_i : L/A \to \mathfrak{O}_i^{n_i}$ as in Section 4.2. Let c' and c be the ranks of the free $\mathfrak{o}$-modules L' and A, respectively. In particular, $n + c = \mathrm{rk}_{\mathfrak{o}} L$.

Given an $\mathfrak{o}$-sublattice $\Lambda \leqslant L/A$ of finite index, the commutator $[\Lambda, L]$ is well-defined, as A is central, and of finite index in L'. Let $\lambda(\Lambda)$ be the $\mathfrak{o}$-elementary divisor type of $[\Lambda, L]$ as $\mathfrak{o}$-submodule of L'.

Definition 4.3. Let $\nu^{(1)} = (\nu_1^{(1)}, \dots, \nu_{n_1}^{(1)})$ and $\nu^{(2)} = (\nu_1^{(2)}, \dots, \nu_{n_2}^{(2)})$ be partitions with n_1 and n_2 parts, respectively. We define $\nu^{(1)} * \nu^{(2)}$ to be the partition whose $n_1 n_2$ parts are obtained from the multiset

$$\left\{ \min\{\nu_k^{(1)}, \nu_\ell^{(2)}\} \right\}_{k \in [n_1],\, \ell \in [n_2]}.$$

Given, in addition, $b \in [n_1]$, we define $(\nu^{(1)})^{*b}$ to be the partition whose $\binom{n_1}{b}$ parts are obtained from the multiset

$$\left\{ \min\{\nu_i^{(1)} \mid i \in I\} \right\}_{I \subseteq [n_1],\, |I| = b}.$$

We observe that $*$ is an associative binary operation on the set of partitions and that $(\nu^{(1)})^{*2} \neq \nu^{(1)} * \nu^{(1)}$.

Definition 4.4. Let $Z \in \mathbb{N}_0$ and fix, for every $k \in [Z]$, a pair $\widetilde{\mathfrak{S}}_k = (\mathfrak{S}_k, \underline{\sigma_k})$, where $\mathfrak{S}_k = \{s_{k1}, \dots, s_{k,\tau_k}\} \subseteq [h]$ is a subset of cardinality τ_k and $\underline{\sigma_k} = (\sigma_{k1}, \dots, \sigma_{k,\tau_k}) \in \mathbb{N}^{\tau_k}$.

Given a family $\widetilde{\boldsymbol{\nu}} = (\nu^{(1)}, \dots, \nu^{(h)})$ of partitions $\nu^{(i)}$, each with n_i parts, define $\lambda(\widetilde{\boldsymbol{\nu}})$ to the be partition obtained from the multiset

$$\bigcup_{k=1}^{Z} \left\{ (\nu^{(s_{k1})})^{*\sigma_{k1}} * \cdots * (\nu^{(s_{k,\tau_k})})^{*\sigma_{k,\tau_k}} \right\},$$

where $\{\nu^{(i)}\}$ denotes the multiset of parts of the partition $\nu^{(i)}$ and the union is a union of multisets.

We will suppose for the rest of Section 4 that the following assumption on (L, A) holds.

Hypothesis 4.5. *The pairs $\widetilde{\mathfrak{S}}_1, \dots, \widetilde{\mathfrak{S}}_Z$ in Definition 4.4 may be chosen so that for any $\mathfrak{o}$-sublattice $\Lambda \leqslant L/A$, the equality of partitions $\lambda(\Lambda) = \lambda(\boldsymbol{\nu}(\Lambda))$ holds.*

Comparing the lengths of the partitions $\lambda(\Lambda)$ and $\lambda(\boldsymbol{\nu}(\Lambda))$, we find that Hypothesis 4.5 implies that

$$c' = \sum_{k=1}^{Z} \binom{n_{s_{k1}}}{\sigma_{k1}} \binom{n_{s_{k2}}}{\sigma_{k2}} \cdots \binom{n_{s_{k,\tau_k}}}{\sigma_{k,\tau_k}}.$$

Definition 4.6. Let $\mathfrak{S} = \bigcup_{k=1}^{Z} \mathfrak{S}_k \subseteq [h]$. Let $m = |\mathfrak{S}|$. Renumbering the components in (4.5) if necessary, we may suppose without loss of generality that $\mathfrak{S} = [m]$.

We briefly discuss the motivation for Hypothesis 4.5. It ensures that the elementary divisor type $\lambda(\Lambda)$ depends only, and only in a combinatorial manner, on the projection data $\boldsymbol{\nu}(\Lambda)$, and that all parts of $\lambda(\Lambda)$ also appear as parts of $\boldsymbol{\nu}(\Lambda)$. This assumption is crucial to our method and enables us to express the $\mathfrak{o}$-ideal zeta function $\zeta_L^{\triangleleft \mathfrak{o}}(s)$ in terms of the generalized Igusa functions of Definition 3.5. A further consequence of Hypothesis 4.5 is a dichotomy among the components of L/A in (4.5). If, on the one hand, $i > m$, then the commutator $[\Lambda, L]$ is independent of the component $\mathfrak{O}_i^{n_i}$; this means that $\mathfrak{O}_i^{n_i}$ lies in the kernel of the projection $\mathrm{pr} : L/A \to L/Z(L)$. If, on the other hand, $i \leqslant m$, then $\mathrm{pr}(\mathfrak{O}_i^{n_i})$ and $\mathfrak{O}_i^{n_i}$ have the same rank as $\mathfrak{o}$-modules, namely $n_i e_i f_i$. In particular,

$$\sum_{i=1}^{m} n_i e_i f_i = \mathrm{rk}_{\mathfrak{o}}(L/Z(L)). \tag{4.6}$$

This consequence of Hypothesis 4.5 is used in a subtle but crucial way in the proof of Corollary 4.22, which establishes the functional equation satisfied by $\zeta_L^{\triangleleft \mathfrak{o}}(s)$. Indeed, Theorem 4.21 expresses $\zeta_L^{\triangleleft \mathfrak{o}}(s)$ as a sum of finitely many summands. The above observation ensures that each summand satisfies a functional equation with the same symmetry factor.

Remark 4.7. We note that, trivially, Hypothesis 4.5 is stable under direct products.

Remark 4.8. Before proceeding, we observe that Hypothesis 4.5 constrains the extensions $\mathfrak{O}_i$ of $\mathfrak{o}$ to be unramified in natural examples, such as the non-abelian examples considered in Section 5. Suppose that $L = \mathcal{L}_1(\mathfrak{O}_1) \times \cdots \times \mathcal{L}_r(\mathfrak{O}_r)$, where $\mathcal{L}_i$ is a class-2-nilpotent Lie ring and $\mathfrak{O}_i$ is a finite extension of $\mathfrak{o}$ for every $i \in [r]$. Suppose that the subalgebra $L' \leqslant A \leqslant Z(L)$ is of the form $A = A_1 \times \cdots \times A_r$, where each A_i is an isolated subalgebra of $\mathcal{L}_i(\mathfrak{O}_i)$; this will be true, for instance, if $A = L'$ or $A = Z(L)$. Then $L/A \simeq \mathcal{L}_1(\mathfrak{O}_1)/A_1 \times \cdots \times \mathcal{L}_r(\mathfrak{O}_r)/A_r$. Suppose, furthermore, that we decompose

$$\begin{array}{rcl} \mathcal{L}_1(\mathfrak{O}_1)/A_1 & \simeq & \mathfrak{O}_1^{n_1} \times \cdots \times \mathfrak{O}_1^{n_{N_1}} \\ \mathcal{L}_2(\mathfrak{O}_2)/A_2 & \simeq & \mathfrak{O}_2^{n_{N_1+1}} \times \cdots \times \mathfrak{O}_2^{n_{N_2}} \\ \vdots & & \vdots \\ \mathcal{L}_r(\mathfrak{O}_r)/A_r & \simeq & \mathfrak{O}_r^{n_{N_{r-1}+1}} \times \cdots \times \mathfrak{O}_r^{n_{N_r}} \end{array}$$

and consider the projection data with respect to the resulting decomposition

$$L/A \simeq \mathfrak{O}_1^{n_1} \times \cdots \times \mathfrak{O}_r^{n_{N_r}}.$$

Here the number of projections is $h = N_r$. Assume that Hypothesis 4.5 is satisfied. We claim that $\mathfrak{O}_i/\mathfrak{o}$ is unramified for all $i \in [r]$ such that $\mathcal{L}_i$ is not abelian.

Indeed, fix uniformizers $\Pi_i \in \mathfrak{O}_i$, let $\tau \in \mathbb{N}$, and consider the lattice

$$\Lambda = \Pi_1^{\tau} \mathfrak{O}_1^{n_1} \times \cdots \times \Pi_1^{\tau} \mathfrak{O}_1^{n_{N_1}} \times \Pi_2^{\tau} \mathfrak{O}_2^{n_{N_1+1}} \times \cdots \times \Pi_r^{\tau} \mathfrak{O}_r^{n_{N_r}}.$$

The projection data are $\nu_j^{(i)} = \tau$ for all $i \in [N_r]$ and all $j \in [n_i]$. Furthermore, it is clear that

$$[\Lambda, L] = \Pi_1^{\tau}[\mathcal{L}_1(\mathfrak{O}_1), \mathcal{L}_1(\mathfrak{O}_1)] \times \cdots \times \Pi_r^{\tau}[\mathcal{L}_r(\mathfrak{O}_r), \mathcal{L}_r(\mathfrak{O}_r)].$$

For every $i \in [r]$, let b_i be the rank of $[\mathcal{L}_i(\mathfrak{O}_i), \mathcal{L}_i(\mathfrak{O}_i)]$ as an $\mathfrak{o}$-module. Then it is immediate from Lemma 2.3 that the partition $\lambda(\Lambda)$ is the disjoint union of the sets $\{\tau\}_{e_i, f_i}$ (cf. Definition 2.4), with respective multiplicities b_i. Suppose that $\mathcal{L}_i$ is not abelian. Then $b_i > 0$.

If, in addition, $e_i \geqslant 2$, then the elements of $\{\tau\}_{e_i,f_i}$ are not all equal to τ. This contradicts Hypothesis 4.5, which implies that all parts of $\lambda(\Lambda)$ are parts of the projection data $\widetilde{\boldsymbol{\nu}}$.

Definition 4.9. Set $\varepsilon = c - c'$. Given partitions λ and μ with c' and c parts, respectively, we say that $\mu \leqslant \lambda$ if $\mu \leqslant \widetilde{\lambda}$, where $\widetilde{\lambda}$ is any partition with c parts whose parts consist of the c' parts of λ together with any ε integers $\xi_1 \geqslant \cdots \geqslant \xi_\varepsilon \geqslant \mu_1$. By $\alpha(\lambda, \mu; Y)$ we will mean $\alpha(\widetilde{\lambda}, \mu; Y)$, the "Birkhoff polynomial" (2.5); note that both definitions are independent of the choice of $\widetilde{\lambda}$.

Our aim, which will be attained with Theorem 4.21, is to compute the $\mathfrak{o}$-ideal zeta function of the $\mathfrak{o}$-Lie algebra L. We maintain the notation from above. Recall, in particular, that $n = \sum_{i=1}^h e_i f_i n_i$ is the $\mathfrak{o}$-rank of L/A. Observe that if $\Lambda \leqslant L/A$ as above, then there exists an $\mathfrak{o}$-sublattice $\mathrm{M} \leqslant A$ of elementary divisor type μ such that $[\Lambda, L] \leqslant \mathrm{M}$ if and only if $\mu \leqslant \lambda(\Lambda)$. Furthermore, as L' is isolated in L, the number of sublattices $\mathrm{M} \leqslant A$ of elementary divisor type μ that contain $[\Lambda, L]$ is given by $\alpha(\lambda(\Lambda), \mu; q)$.

Recall m from Definition 4.6. Given projection data $\widetilde{\boldsymbol{\nu}} = (\nu^{(1)}, \dots, \nu^{(h)})$, the partition $\lambda(\widetilde{\boldsymbol{\nu}})$ depends only on the m-tuple $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(m)})$. Thus we may, and will, write $\lambda(\boldsymbol{\nu})$ for $\lambda(\widetilde{\boldsymbol{\nu}})$.

Proposition 4.10. *Assuming Hypothesis 4.5, the $\mathfrak{o}$-ideal zeta function of L is given by*

$$\zeta_L^{\triangleleft \mathfrak{o}}(s) = \frac{\zeta_{\mathfrak{o}^n}(s)}{\prod_{i=1}^m \zeta_{\mathfrak{O}_i^{n_i}}(s)} \sum_{\substack{\boldsymbol{\nu}, \mu \\ \mu \leqslant \lambda(\boldsymbol{\nu})}} \left(\prod_{i=1}^m \beta(\nu^{(i)}; q_i) \right) \alpha(\lambda(\boldsymbol{\nu}), \mu; q) (q^n t)^{\sum_{k=1}^c \mu_k} t^{\sum_{i=1}^m (\sum_{j=1}^{n_i} \nu_j^{(i)}) f_i}.$$

Here $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(m)})$ runs over all m-tuples of partitions with $n_1, \dots, n_m$ parts, respectively, whereas μ runs over all partitions with c parts. The condition $\mu \leqslant \lambda(\boldsymbol{\nu})$ is to be understood as in Definition 4.9.

Proof. The quotient L/A has $\mathfrak{o}$-rank n, so it follows from [11, Lemma 6.1] that

$$\zeta_L^{\triangleleft \mathfrak{o}}(s) = \sum_{\Lambda \leqslant L/A} |L/A : \Lambda|^{-s} \sum_{[\Lambda, L] \leqslant \mathrm{M} \leqslant A} |A : \mathrm{M}|^{n-s}.$$

Grouping the lattices $\Lambda \leqslant L/A$ by their projection data $\boldsymbol{\nu}(\Lambda)$, we obtain

$$\zeta_L^{\triangleleft \mathfrak{o}}(s) = \sum_{\widetilde{\boldsymbol{\nu}}} \sum_{\substack{\Lambda \leqslant L/A \\ \boldsymbol{\nu}(\Lambda) = \widetilde{\boldsymbol{\nu}}}} |L/A : \Lambda|^{-s} \sum_{[\Lambda, L] \leqslant \mathrm{M} \leqslant A} |A : \mathrm{M}|^{n-s}.$$

Setting μ to be the elementary divisor type of M, it now follows from Proposition 4.2 and the Definition 4.4 of $\lambda(\widetilde{\boldsymbol{\nu}})$ that

$$\zeta_L^{\triangleleft \mathfrak{o}}(s) = \frac{\zeta_{\mathfrak{o}^n}(s)}{\prod_{i=1}^h \zeta_{\mathfrak{O}_i^{n_i}}(s)} \sum_{\substack{\widetilde{\boldsymbol{\nu}}, \mu \\ \mu \leqslant \lambda(\widetilde{\boldsymbol{\nu}})}} \left(\prod_{i=1}^h \beta(\nu^{(i)}; q_i) \right) \alpha(\lambda(\widetilde{\boldsymbol{\nu}}), \mu; q) (q^n t)^{\sum_{k=1}^c \mu_k} t^{\sum_{i=1}^h (\sum_{j=1}^{n_i} \nu_j^{(i)}) f_i}.$$

As we observed above, $\alpha(\lambda(\widetilde{\boldsymbol{\nu}}), \mu; q)$ depends only on the first m components of the h-tuple $\widetilde{\boldsymbol{\nu}}$. Hence the sum in the previous displayed formula may be expressed as

$$\sum_{\substack{\boldsymbol{\nu}, \mu \\ \mu \leqslant \lambda(\boldsymbol{\nu})}} \left(\prod_{i=1}^m \beta(\nu^{(i)}; q_i) \right) \alpha(\lambda(\boldsymbol{\nu}), \mu; q) (q^n t)^{\sum_{k=1}^c \mu_k} t^{\sum_{i=1}^m (\sum_{j=1}^{n_i} \nu_j^{(i)}) f_i} \times \sum_{(\nu^{(m+1)}, \dots, \nu^{(h)})} \left(\prod_{i=m+1}^h \beta(\nu^{(i)}; q_i) \right) t^{\sum_{i=m+1}^h (\sum_{j=1}^{n_i} \nu_j^{(i)}) f_i}. \tag{4.7}$$

Observing that

$$\sum_{\nu^{(i)}} \beta(\nu^{(i)}; q_i) t^{\sum_{j=1}^{n_i} \nu_j^{(i)} f_i} = \sum_{\mathrm{M} \leqslant \mathfrak{O}_i^{n_i}} [\mathfrak{O}_i^{n_i} : \mathrm{M}]^{-s} = \zeta_{\mathfrak{O}_i^{n_i}}(s),$$

we see that the second sum in (4.7) is equal to $\prod_{i=m+1}^h \zeta_{\mathfrak{O}_i^{n_i}}(s)$. The claim follows. $\square$

Let $w \in \mathcal{D}_{2c}$ be a Dyck word. Recall, from Section 2.2, that w is specified by two r-tuples $(L_1, L_2, \ldots, L_r)$ and $(M_1, M_2, \ldots, M_r)$ satisfying $L_i - M_i \geqslant 0$ for all $i \in [r]$ and $L_r = M_r = c$. Recall further that $\varepsilon = c - c'$ and define $\widetilde{L}_j = L_j - \varepsilon$ for all $j \in [r]$.

Definition 4.11. Let λ and μ be partitions with c' and c parts, respectively, and let $w \in \mathcal{D}_{2c}$ such that $L_1 \geqslant \varepsilon$. Fix a partition $\widetilde{\lambda}$ with c parts as above; without loss of generality we may take $\xi_\varepsilon \geqslant \max\{\lambda_1, \mu_1\}$. We say that λ and μ have *overlap type* w, written $w(\lambda, \mu) = w$, if their parts satisfy the following inequalities:

$$\xi_1 \geqslant \cdots \geqslant \xi_\varepsilon \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_{\widetilde{L}_1} \geqslant \mu_1 \geqslant \cdots \geqslant \mu_{M_1} > \lambda_{\widetilde{L}_1+1} \geqslant \cdots \geqslant \lambda_{\widetilde{L}_2} \geqslant \\ \mu_{M_1+1} \geqslant \cdots \geqslant \mu_{M_2} > \cdots > \lambda_{\widetilde{L}_{r-1}+1} \geqslant \cdots \geqslant \lambda_{\widetilde{L}_r} \geqslant \mu_{M_{r-1}+1} \geqslant \cdots \geqslant \mu_{M_r}.$$

In other words, $w(\lambda, \mu) = w$ if $w(\widetilde{\lambda}, \mu) = w$ in the sense of (2.6). Note that $\widetilde{L}_1 = 0$ may occur, if $\varepsilon > 0$. Moreover, the set $\mathcal{D}_{2c}$ depends on c and so on the choice of A.

Observe that $\lambda \geqslant \mu$ if and only if there exists a Dyck word $w \in \mathcal{D}_{2c}$, necessarily unique, such that $w(\lambda, \mu) = w$. Given $w \in \mathcal{D}_{2c}$, we define the function

$$D_w(q,t) = \sum_{\boldsymbol{\nu}} \sum_{\substack{\mu \leqslant \lambda(\boldsymbol{\nu}) \\ w(\lambda,\mu)=w}} \left(\prod_{i=1}^{m} \beta(\nu^{(i)}; q_i) \right) \alpha(\lambda(\boldsymbol{\nu}), \mu; q)(q^n t)^{\sum_{k=1}^{c} \mu_k} t^{\sum_{i=1}^{m} (\sum_{j=1}^{n_i} \nu_j^{(i)}) f_i}. \tag{4.8}$$

Remark 4.12. If w is such that $L_1 < \varepsilon$, then the above sum is empty and so $D_w(p,t) = 0$. In addition, the definition of the partition $\lambda(\boldsymbol{\nu})$ will usually impose some equalities among its parts. Thus, it may happen that the set of projection data $\boldsymbol{\nu}$ whose associated partition $\lambda(\boldsymbol{\nu})$ is compatible with a given Dyck word w is empty even if w satisfies the condition $L_1 \geqslant \varepsilon$ of Definition 4.11. We will see examples of this phenomenon below, e.g. in Section 5.3.2.

Proposition 4.10 now tells us that

$$\zeta_L^{\triangleleft \mathfrak{o}}(s) = \frac{\zeta_{\mathfrak{o}^n}(s)}{\prod_{i=1}^{m} \zeta_{\mathfrak{O}_i^{n_i}}(s)} \sum_{w \in \mathcal{D}_{2c}} D_w(q,t). \tag{4.9}$$

4.4. **An explicit expression for $\zeta_L^{\triangleleft \mathfrak{o}}(s)$.** Our aim in this section is to give explicit formulae for the terms $D_w(q,t)$ in (4.9). We will achieve it with Proposition 4.20—a result whose proof will be given in Section 4.5—leading to a fully explicit formula for the relevant $\mathfrak{o}$-ideal zeta functions in Theorem 4.21.

We maintain the notation of Section 4.3 and resume some introduced in Section 3. Consider the composition $\underline{n} = (n_1, \ldots, n_m)$ and a family $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(m)})$ of partitions $\nu^{(i)}$, each with n_i parts. The natural ordering of the elements of the multiset

$$S = \bigcup_{i=1}^{m} \left\{ \nu_j^{(i)} \mid j \in [n_i] \right\}$$

gives rise to an element $V(\boldsymbol{\nu}) \in \mathrm{WO}_{\underline{n}}$. Indeed, the word $v = \prod_{i=1}^{m} a_i^{\alpha_i} \in C_{\underline{n}}$ appears in the flag $V(\boldsymbol{\nu})$ if and only if any element of the multiset

$$S_v = \bigcup_{i=1}^{m} \left\{ \nu_j^{(i)} \mid j \in [\alpha_i] \right\} \tag{4.10}$$

is larger than any element of the complement $S \backslash S_v = \bigcup_{i=1}^{m} \left\{ \nu_j^{(i)} \mid j \in [\alpha_i + 1, n_i] \right\}$. Given a word $v \in C_{\underline{n}}$, let $m(v)$ denote a minimal element of the multiset S_v. Since, by virtue of Definition 4.4, all parts of $\lambda(\boldsymbol{\nu})$ appear in S, we see that if $k \in \mathbb{N}$ satisfies $\lambda'_k > \lambda'_{k+1}$, then necessarily $k = m(v)$ for some $v \in C_{\underline{n}}$. Here we denote the dual partition of $\lambda(\boldsymbol{\nu})$ by λ' for brevity. Moreover, Hypothesis 4.5 implies that $\lambda'_{m(v)}$ depends only on v and not on the flag $V(\boldsymbol{\nu})$ or on the actual values of the parts $\nu_j^{(i)}$.

Definition 4.13. Let $v \in C_{\underline{n}}$.

(1) Set $\ell(v) = \lambda'_{m(v)}$. In particular, $\ell(v') \leqslant \ell(v)$ if $v' \leqslant v$.
(2) We say that v is *radical* if $\ell(v') < \ell(v)$ for all proper subwords $v' < v$.

Note the following explicit formula for $\ell(v)$.

Lemma 4.14. *Let $v = \prod_{i=1}^m a_i^{\alpha_i} \in C_{\underline{n}}$. Then*

$$\ell(v) = \lambda(\boldsymbol{\nu})'_{m(v)} = \sum_{k=1}^{Z} \prod_{j=1}^{\tau_k} \binom{\alpha_{s_{kj}}}{\sigma_{kj}}.$$

Proof. This is a straightforward consequence of Definition 4.4. □

Definition 4.15. Let $w \in \mathcal{D}_{2c}$ be a Dyck word; cf. Section 2.2. A flag $V = \{v_1 < \cdots < v_t\}$ of elements of $C_{\underline{n}}$ is said to be *compatible* with w, or simply *w-compatible*, if

- $t = r$,
- for all $j \in [r]$, the word v_j is radical and satisfies $\ell(v_j) = \widetilde{L}_j$.

Remark 4.16. It follows from Hypothesis 4.5 that all parts of $\boldsymbol{\nu}$ participate in the minima that determine the parts of $\lambda(\boldsymbol{\nu})$. Therefore, the maximal word $\prod_{i=1}^m a_i^{n_i}$ is always radical, and $v_r = \prod_{i=1}^m a_i^{n_i}$ for any w-compatible flag V.

In addition, note that if $\varepsilon > 0$, i.e. if $L' < A$, then some Dyck words $w \in \mathcal{D}_{2c}$ for which there exist w-compatible flags will satisfy $\tilde{L}_1 = 0$. In this case, $v_1 = \varnothing$ for any such flag.

For $w \in \mathcal{D}_{2c}$, let $\mathcal{F}_w$ denote the set of w-compatible flags. It will be convenient to organize the information carried by an element of $\mathcal{F}_w$ in matrix form. Given an element $V = \{v_1 < \cdots < v_r\} \in \mathcal{F}_w$, we let v_0 be the empty word and define ρ_{ij}, for $i \in [m]$ and $j \in [r]$, by

$$\frac{v_j}{v_{j-1}} = \prod_{i=1}^m a_i^{\rho_{ij}}.$$

In this way, the flag V gives rise to a matrix $\rho(V) \in \mathrm{Mat}_{m,r}(\mathbb{N}_0)$. Conversely, given a matrix $\rho \in \mathrm{Mat}_{m,r}(\mathbb{N}_0)$, we consider the cumulative sums of its rows: for $i \in [m]$ and $j \in [r]$, define

$$\mathrm{P}_{ij} = \sum_{k=1}^{j} \rho_{ik}. \tag{4.11}$$

Definition 4.17. Let $\mathcal{M}_{\underline{n},w} \subseteq \mathrm{Mat}_{m,r}(\mathbb{N}_0)$ be the set of $(\underline{n}, w)$-*admissible compositions*, namely of matrices ρ satisfying the following two properties:

(1) $\ell(\prod_{i=1}^m a_i^{\mathrm{P}_{ij}}) = \widetilde{L}_j$ for all $j \in [r]$.
(2) The word $\prod_{i=1}^m a_i^{\mathrm{P}_{ij}}$ is radical for all $j \in [r]$.

By Remark 4.16, these properties imply that $\mathrm{P}_{ir} = n_i$ for all $i \in [m]$. Set $w_j = \prod_{i=1}^m a_i^{\mathrm{P}_{ij}}$ for all $j \in [r]$. It is easy to see that the map $\mathcal{F}_w \to \mathcal{M}_{\underline{n},w}$ given by $V \mapsto \rho$ is a bijection, with inverse $\rho \mapsto \{w_1 < \cdots < w_r\}$. Denote

$$\mathrm{P}_i = \{\mathrm{P}_{ij} \mid j \in [r]\}$$

for all $i \in [m]$. For $j \in [r]$, we denote by $\underline{\rho_j}$ the following composition with m parts:

$$\underline{\rho_j} = (\rho_{1j}, \ldots, \rho_{mj}). \tag{4.12}$$

Recall from Definition 2.2 the notion of Igusa function and from Definition 3.5 the notion of generalized Igusa function $I^{\mathrm{wo}}_{\underline{\rho_j}}(\mathbf{Y}; \mathbf{X}) \in \mathbb{Q}(Y_1, \ldots, Y_m; (X_v)_{v \leqslant w_j})$.

Definition 4.18. Let $\rho \in \mathcal{M}_{\underline{n},w}$. We define

$$D_{w,\rho}(q,t) = \left(\prod_{i=1}^{m} \binom{n_i}{\mathrm{P}_i}_{q_i^{-1}}\right) \prod_{j=1}^{r} \left(\binom{L_j - M_{j-1}}{L_j - M_j}_{q^{-1}} I^{\mathrm{wo}}_{\underline{\rho_j}}(q_1^{-1},\dots,q_m^{-1};\mathbf{y}^{(j)}) \right) \cdot \prod_{j=1}^{r-1} I^{\circ}_{M_j - M_{j-1}}(q^{-1}; x_{M_{j-1}+1},\dots,x_{M_j}) I_{M_r - M_{r-1}}(q^{-1}; x_{M_{r-1}+1},\dots,x_{M_r}),$$

with numerical data defined as follows. For a subword $v = \prod_{i=1}^m a_i^{\alpha_i}$ of $\prod_{i=1}^m a_i^{\rho_{ij}}$ we set $\alpha_i^{(j)} = \mathrm{P}_{i,j-1} + \alpha_i$ and $v^{(j)} = v \cdot w_{j-1} = \prod_{i=1}^m a_i^{\alpha_i^{(j)}}$. Set

$$\delta_v^{(j)} = \begin{cases} 0, & \text{if } \ell(v^{(j)}) = \ell(w_{j-1}),\\ 1, & \text{otherwise,}\end{cases}$$

and define

$$B_v^{(j)} = \begin{cases} \sum_{i=1}^m f_i \alpha_i (n_i - \alpha_i), & \text{if } \delta_v^{(j)} = 0,\\ \sum_{i=1}^m f_i \alpha_i^{(j)} (n_i - \alpha_i^{(j)}), & \text{if } \delta_v^{(j)} = 1.\end{cases}$$

Finally, we set

$$y_v^{(j)} = q^{\delta_v^{(j)} M_{j-1}(n+\ell(v^{(j)})+\varepsilon - M_{j-1}) + B_v^{(j)}} t^{\sum_{i=1}^m \alpha_i f_i + \delta_v^{(j)}(M_{j-1} + \sum_{i=1}^m \mathrm{P}_{i,j-1} f_i)},$$

where $\ell(v^{(j)})$ is given explicitly by Lemma 4.14. For $k \in [M_{j-1}+1, M_j]$, we set

$$x_k = q^{k(n+L_j-k) + \sum_{i=1}^m f_i \mathrm{P}_{ij}(n_i - \mathrm{P}_{ij})} t^{k + \sum_{i=1}^m f_i \mathrm{P}_{ij}}.$$

Proposition 4.19. *The following functional equation holds:*

$$D_{w,\rho}(q^{-1},t^{-1}) = (-1)^{c+\sum_{i=1}^m n_i} q^{\binom{n+c}{2} - \binom{n}{2} + \sum_{i=1}^m f_i \binom{n_i}{2}} t^{c + 2\sum_{i=1}^m n_i f_i} D_{w,\rho}(q,t).$$

Proof. The proof is a straightforward computation using the functional equations of

(1) Gaussian binomials (2.1),
(2) classical Igusa functions (2.3), (2.4), and
(3) generalized Igusa functions,

as well as the definition (4.11) of P_{ij} and the observation that $\mathrm{P}_{ir} = n_i$ for all $i \in [m]$. □

Recall the functions $D_w(q,t)$ introduced in (4.8) and used to describe the $\mathfrak{o}$-ideal zeta function of L in (4.9). The following result, which constitutes the technical heart of the computation of the ideal zeta function $\zeta_L^{\triangleleft \mathfrak{o}}(s)$, relates $D_w(q,t)$ with the explicit functions $D_{w,\rho}(q,t)$ of Definition 4.18. We defer its proof to the next section.

Proposition 4.20. *Let $w \in \mathcal{D}_{2c}$ be a Dyck word. Then*

$$D_w(q,t) = \sum_{\rho \in \mathcal{M}_{\underline{n},w}} D_{w,\rho}(q,t).$$

Theorem 4.21. *The $\mathfrak{o}$-ideal zeta function of L is*

$$\zeta_L^{\triangleleft \mathfrak{o}}(s) = \frac{\zeta_{\mathfrak{o}^n}(s)}{\prod_{i=1}^m \zeta_{\mathfrak{O}_i^{n_i}}(s)} \sum_{w \in \mathcal{D}_{2c}} \sum_{\rho \in \mathcal{M}_{\underline{n},w}} D_{w,\rho}(q,t).$$

Proof. The claim is immediate from (4.9) and Proposition 4.20. □

Corollary 4.22. *Suppose that the extension $\mathfrak{O}_i/\mathfrak{o}$ is unramified for all $i \in [m]$. Then the $\mathfrak{o}$-ideal zeta function of L satisfies the functional equation*

$$\zeta_L^{\triangleleft \mathfrak{o}}(s)\big|_{q \to q^{-1}} = (-1)^{\mathrm{rk}_{\mathfrak{o}}(L)} q^{\binom{\mathrm{rk}_{\mathfrak{o}}(L)}{2}} t^{\mathrm{rk}_{\mathfrak{o}}(L) + \mathrm{rk}_{\mathfrak{o}}(L/Z(L))} \zeta_L^{\triangleleft \mathfrak{o}}(s).$$

Proof. Recall that $n+c = \mathrm{rk}_{\mathfrak{o}}(L/A) + \mathrm{rk}_{\mathfrak{o}}(A) = \mathrm{rk}_{\mathfrak{o}}(L)$. Observe that the symmetry factor in Proposition 4.19 is independent of w and ρ. Consequently, the sum $\sum_{w\in\mathcal{D}_{2c}}\sum_{\rho\in\mathcal{M}_{\underline{n},w}} D_{w,\rho}(q,t)$ itself satisfies a functional equation with the same symmetry factor. The remaining factors in Theorem 4.21 satisfy

$$\left.\frac{\zeta_{\mathfrak{o}^n}(s)}{\prod_{i=1}^m \zeta_{\mathfrak{O}_i^{n_i}}(s)}\right|_{q\to q^{-1}} = \frac{(-1)^n q^{\binom{n}{2}} t^n}{\prod_{i=1}^m (-1)^{n_i} q^{f_i\binom{n_i}{2}} t^{n_i f_i}} \frac{\zeta_{\mathfrak{o}^n}(s)}{\prod_{i=1}^m \zeta_{\mathfrak{O}_i^{n_i}}(s)}.$$

This yields the functional equation

$$\zeta_L^{\triangleleft \mathfrak{o}}(s)|_{q\to q^{-1}} = (-1)^{\mathrm{rk}_{\mathfrak{o}}(L)} q^{\binom{\mathrm{rk}_{\mathfrak{o}}(L)}{2}} t^{\mathrm{rk}_{\mathfrak{o}}(L)+\sum_{i=1}^m n_i f_i} \zeta_L^{\triangleleft \mathfrak{o}}(s).$$

Since we have assumed that all the extensions $\mathfrak{O}_i/\mathfrak{o}$ are unramified, our claim is now immediate from (4.6). □

Remark 4.23. The explicit formula given in Theorem 4.21 allows one to determine, in principle, the (local) *abscissa of convergence* $\alpha_L^{\triangleleft \mathfrak{o}}$ of the $\mathfrak{o}$-ideal zeta function $\zeta_L^{\triangleleft \mathfrak{o}}(s)$, viz.

$$\alpha_L^{\triangleleft \mathfrak{o}} := \inf\{\alpha\in\mathbb{R}_{>0} \mid \zeta_L^{\triangleleft \mathfrak{o}}(s) \text{ converges on } \{s\in\mathbb{C}\mid \Re(s)>\alpha\}\} \in \mathbb{Q}_{>0}.$$

Indeed, if one writes the rational function $\zeta_L^{\triangleleft \mathfrak{o}}(s)$ over a common denominator of the form $\prod_{(a,b)\in I}(1-q^a t^b)$, with a,b given by the numerical data given in Definition 4.18, then

$$\alpha_L^{\triangleleft \mathfrak{o}} = \max\left\{n, \frac{a}{b} \mid (a,b)\in I\right\}.$$

This reflects the facts that a/b is the abscissa of convergence of the geometric progression $q^{a-bs}/(1-q^{a-bs})$ and that each of the $D_w(q,t)$ is a non-negative linear combination of products of such geometric progressions.

4.5. **Proof of Proposition 4.20.** We start with a lemma involving the notions of Definition 4.13. This observation is simple but crucial to the method of the article.

Lemma 4.24. *Let $v\in C_{\underline{n}}$. There is a unique radical subword $\sqrt{v}\leqslant v$ such that $\ell(\sqrt{v})=\ell(v)$.*

Proof. Suppose $v=\prod_{i=1}^m a_i^{\alpha_i}$. If a binomial coefficient $\binom{\alpha}{\sigma}$ is positive, then it will decrease if α is decreased. It follows that if the k-th term in the sum in the statement of Lemma 4.14 is positive, then in any subword $v'\leqslant v$ satisfying $\ell(v')=\ell(v)$ all the variables $a_{s_{kj}}$ must appear with exponent $\alpha_{s_{kj}}$. Hence we are led to define the set

$$\mathcal{K}_v = \left\{k\in[Z] \mid \alpha_{s_{kj}} \geqslant \sigma_{kj} \text{ for all } j\in[\tau_k]\right\}.$$

Furthermore, we put $\mathfrak{S}_v = \bigcup_{k\in\mathcal{K}_v}\mathfrak{S}_k$ and finally define $\sqrt{v}=\prod_{i\in\mathfrak{S}_v} a_i^{\alpha_i}$. It is clear from the preceding discussion that a subword $v'\leqslant v$ satisfies $\ell(v')=\ell(v)$ if and only if $\sqrt{v}\leqslant v'\leqslant v$. The claimed existence and uniqueness follow. □

Corollary 4.25. *Suppose that $v_1<v_2$ are two elements of $C_{\underline{n}}$ such that $\ell(v_1)=\ell(v_2)$. Then $\sqrt{v_1}=\sqrt{v_2}$.*

Proof. This is immediate from the construction of $\sqrt{v}$ in the proof of Lemma 4.24. □

Fix a Dyck word $w\in\mathcal{D}_{2c}$. We aim to evaluate the function $D_w(q,t)$ of (4.8). Let $\boldsymbol{\nu}=(\nu^{(1)},\dots,\nu^{(m)})$ be an m-tuple of partitions, where, for each $i\in[m]$, the partition $\nu^{(i)}$ has n_i parts. Let μ be a partition with c parts such that $\mu\leqslant\lambda(\boldsymbol{\nu})$ and $w(\lambda(\boldsymbol{\nu}),\mu)=w$, in the sense of Definitions 4.9 and 4.11. To simplify the notation, we write λ for $\lambda(\boldsymbol{\nu})$.

Now let $\{L_j,M_j\}_{j\in[r]}$ be the parameters associated with the Dyck word w. Recall that we have set $L_0=M_0=0$. It follows from the assumption $w(\lambda,\mu)=w$ that $\lambda_{\widetilde{L}_j}>\lambda_{\widetilde{L}_j+1}$ for all $j\in[r-1]$, hence that all the positive $\widetilde{L}_j$ appear as parts of the dual partition λ'. By the observations before Definition 4.13, there exists a subflag $\kappa_1<\kappa_2\cdots<\kappa_r$ of $V(\boldsymbol{\nu})$ such that $\ell(\kappa_j)=\widetilde{L}_j$ for every $j\in[r]$; if $\widetilde{L}_1=0$, then we may take $\kappa_1=\varnothing$. This subflag need not be unique, and its constituent words need not be radical. However, the flag $\sqrt{\kappa_1}<\cdots<\sqrt{\kappa_r}$

is well-defined by Corollary 4.25. Moreover, it is clear that this flag is an element of $\mathcal{F}_w$ and thus corresponds to an $(\underline{n}, w)$-admissible composition $\rho(\boldsymbol{\nu}) \in \mathcal{M}_{\underline{n},w}$.

For every $\rho \in \mathcal{M}_{\underline{n},w}$ we define the function

$$\Delta_{w,\rho}(q,t) = \sum_{\substack{\boldsymbol{\nu} \\ \rho(\boldsymbol{\nu})=\rho}} \sum_{\substack{\mu \leqslant \lambda(\boldsymbol{\nu}) \\ w(\lambda,\mu)=w}} \left(\prod_{i=1}^{m} \beta(\nu^{(i)}; q_i) \right) \alpha(\lambda(\boldsymbol{\nu}), \mu; q)(q^n t)^{\sum_{k=1}^{c} \mu_k} t^{\sum_{i=1}^{m} \sum_{j=1}^{n_i} \nu_j^{(i)}}. \tag{4.13}$$

Clearly, $D_w(q,t) = \sum_{\rho \in \mathcal{M}_{\underline{n},w}} \Delta_{w,\rho}(q,t)$. Hence, to prove Proposition 4.20 it suffices to show the following:

Lemma 4.26. *The equality $\Delta_{w,\rho}(q,t) = D_{w,\rho}(q,t)$ holds for all $\rho \in \mathcal{M}_{\underline{n},w}$.*

Proof. Fix $\rho \in \mathcal{M}_{\underline{n},w}$. For each $j \in [r]$ we define a multiset

$$\mathcal{S}_j = \bigcup_{i=1}^{m} \left\{ \nu_k^{(i)} \mid k \in [\mathrm{P}_{i,j-1}+1, \mathrm{P}_{ij}] \right\}.$$

Recall the compositions $\underline{\rho_j}$ defined in (4.12) above, which depend only on ρ. For each $j \in [r]$, the natural ordering of the elements of $\mathcal{S}_j$ provides a weak order $v_j \in \mathrm{WO}_{\underline{\rho_j}}$. Again, these depend only on the projection data $\boldsymbol{\nu}$, so we denote them $v_j(\boldsymbol{\nu})$ and set $\boldsymbol{v}(\boldsymbol{\nu}) = (v_1(\boldsymbol{\nu}), \ldots, v_r(\boldsymbol{\nu}))$. As in the previous section, we define $w_j = \prod_{i=1}^{m} a_i^{\mathrm{P}_{ij}} \in C_{\underline{n}}$.

Now fix an r-tuple $(v_1, \ldots, v_r) \in \prod_{j=1}^{r} \mathrm{WO}_{\underline{\rho_j}}$. For every $j \in [r]$, suppose that v_j includes the word $\prod_{i=1}^{m} a_i^{\rho_{ij}}$ (except when $\underline{\rho_1}$ is the zero composition, in which case v_1 is empty). Write

$$v_j = \{v_{j1} < v_{j2} < \cdots < v_{j,\ell_j}\}$$

for some $\ell_j \in \mathbb{N}_0$. We define $\widetilde{v}_{jk} = w_{j-1} \cdot v_{jk} \in C_{\underline{n}}$. Consider the set $S_{\widetilde{v}_{jk}}$ and its minimal element $m(\widetilde{v}_{jk})$ as in (4.10). Note that $v_{j,\ell_j} = \prod_{i=1}^{m} a_i^{\rho_{ij}}$ and that consequently $m(\widetilde{v}_{j,\ell_j}) = \lambda_{\widetilde{L}_j}$. Let $\varepsilon_j \in \mathbb{N}$ be the minimal positive integer such that $\ell(\widetilde{v}_{j,\varepsilon_j}) > \widetilde{L}_{j-1}$. Then $m(\widetilde{v}_{j,\varepsilon_j}) = \lambda_{\widetilde{L}_{j-1}+1}$. Observe that $\delta_{v_{jk}}^{(j)} = 0$ in Definition 4.18 if and only if $k < \varepsilon_j$; in this case, $m(\widetilde{v}_{jk})$ is equal to a part of $\boldsymbol{\nu}$ that does not appear in the partition $\lambda(\boldsymbol{\nu})$.

For every element $v_{jk} = \prod_{i=1}^{m} a_i^{\gamma_i} \in C_{\underline{\rho_j}}$, define

$$m(v_{jk}) = \min\{\nu_u^{(i)} \mid u \in [\mathrm{P}_{i,j-1}+1, \mathrm{P}_{i,j-1}+\gamma_i]\}.$$

Note that the elements of the set $\{m(v_{jk}) \mid j \in [r], k \in [\ell_j]\}$ are exactly the parts of the projection data $\boldsymbol{\nu}$. Moreover, if $\delta_{v_{jk}}^{(j)} = 1$, then $m(v_{jk}) = m(\widetilde{v}_{jk})$. Otherwise, it may happen that $m(v_{jk}) > m(\widetilde{v}_{jk})$, as the set defining $m(v_{jk})$ consists entirely of parts of $\boldsymbol{\nu}$ that do not appear in $\lambda(\boldsymbol{\nu})$ and may all be larger than the minimal element of the disjoint set $S_{w_{j-1}}$.

We now define a collection of differences that will provide a convenient parametrization of the pairs $(\boldsymbol{\nu}, \mu)$ that we are considering:

$$s_{jk} = \begin{cases} m(v_{jk}) - m(v_{j,k+1}), & \text{for } k < \ell_j, \\ m(v_{jk}) - \mu_{M_{j-1}+1}, & \text{for } k = \ell_j, \end{cases}$$

$$r_k = \begin{cases} \mu_k - m(v_{j+1,\varepsilon_{j+1}}), & \text{for } k \in \{M_1, \ldots, M_{r-1}\}, \\ \mu_k, & \text{for } k = M_r, \\ \mu_k - \mu_{k+1} & \text{otherwise.} \end{cases}$$

Here the indices of the r_k run over the set $[M_r] = [c]$, whereas the indices of the s_{jk} satisfy $j \in [r]$ and $k \in [\ell_j]$. We emphasize that the r_k have no connection with the parameter r defined earlier. Observe that the s_{jk} and the r_k are all non-negative integers. Moreover, if we allow all the s_{jk} to run over $\mathbb{N}_0$ and all the r_k to run over $\mathbb{N}$ if $k \in \{M_1, \ldots, M_{r-1}\}$ and over $\mathbb{N}_0$ otherwise, then we exactly obtain all the pairs $(\boldsymbol{\nu}, \mu)$ satisfying the following three conditions:

(1) $w(\lambda(\boldsymbol{\nu}), \mu) = w$
(2) $\rho(\boldsymbol{\nu}) = \rho$

(3) $\boldsymbol{v}(\boldsymbol{\nu}) = (v_1, \dots, v_r)$.

Let $\Delta_{w,\rho,\boldsymbol{v}}(q,t)$ be the function defined by the right-hand side of (4.13), except that the sum runs only over the data $\boldsymbol{\nu}$ satisfying $\boldsymbol{v}(\boldsymbol{\nu}) = (v_1, \dots, v_r)$. Our task is now to rewrite the ingredients of $\Delta_{w,\rho,\boldsymbol{v}}(q,t)$, and hence the function itself, in terms of the parameters s_{jk} and r_k. Consider the following collection of intervals:

$$\begin{aligned} &[\mu_k - r_k + 1, \mu_k], && k \in [c], \\ &[m(v_{jk}) - s_{jk} + 1, m(v_{jk})], && j \in [2,r],\ k \in [\varepsilon_j, \ell_j]. \end{aligned} \tag{4.14}$$

The reader will easily verify that these intervals are disjoint and that their union is the interval $[1, \mu_1]$. It follows from this observation that

$$\mu_k = \sum_{b=k}^{c} r_b + \sum_{b=j+1}^{r} \sum_{u=\varepsilon_b}^{\ell_b} s_{bu} \tag{4.15}$$

if $k \in [M_{j-1}+1, M_j]$, whereas if $\nu_d^{(i)} = m(v_{jk})$, then

$$\nu_d^{(i)} = \sum_{u=k}^{\ell_j} s_{ju} + \sum_{b=j+1}^{r} \sum_{u=\varepsilon_b}^{\ell_b} s_{bu} + \sum_{b=M_{j-1}+1}^{c} r_b. \tag{4.16}$$

We now treat the ingredients of $\Delta_{w,\rho,\boldsymbol{\nu}}(q,t)$, starting with the $\beta(\nu^{(i)}; q_i)$. Since $\rho(\boldsymbol{\nu}) = \rho$, it follows that $\{\mathrm{P}_{ij} \mid j \in [r-1]\} \subseteq J_{\nu^{(i)}}$ for all $i \in [m]$. For every $j \in [r]$ define the set

$$J^{(j)}_{\nu^{(i)}} = \{k - \mathrm{P}_{i,j-1} \mid k \in J_{\nu^{(i)}} \cap (\mathrm{P}_{i,j-1}, \mathrm{P}_{ij})\}.$$

Lemma 2.1 implies that

$$\binom{n_i}{J_{\nu^{(i)}}}_{q_i^{-1}} = \binom{n_i}{\mathrm{P}_i}_{q_i^{-1}} \prod_{j=1}^{r} \binom{\rho_{ij}}{J^{(j)}_{\nu^{(i)}}}_{q_i^{-1}}. \tag{4.17}$$

Using (4.15) and (4.16), the differences $\nu_d^{(i)} - \nu_{d+1}^{(i)}$ appearing in the exponents in $\beta(\nu^{(i)}; q_i)$, as defined in (4.1), can be expressed as sums of distinct parameters s_{jk} and r_k. In particular, we observe that the elements of $J^{(j)}_{\nu^{(i)}}$ are precisely the exponents of the variable a_i that occur in the weak order v_j. It then follows from (4.17) that

$$\prod_{i=1}^{m} \binom{n_i}{J_{\nu^{(i)}}}_{q_i^{-1}} = \prod_{i=1}^{m} \binom{n_i}{\mathrm{P}_i}_{q_i^{-1}} \prod_{j=1}^{r} \binom{\rho_j}{v_j}_{\mathbf{Y}},$$

where $\mathbf{Y} = (q_1^{-1}, \dots, q_m^{-1})$ and $\binom{\rho_j}{r_j}_{\mathbf{Y}}$ is as in Definition 3.3. This completes our analysis of the factors $\beta(\nu^{(i)}; q_i)$.

We now consider the factors $\alpha(\lambda(\boldsymbol{\nu}), \mu; q)$, using the idea behind the proofs of [24, Lemmata 2.16 and 2.17]. The range of parameters k over which the infinite product in Definition 2.5 giving $\alpha(\lambda(\boldsymbol{\nu}), \mu; q) = \alpha(\widetilde{\lambda}, \mu; q)$ may have non-trivial factors is precisely $[1, \mu_1]$. Recall that $\widetilde{\lambda}'_k = \lambda'_k + \varepsilon$ for all k and observe that the dual partitions $\widetilde{\lambda}'$ and μ' are constant on each of the intervals of (4.14). Indeed, if $d \in [\mu_k - r_k + 1, \mu_k]$, where $k \in [M_{j-1}+1, M_j]$, then $\widetilde{\lambda}'_d = L_j$ and $\mu'_d = k$. Similarly, if $d \in [m(v_{jk}) - s_{jk} + 1, m(v_{jk})]$ with $k \in [\varepsilon_j, \ell_j]$, then $\lambda'_d = \ell(\widetilde{v}_{jk})$, hence $\widetilde{\lambda}'_d = \ell(\widetilde{v}_{jk}) + \varepsilon$, and $\mu'_d = M_{j-1}$. By manipulations with Gaussian binomials analogous to those above we find that

$$\prod_{k=1}^{\infty} \binom{\widetilde{\lambda}'_k - \mu'_{k+1}}{\widetilde{\lambda}'_k - \mu'_k}_{q^{-1}} = \prod_{j=1}^{r} \binom{L_j - M_{j-1}}{L_j - M_j}_{q^{-1}} \binom{M_j - M_{j-1}}{I^{(j)}_{\mu}}_{q^{-1}},$$

where $I_\mu^{(j)} = \{k - M_{j-1} \mid k \in J_\mu \cap (M_{j-1}, M_j)\} \subset [M_j - M_{j-1} - 1]$. Combining these observations, we obtain

$$\alpha(\lambda(\boldsymbol{\nu}), \mu; q) =$$
$$\prod_{j=1}^{r} \left(\binom{L_j - M_{j-1}}{L_j - M_j}_{q^{-1}} \binom{M_j - M_{j-1}}{I_\mu^{(j)}}_{q^{-1}} \prod_{k=M_{j-1}+1}^{M_j} q^{k(L_j-k)r_k} \prod_{k=\varepsilon_j}^{\ell_j} q^{M_{j-1}(\ell(\tilde{v}_{jk})+\varepsilon-M_{j-1})s_{jk}} \right).$$

The exponents in the remaining factor $(q^n t)^{\sum_{k=1}^{c}\mu_k} t^{\sum_{i=1}^{m}\sum_{j=1}^{n_i}\nu_j^{(i)}}$ of the right-hand side of (4.13) are again readily expressed as sums of parameters r_k and s_{jk} using (4.15) and (4.16). We leave the final assembly as an exercise for the reader. Summing the parameters r_k and s_{jk} over the ranges indicated above, we obtain

$$\Delta_{w,\rho,\boldsymbol{v}}(q,t) = \left(\prod_{i=1}^{m} \binom{n_i}{\mathrm{P}_i}_{q_i^{-1}} \right) \prod_{j=1}^{r} \left(\binom{L_j - M_{j-1}}{L_j - M_j}_{q^{-1}} \binom{\rho_j}{v_j}_{\mathbf{Y}} \prod_{k=1}^{\ell_j} \frac{y^{(j)}_{v_{jk}}}{1 - y^{(j)}_{v_{jk}}} \right) \times$$
$$\prod_{j=1}^{r-1} I^{\circ}_{M_j - M_{j-1}}(q^{-1}; x_{M_{j-1}+1}, \dots, x_{M_j}) \cdot I_{M_r - M_{r-1}}(q^{-1}; x_{M_{t-1}+1}, \dots, x_{M_t}),$$

where the numerical data x_k and $y^{(j)}_{v_{jk}}$ are as given in Definition 4.18. In particular, note that $y^{(j)}_{v_{jk}}$ depends only on the word $\tilde{v}_{jk}$ and not on the weak order v_j. Summation over all r-tuples $\boldsymbol{v} = (v_1, \dots, v_r) \in \prod_{j=1}^{r} \mathrm{WO}_{\underline{\rho_j}}$ now completes the proof of Lemma 4.26, and hence of Proposition 4.20. □

5. Application to the class $\mathfrak{L}$ – proof of Theorem 1.3

In order to deduce Theorem 1.3 from the results of the previous section, namely Theorem 4.21 and Corollary 4.22, it remains to show that Hypothesis 4.5 is satisfied for $\mathfrak{o}$-Lie algebras L as in the statement of Theorem 1.3. We noted in Remark 4.7 that the hypothesis is stable under direct products. Hence it suffices to verify the hypothesis in the case $L = \mathcal{L}(\mathfrak{O}_1) \times \cdots \times \mathcal{L}(\mathfrak{O}_g)$, where $\mathcal{L}$ is a Lie ring from one of the three defining subclasses in Definition 1.1 and $\mathfrak{O}_i$ is a finite extension of $\mathfrak{o}$, for each $i \in [g]$. It is enough to compute the $\mathfrak{o}$-ideal zeta function of L; indeed, the $\mathfrak{O}$-ideal zeta function of $L(\mathfrak{O})$ is obtained from the $\mathfrak{o}$-ideal zeta function of L by substituting q^f for q, where f is the inertia degree of $\mathfrak{O}/\mathfrak{o}$. This verification (and more) is done in Sections 5.2, 5.3, and 5.4. We recover, *en passant*, the results of previous work by several authors.

5.1. **Abelian Lie rings.** It is instructive to consider the output of Theorem 4.21 for the basic example of the abelian $\mathfrak{o}$-Lie algebra $L = \mathfrak{o}^b$. Its zeta function is well-known; cf. (1.3). Let $A \leqslant L$ be an $\mathfrak{o}$-sublattice of rank c with a torsion-free quotient $L/A \simeq \mathfrak{o}^n$; here $n = b - c$. Now, let $h \in \mathbb{N}$ and n_i, e_i, f_i, for $i \in [h]$, be natural numbers such that $\sum_{i=1}^{h} n_i e_i f_i = n$, and let $\mathfrak{O}_1, \dots, \mathfrak{O}_h$ be arbitrary finite extensions of $\mathfrak{o}$ with ramification indices e_i and inertia degrees f_i. Then we may express $L/A \simeq \mathfrak{O}_1^{n_1} \times \cdots \times \mathfrak{O}_h^{n_h}$ as in (4.5). Hypothesis 4.5 is satisfied vacuously, as $c' = 0$. Moreover, $m = 0$ in the sense of Definition 4.6. As $\varepsilon = c$, it follows from Remark 4.12 that the only Dyck word $w \in \mathcal{D}_{2c}$ for which $D_w(q,t) \neq 0$ is the "trivial" word $w = \mathbf{0}^c\mathbf{1}^c$. Since the composition $\underline{n}$ is empty, the only $(\underline{n}, w)$-admissible partition is the empty one. We then read off from Theorem 4.21 that

$$\zeta_L^{\triangleleft \mathfrak{o}}(s) = \zeta_{\mathfrak{o}^n}(s) I_c(q^{-1}; x_1, \dots, x_c),$$

where the numerical data are given by $x_k = q^{k(n+c-k)}t^k = q^{k(b-k)}t^k$. Indeed, it is immediate from (1.3) and (1.4) that

$$I_c(q^{-1}; x_1, \dots, x_c) = \zeta_{\mathfrak{o}^c}(s-n) = \prod_{i=n}^{b-1} \frac{1}{1-q^i t} = \frac{\zeta_{\mathfrak{o}^b}(s)}{\zeta_{\mathfrak{o}^n}(s)}.$$

5.2. **Free class-2-nilpotent Lie rings.** Let $\mathfrak{f}_{2,d}$ denote the free class-2-nilpotent Lie ring on d generators. If $\mathfrak{O}$ is a finite extension of $\mathfrak{o}$ with ramification index e and inertia degree f, then the derived subalgebra of $\mathfrak{f}_{2,d}(\mathfrak{O})$ is isolated and has $\mathfrak{o}$-rank $\binom{d}{2}ef$ and abelianization of $\mathfrak{o}$-rank def. We will now implement the general framework developed in Section 4 to compute the $\mathfrak{o}$-ideal zeta function of the direct product

$$L = \mathfrak{f}_{2,d_1}(\mathfrak{O}_1) \times \cdots \times \mathfrak{f}_{2,d_m}(\mathfrak{O}_m),$$

where $d_i \in \mathbb{N}$ and $\mathfrak{O}_i$ is a finite extension of $\mathfrak{o}$ for all $i \in [m]$. The abelianization of $\mathfrak{f}_{2,d_i}(\mathfrak{O}_i)$ is isomorphic to $\mathfrak{O}_i^{d_i}$ as an $\mathfrak{o}$-module. Thus L satisfies (4.5), with $A = L' = Z(L)$ and $n_i = d_i$ for every $i \in [m]$. We set $\overline{L} = L/L'$ and let $\pi_i : \overline{L} \to \mathfrak{O}_i^{d_i}$ be the projections as in Section 4.2. Let $\Lambda \leqslant \overline{L}$ be a finite-index $\mathfrak{o}$-sublattice and $\nu(\pi_i(\Lambda))$ be the elementary divisor type of the $\mathfrak{O}_i$-sublattice of $\mathfrak{O}_i^{d_i}$ generated by $\pi_i(\Lambda)$. To use the method of the previous section, we must compute the elementary divisor type of the commutator lattice $[\Lambda, L]$.

Lemma 5.1. *Let $L = \mathfrak{f}_{2,d_1}(\mathfrak{O}_1) \times \cdots \times \mathfrak{f}_{2,d_m}(\mathfrak{O}_m)$ and let $\Lambda \leqslant \overline{L}$ be an $\mathfrak{o}$-sublattice. For every $i \in [m]$, let $\nu^{(i)} = \nu(\pi_i(\Lambda)) = (\nu_1^{(i)}, \dots, \nu_{d_i}^{(i)})$. Then the $\mathfrak{o}$-elementary divisor type $\lambda(\Lambda)$ of the commutator $[\Lambda, L] \leqslant L'$ is obtained from the following multiset with $c = \sum_{i=1}^m \binom{d_i}{2} e_i f_i$ elements:*

$$\coprod_{i=1}^{m} \coprod_{1 \leqslant j < k \leqslant d_i} \{\min\{\nu_j^{(i)}, \nu_k^{(i)}\}\}_{e_i, f_i}.$$

Proof. Let $(x_1^{(i)}, \dots, x_{d_i}^{(i)})$ be an $\mathfrak{O}_i$-basis of $\mathfrak{f}_{2,d_i}(\mathfrak{O}_i)$ with respect to which $\pi_i(\Lambda)$ is diagonal:

$$\pi_i(\Lambda) = \langle \Pi_i^{\nu_1^{(i)}} x_1^{(i)}, \dots, \Pi_i^{\nu_{d_i}^{(i)}} x_{d_i}^{(i)} \rangle_{\mathfrak{O}_i},$$

where $\Pi_i \in \mathfrak{O}_i$ is a uniformizer. Observe that the collection of commutators

$$\left\{ [x_j^{(i)}, x_k^{(i)}] \right\}_{1 \leqslant j < k \leqslant d_i}$$

provides an $\mathfrak{O}_i$-basis of the derived subalgebra of $\mathfrak{f}_{2,d_i}(\mathfrak{O}_i)$. Clearly, the commutator subalgebra $[\pi_i(\Lambda), \pi_i(L)]$ is the $\mathfrak{O}_i$-lattice spanned by the elements $\{\Pi_i^{\nu_j^{(i)}} [x_j^{(i)}, x_k^{(i)}]\}_{j \neq k}$. The $\mathfrak{O}_i$-elementary divisor type of this lattice is the partition with parts $\min\{\nu_j^{(i)}, \nu_k^{(i)}\}$, as observed already just before [11, Lemma 5.2]. The elementary divisor type of the same object, viewed as a lattice over $\mathfrak{o}$, is given by the multiset

$$\coprod_{1 \leqslant j < k \leqslant d_i} \{\min\{\nu_j^{(i)}, \nu_k^{(i)}\}\}_{e_i, f_i}$$

by Lemma 2.3. To complete the proof, we observe that the direct product structure of L implies that $[\Lambda, L] = \bigoplus_{i=1}^m [\pi_i(\Lambda), \pi_i(L)]$. □

Remark 5.2. Observe that $\{\nu\}_{1,f}$ is simply the multiset consisting of the element ν with multiplicity f. Therefore, if the extensions $\mathfrak{O}_i/\mathfrak{o}$ are all unramified (i.e. $e_i = 1$ for all i) then it is immediate from Lemma 5.1 that L satisfies Hypothesis 4.5. Indeed, we may set $Z = \sum_{i=1}^m f_i$ and let the collection $\widetilde{\mathfrak{S}}_1, \dots, \widetilde{\mathfrak{S}}_Z$ consist of f_i copies of the pair $(\{i\}, 2)$ for every $i \in [m]$. Moreover, our decomposition of L/A satisfies the conditions of Remark 4.8. Therefore, Hypothesis 4.5 necessarily fails if any of the extensions $\mathfrak{O}_i/\mathfrak{o}$ are ramified, and the method of Section 4 is inapplicable. *We therefore assume for the remainder of Section 5.2 that all the $\mathfrak{O}_i$ are unramified over $\mathfrak{o}$.*

As at the beginning of Section 4.4, the possible orderings of the projection data $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(m)})$ are parametrized by the the chain complex $\mathrm{WO}_{\underline{n}}$ of $C_{\underline{n}}$. Recall the function $\ell(v)$ of Definition 4.13.

Lemma 5.3. *Let $v = \prod_{i=1}^m a_i^{\alpha_i} \in C_{\underline{n}}$. Then $\ell(v) = \sum_{i=1}^m \binom{\alpha_i}{2} f_i$.*

Proof. Let $i \in [m]$. There are exactly α_i parts of the partition $\nu(\pi_i(\Lambda))$ that are not less than $m(v)$, and hence there are $\binom{\alpha_i}{2}$ pairwise minima that are not less than $m(v)$. Each of these minima appears in $\lambda(\Lambda)$ with multiplicity f_i. Alternatively, apply Lemma 4.14 and the description of the sets $\widetilde{\mathfrak{S}}_1, \dots, \widetilde{\mathfrak{S}}_Z$ given in Remark 5.2 above. □

We now have all the ingredients necessary to apply Definition 4.18 and Theorem 4.21 to obtain an explicit expression for $\zeta_L^{\triangleleft\,\mathfrak{o}}(s)$.

Example 5.4. We recover an expression for the $\mathbb{Z}_p$-ideal zeta function of $\mathfrak{f}_{2,d}(\mathbb{Z}_p)$, where $d \geqslant 2$, which was computed by the third author in [31]. The expressions of Theorem 4.21 reduce to a particularly simple form in this case. Here $m = 1$ and $\mathfrak{o} = \mathbb{Z}_p$, and, given a $\mathbb{Z}_p$-sublattice $\Lambda \leqslant \overline{L}$, there is only one relevant projection datum, namely the elementary divisor type $\nu = (\nu_1, \dots, \nu_d)$ of Λ itself. The derived subalgebra has rank $c = \binom{d}{2}$. In view of Lemma 5.3, the parts of the dual partition $\lambda(\Lambda)' = \lambda(\nu)'$ are all triangular numbers. In particular, if $w \in \mathcal{D}_{2c} = \mathcal{D}_{d(d-1)}$ is a Dyck word, then $D_w(p,t) = 0$ unless all the parameters $L_1, \dots, L_r$ associated to w are triangular numbers.

So suppose that $w \in \mathcal{D}_{d(d-1)}$ is such that $L_j = \binom{\gamma_j}{2}$ for all $j \in [r]$. It is easy to see from Definition 4.17 that there is only one (d,w)-admissible composition, namely $\rho_{1j} = \gamma_j - \gamma_{j-1}$ for all $j \in [r]$ (where we have set $\gamma_0 = 0$). Thus $\mathrm{P}_{1j} = \gamma_j$ for all j. Noting from Example 3.6 that the generalized Igusa function associated to a composition with one part is a classical Igusa function in the sense of Definition 2.2, we read off from Definition 4.18 that

$$D_w(p,t) = \prod_{j=1}^{r} \left(\binom{L_j - M_{j-1}}{L_j - M_j}_{p^{-1}} \binom{d}{\gamma_j}_{p^{-1}} I_{\gamma_j - \gamma_{j-1}}(p^{-1}; y_1^{(j)}, \dots, y_{\gamma_j-\gamma_{j-1}}^{(j)}) \right) \cdot$$
$$\prod_{j=1}^{r-1} I^{\circ}_{M_j - M_{j-1}}(p^{-1}; x_{M_{j-1}+1}, \dots, x_{M_j}) \cdot I_{M_r - M_{r-1}}(p^{-1}; x_{M_{r-1}+1}, \dots, x_{M_r}),$$

where

$$y_k^{(j)} = p^{M_{j-1}(d + \binom{\gamma_{j-1}+k}{2} - M_{j-1}) + (\gamma_{j-1}+k)(d - \gamma_{j-1} - k)} t^{\gamma_{j-1}+k+M_{j-1}}$$
$$x_k = p^{k(d + \binom{\gamma_j}{2} - k) + \gamma_j(d-\gamma_j)} t^{k+\gamma_j}.$$

Here, as usual, we have $k \in [M_{j-1}+1, M_j]$ in the definition of x_k. Indeed, observe that the only instance of two distinct subwords $v_1, v_2 \leqslant a_1^d$ satisfying $\ell(v_1) = \ell(v_2)$ is $\ell(\varnothing) = \ell(a_1) = 0$. Thus we always have $\delta_v^{(j)} = 1$ except in the case $\delta_{a_1}^{(1)} = 0$, but it is easy to verify that the uniform expressions given above for the numerical data hold. Finally, by Theorem 4.21,

$$\zeta^{\triangleleft}_{\mathfrak{f}_{2,d}(\mathbb{Z}_p)}(s) = \sum_{w \in \mathcal{D}_{d(d-1)}} D_w(p,t).$$

We leave it as an exercise for the reader to unwind the definitions of [31] and verify that this formula matches [31, Theorem 4].

5.3. **Free class-2-nilpotent products of abelian Lie rings.** Let L_1 and L_2 be abelian Lie rings of ranks d and d', respectively. We denote by $\mathfrak{g}_{d,d'}$ the free class-2-nilpotent product of L_1 and L_2 of nilpotency class at most two. This is the Lie ring version of a group-theoretical construction considered by Levi [16] (see also [10]), which is itself a special case of a varietal product as in [20, Section 1.8]. Concretely, a presentation of $\mathfrak{g}_{d,d'}$ is given by

$$\mathfrak{g}_{d,d'} = \langle x_1, \dots, x_d, y_1, \dots, y_{d'}, (z_{ij})_{i \in [d], j \in [d']} \mid [x_i, y_j] = z_{ij} \rangle,$$

where all Lie brackets not following from the relations above vanish.

Example 5.5.

(1) $\mathfrak{g}_{1,1}$ is the Heisenberg Lie ring.
(2) $\mathfrak{g}_{d,1}$ is the Grenham Lie ring of degree d.
(3) $\mathfrak{g}_{d,0} = \mathbb{Z}^d$ is the abelian Lie ring of rank d.

(4) $\mathfrak{g}_{d,d} = \mathcal{G}_d$ is the Lie ring featuring in [28, Definition 1.2].

We fix $g \in \mathbb{N}$ and g-tuples $\underline{d} = (d_1, \dots, d_g)$ and $\underline{d}' = (d'_1, \dots, d'_g)$ of natural numbers. Let $\mathfrak{O}_1, \dots, \mathfrak{O}_g$ be finite extensions of $\mathfrak{o}$ with ramification indices e_i and inertia degrees f_i, respectively. Consider the $\mathfrak{o}$-Lie algebra

$$L = \mathfrak{g}_{d_1,d'_1}(\mathfrak{O}_1) \times \cdots \times \mathfrak{g}_{d_g,d'_g}(\mathfrak{O}_g).$$

Define $d = \sum_{i=1}^{g} d_i e_i f_i$ and $d' = \sum_{i=1}^{g} d'_i e_i f_i$, and set $c = \sum_{i=1}^{g} d_i d'_i e_i f_i$. Observe that, as an $\mathfrak{o}$-module, L is free of rank $d + d' + c$. Let L' denote the derived subalgebra of L, and let

$$\overline{L} = L/L' \simeq (\mathfrak{O}_1^{d_1} \times \mathfrak{O}_1^{d'_1}) \times (\mathfrak{O}_2^{d_2} \times \mathfrak{O}_2^{d'_2}) \times \cdots \times (\mathfrak{O}_g^{d_g} \times \mathfrak{O}_g^{d'_g})$$

be its abelianization. For each $i \in [g]$, consider the usual basis $\left\{x_k^{(i)}, y_\ell^{(i)}, z_{k\ell}^{(i)}\right\}_{\substack{k\in[d_i]\\ \ell\in[d'_i]}}$ of $\mathfrak{g}_{d_i,d'_i}(\mathfrak{O}_i)$ as an $\mathfrak{O}_i$-module. Consider the natural linear projections

$$\pi_i : \overline{L} \to \langle x_1^{(i)}, \dots, x_{d_i}^{(i)}\rangle_{\mathfrak{O}_i} \simeq \mathfrak{O}_i^{d_i}$$
$$\pi'_i : \overline{L} \to \langle y_1^{(i)}, \dots, y_{d'_i}^{(i)}\rangle_{\mathfrak{O}_i} \simeq \mathfrak{O}_i^{d'_i}.$$

For each $i \in [g]$, fix a $\mathfrak{o}$-basis $(\alpha_1^{(i)}, \dots, \alpha_{e_i f_i}^{(i)})$ of $\mathfrak{O}_i$. Then $\left\{\alpha_j^{(i)} x_k^{(i)}, \alpha_j^{(i)} y_\ell^{(i)}, \alpha_j^{(i)} z_{k\ell}^{(i)}\right\}_{\substack{k\in[d_i],\ell\in[d'_i]\\ j\in[e_i f_i]}}$ is a $\mathfrak{o}$-basis of $\mathfrak{g}_{d_i,d'_i}(\mathfrak{O}_i)$ and the union of these bases is an $\mathfrak{o}$-basis of L.

Let $\Lambda \leqslant \overline{L}$ be an $\mathfrak{o}$-sublattice. For each $i \in [g]$, we let $\nu^{(i)}$, a partition with d_i parts, be the elementary divisor type of the $\mathcal{O}_i$-sublattice of $\mathfrak{O}_i^{d_i}$ generated by $\pi_i(\Lambda)$. Similarly, we set $\nu^{(i+g)}$ to be the elementary divisor type of the $\mathfrak{O}_i$-sublattice of $\mathfrak{O}_i^{d'_i}$ generated by $\pi'_i(\Lambda)$. In other words,

$$\boldsymbol{\nu} = \boldsymbol{\nu}(\Lambda) = (\nu^{(1)}, \nu^{(1+g)}, \nu^{(2)}, \nu^{(2+g)}, \dots, \nu^{(g)}, \nu^{(2g)}) \tag{5.1}$$

is the projection data of Λ as $\mathfrak{o}$-sublattice of $\overline{L}$.

Lemma 5.6. *Let $L = \mathfrak{g}_{d_1,d'_1}(\mathfrak{O}_1) \times \cdots \times \mathfrak{g}_{d_g,d'_g}(\mathfrak{O}_g)$ and let $\Lambda \leqslant \overline{L}$ be an $\mathfrak{o}$-sublattice. Let $\boldsymbol{\nu}(\Lambda)$ be as in* (5.1) *above. Then the $\mathfrak{o}$-elementary divisor type $\lambda(\Lambda)$ of the commutator $[\Lambda, L] \leqslant L'$ is obtained from the following multiset with $c = \sum_{i=1}^{g} d_i d'_i e_i f_i$ elements:*

$$\coprod_{i=1}^{g} \coprod_{k=1}^{d_i d'_i} \{(\nu^{(i)} * \nu^{(i+g)})_k\}_{e_i, f_i},$$

where the operation $*$ is explained in Definition 4.3 and the sets $\{a\}_{e_i,f_i}$, for $a \in \mathbb{N}$, are as in Definition 2.4.

Proof. For every $i \in [g]$, let Π_i denote a uniformizer of $\mathfrak{O}_i$. Let $(\xi_1^{(i)}, \dots, \xi_{d_i}^{(i)})$ and $(\upsilon_1^{(i)}, \dots, \upsilon_{d'_i}^{(i)})$ be bases of $\mathfrak{O}_i^{d_i}$ and $\mathfrak{O}_i^{d'_i}$, respectively, such that

$$\langle \pi_i(\Lambda)\rangle_{\mathfrak{O}_i} = \langle \Pi_i^{\nu_1^{(i)}} \xi_1^{(i)}, \dots, \Pi_i^{\nu_{d_i}^{(i)}} \xi_{d_i}^{(i)}\rangle_{\mathfrak{O}_i}$$
$$\langle \pi'_i(\Lambda)\rangle_{\mathfrak{O}_i} = \langle \Pi_i^{\nu_1^{(i+g)}} \upsilon_1^{(i)}, \dots, \Pi_i^{\nu_{d'_i}^{(i+g)}} \upsilon_{d'_i}^{(i)}\rangle_{\mathfrak{O}_i}.$$

Observe that the commutators $[\xi_k^{(i)}, \upsilon_\ell^{(i)}]$ form an $\mathfrak{O}_i$-basis of the subspace $\langle z_{k\ell}^{(i)}\rangle_{\mathfrak{O}_i}$ of L'. Fixing $k \in [d_i]$, we find that

$$[\Pi_i^{\nu_k^{(i)}} \xi_k^{(i)}, \overline{L}] = \bigoplus_{\ell\in[d'_i]} \Pi_i^{\nu_k^{(i)}} \mathfrak{O}_i[\xi_k^{(i)}, y_\ell^{(i)}] = \bigoplus_{\ell\in[d'_i]} \Pi_i^{\nu_k^{(i)}} \mathfrak{O}_i[\xi_k^{(i)}, \upsilon_\ell^{(i)}].$$

Similarly, for a fixed $\ell \in [d_i']$ we obtain

$$[\Pi_i^{\nu_\ell^{(i+g)}} \upsilon_\ell^{(i)}, \overline{L}] = \bigoplus_{k\in[d_i]} \Pi_i^{\nu_\ell^{(i+g)}} \mathfrak{O}_i[x_k^{(i)}, \upsilon_\ell^{(i)}] = \bigoplus_{k\in[d_i]} \Pi_i^{\nu_\ell^{(i+g)}} \mathfrak{O}_i[\xi_k^{(i)}, \upsilon_\ell^{(i)}].$$

From this we conclude that

$$[\overline{\mathfrak{g}_{d_i,d_i'}(\mathfrak{O}_i)}, \Lambda] = \bigoplus_{k\in[d_i],\ell\in[d_i']} \Pi_i^{\min\{\nu_k^{(i)},\nu_\ell^{(i+g)}\}} \mathfrak{O}_i[\xi_k^{(i)}, \upsilon_\ell^{(i)}] = \bigoplus_{k\in[d_i],\ell\in[d_i']} \Pi_i^{\min\{\nu_k^{(i)},\nu_\ell^{(i+g)}\}} \mathfrak{O}_i z_{k\ell}^{(i)}$$

as $\mathfrak{O}_i$-modules, where $\overline{\mathfrak{g}_{d_i,d_i'}(\mathfrak{O}_i)}$ is the abelianization of $\mathfrak{g}_{d_i,d_i'}(\mathfrak{O}_i)$. Therefore,

$$[\overline{L}, \Lambda] = \bigoplus_{i\in[g],k\in[d_i],\ell\in[d_i']} \Pi_i^{\min\{\nu_k^{(i)},\nu_\ell^{(i+g)}\}} \mathfrak{O}_i z_{k\ell}^{(i)}$$

as $\mathfrak{o}$-modules. The claim follows. □

Set $m = 2g$. For $i \in [g]$, set $\mathfrak{O}_{i+g} = \mathfrak{O}_i$ and define $n_i = d_i$ and $n_{i+g} = d_i'$. It is clear from Lemma 5.6 that the Lie ring L fits the general framework of the beginning of Section 4.3. Moreover, we see analogously to Remark 5.2 that if all the $\mathfrak{O}_i$ are unramified over $\mathfrak{o}$, then Hypothesis 4.5 is satisfied. In this case, we take $Z = \sum_{i=1}^g f_i$; the collection $\widetilde{\mathfrak{S}}_1, \ldots, \widetilde{\mathfrak{S}}_Z$ consists of f_i copies of the pair $((i, i+g), (1,1))$ for every $i \in [g]$. *Thus we assume for the remainder of this section that all the $\mathfrak{O}_i$ are unramified over $\mathfrak{o}$.*

Consider the composition $\underline{n} = (n_1, \ldots, n_{2g})$. Then the natural ordering among all the parts of the projection data $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(2g)})$ corresponds to an element of $\mathrm{WO}_{\underline{n}}$.

Lemma 5.7. *Let $v = \prod_{i=1}^{2g} a_i^{\alpha_i} \in C_{\underline{n}}$. Then $\ell(v) = \sum_{i=1}^g \alpha_i\alpha_{i+g} f_i$.*

Proof. Let $v \in C_{\underline{n}}$ as above. For any $i \in [g]$, the $d_i d_i'$ parts of $\nu^{(i)} * \nu^{(i+g)}$ are, by definition, the minima $\min\{\nu_k^{(i)}, \nu_\ell^{(i+g)}\}_{k\in[d_i],\ell\in[d_i']}$. Clearly, $\min\{\nu_k^{(i)}, \nu_\ell^{(i+g)}\} \geqslant m(v)$ if and only if both elements of the pair $(\nu_k^{(i)}, \nu_\ell^{(i+g)})$ are contained in S_v, and it is clear from (4.10) that there are $\alpha_i\alpha_{i+g}$ such pairs. Finally, since we have assumed all $\mathfrak{O}_i/\mathfrak{o}$ to be unramified, every part of $\nu^{(i)} * \nu^{(i+g)}$ appears in $\lambda(\boldsymbol{\nu})$ with multiplicity f_i. Alternatively, use Lemma 4.14. □

The $\mathfrak{o}$-ideal zeta function $\zeta_L^{\triangleleft\mathfrak{o}}(s)$ may now be read off from Theorem 4.21.

5.3.1. *Grenham Lie rings over unramified extensions.* As an example, we will treat the case $L = \mathfrak{g}_{d,1}(\mathfrak{O})$, where $\mathfrak{g}_{d,1}$ is the Grenham Lie ring of degree d and $\mathfrak{O}/\mathfrak{o}$ is unramified of degree f. In the case $d = f = 2$, this zeta function was computed previously by Bauer, using methods analogous to those of [30] and quite different from the current paper's approach.

Observe that $L' = Z(L)$, so necessarily we have $A = L'$ and thus $c = c' = df$ and $\varepsilon = 0$ in the notation of Section 4.3. The non-empty radical words $v \in C_{(d,1)}$ are exactly those of the form $v = a_1^{\alpha_1} a_2$ with $\alpha_1 > 0$. If $w \in \mathcal{D}_{2c}$ is a Dyck word with associated parameters $L_1, \ldots, L_r$ and $M_1, \ldots, M_r$, then clearly there are no $((d,1), w)$-admissible compositions (recall Definition 4.17) unless all the L_i are divisible by f. Otherwise, there is a unique $((d,1), w)$-admissible composition $\rho \in \mathrm{Mat}_{2,r}$; it satisfies $\mathrm{P}_{1j} = L_j/f$ and $\mathrm{P}_{2j} = 1$ for all $j \in [r]$. Equivalently, $\rho_{1j} = (L_j - L_{j-1})/f$ for all $j \in [r]$, while $\rho_{21} = 1$ and $\rho_{2j} = 0$ for all $j > 1$.

Let $\mathcal{D}_{2c}(f)$ be the set of Dyck words $w \in \mathcal{D}_{2c}$ such that $f | L_i$ for all $i \in [r]$. Given $w \in \mathcal{D}_{2c}(f)$, set $\boldsymbol{L}_w/f = \{L_i/f \mid i \in [r-1]\}$. The following explicit statement is now immediate from Theorem 4.21.

Proposition 5.8. *Let $L = \mathfrak{g}_{d,1}(\mathfrak{O})$, where $\mathfrak{O}/\mathfrak{o}$ is an unramified extension of degree f. Then*

$$\zeta_L^{\triangleleft\mathfrak{o}}(s) = \frac{\zeta_{\mathfrak{o}^{(d+1)f}}(s)}{\zeta_{\mathfrak{O}}(s)\zeta_{\mathfrak{O}^d}(s)} \sum_{w\in\mathcal{D}_{2c}(f)} D_w(q,t),$$

where

$$D_w(q,t) = \binom{d}{\boldsymbol{L}_w/f}_{q^{-f}} \prod_{j=1}^{r} \binom{L_j - M_{j-1}}{L_j - M_j}_{q^{-1}} I^{\mathrm{wo}}_{(L_1/f,1)}(q^{-f}, q^{-f}; \boldsymbol{y}^{(1)}) \cdot \prod_{j=2}^{r} I_{(L_j-L_{j-1})/f}(q^{-f}; y_{(L_{j-1}/f)+1}, \ldots, y_{L_j/f}) \cdot \prod_{j=1}^{r-1} I^{\circ}_{M_j-M_{j-1}}(q^{-1}; x_{M_{j-1}+1}, \ldots, x_{M_j}) I_{M_r-M_{r-1}}(q^{-1}; x_{M_{r-1}+1}, \ldots, x_{M_r}).$$

Here the numerical data are given by

$$x_k = q^{k((d+1)f+L_j-k)+L_j(d-L_j/f)} t^{k+f+L_j},\ k \in [M_{j-1}+1, M_j]$$
$$y^{(1)}_{a_1^{\alpha_1} a_2^{\alpha_2}} = q^{f\alpha_1(d-\alpha_1)} t^{f(\alpha_1+\alpha_2)}$$
$$y_k = q^{M_{j-1}((d+k+k(d-k)+1)f-M_{j-1})} t^{f(k+1)+M_{j-1}},\ k \in [(L_{j-1}/f)+1, L_j/f].$$

5.3.2. *The Lie ring* $\mathfrak{g}_{2,2}$. Paajanen [21, Theorem 11.1] computed the ideal zeta function of the $\mathfrak{o}$-Lie algebra $L = \mathfrak{g}_{2,2}(\mathfrak{o})$. We recover this computation as a special case of our results. By Theorem 4.21 we have

$$\zeta^{\triangleleft \mathfrak{o}}_L(s) = \frac{\zeta_{\mathfrak{o}^4}(s)}{(\zeta_{\mathfrak{o}^2}(s))^2} \sum_{w \in \mathcal{D}_8} \sum_{\rho \in \mathcal{M}_{(2,2),w}} D_{w,\rho}(q,t) = \frac{(1-t)(1-qt)}{(1-q^2t)(1-q^3t)} \sum_{w \in \mathcal{D}_8} \sum_{\rho \in \mathcal{M}_{(2,2),w}} D_{w,\rho}(q,t).$$

There are fourteen Dyck words of length 8, but it is easy to check that there are only five Dyck words $w \in \mathcal{D}_8$ for which there exist w-compatible flags of subwords of the word $a_1^2 a_2^2$. For simplicity, for the rest of this example we will write a instead of a_1 and b instead of a_2. We tabulate these Dyck words, together with the associated functions $D_{w,\rho}(q,t)$. Observe that there are three Dyck words with two compatible flags, and that in each of these cases both flags give rise to the same function $D_{w,\rho}(q,t)$. This is a consequence of the symmetries of $L = \mathfrak{g}_{2,2}(\mathfrak{o})$ and is not a general phenomenon.

For brevity, we use the notation $\mathrm{gp}(x) = \frac{x}{1-x}$ and $\mathrm{gp}_0(x) = \frac{1}{1-x}$.

Dyck word	Flag	$D_{w,\rho}(q,t)$
00001111	a^2b^2	$I^{\mathrm{wo}}_{(2,2)}(q^{-1}; \mathbf{y}) I_4(q^{-1}; q^7t^5, q^{12}t^6, q^{15}t^7, q^{16}t^8)$
00100111	$a^2b < a^2b^2$ $ab^2 < a^2b^2$	$\binom{2}{1}^2_{q^{-1}} I^{\mathrm{wo}}_{(2,1)}(q^{-1}; \mathbf{y}) \mathrm{gp}(q^6t^4) \mathrm{gp}_0(q^7t^5) I_3(q^{-1}; q^{12}t^6, q^{15}t^7, q^{16}t^8)$
00110011	$a^2b < a^2b^2$ $ab^2 < a^2b^2$	$\binom{2}{1}_{q^{-1}} I^{\mathrm{wo}}_{(2,1)}(q^{-1}; \mathbf{y}) I^{\circ}_2(q^{-1}; q^6t^4, q^9t^5) \mathrm{gp}_0(q^{12}t^6) I_2(q^{-1}; q^{15}t^7, q^{16}t^8)$
01000111	$ab < a^2b^2$	$\binom{2}{1}^2_{q^{-1}} I^{\mathrm{wo}}_{(1,1)}(q^{-1}, \mathbf{y}) \mathrm{gp}(q^6t^3) I^{\mathrm{wo}}_{(1,1)}(q^{-1}; \mathbf{z}) I_3(q^{-1}; q^{12}t^6, q^{15}t^7, q^{16}t^8)$
01010011	$ab < a^2b < a^2b^2$ $ab < ab^2 < a^2b^2$	$\binom{2}{1}^2_{q^{-1}} I^{\mathrm{wo}}_{(1,1)}(q^{-1}; \mathbf{y}) \mathrm{gp}(q^6t^3) \mathrm{gp}_0(q^6t^4) \mathrm{gp}(q^9t^5) \mathrm{gp}_0(q^{12}t^6) \times I_2(q^{-1}; q^{15}t^7, q^{16}t^8)$

Here the numerical data $\mathbf{y}$ and $\mathbf{z}$ are defined as follows:

$$y_a = y_b = qt \quad y_{a^2} = y_{b^2} = t^2 \quad y_{ab} = q^2t^2 \quad y_{a^2b} = y_{ab^2} = qt^3 \quad y_{a^2b^2} = t^4,$$
$$z_a = z_b = q^6t^4 \quad z_{ab} = q^7t^5.$$

5.3.3. *The Heisenberg Lie ring.* The relatively free product $\mathfrak{g}_{1,1}$ is the Heisenberg Lie ring $\mathfrak{h}$. This ring is spanned over $\mathbb{Z}$ by three generators x, y, z, with the relations $[x,y] = z$, $[x,z] = [y,z] = 0$. It is among the smallest non-abelian nilpotent Lie rings. It was studied by two of the authors in [24], in the case $\mathfrak{o} = \mathbb{Z}_p$; the zeta functions computed there can be recovered as special cases of the analysis in this section. Indeed, consider

$$L = \mathfrak{h}(\mathfrak{O}_1) \times \cdots \times \mathfrak{h}(\mathfrak{O}_g),$$

where the $\mathfrak{O}_i$ are unramified over $\mathfrak{o}$ so that Hypothesis 4.5 holds. Then $c = \sum_{i=1}^g f_i$, while $n = 2c$. Note that the quantity denoted n in [24] is called c in the current paper. The composition $\underline{n}$ defined just before the statement of Lemma 5.7 is $\underline{n} = (1, 1, \ldots, 1)$, with $2g$ parts. Thus the elements of $C_{\underline{n}}$ correspond to subwords of the word $a_1 \cdots a_{2g}$. The radical subwords are the words of the form $\prod_{i \in J} a_i a_{i+g}$ for some $J \subseteq [g]$. Thus radical subwords are in bijection with subsets of $[g]$. Moreover, if $w \in \mathcal{D}_{2c}$ is a Dyck word, then a w-compatible flag $V = (v_1 < \cdots < v_r) \in \mathcal{F}_w$ corresponds to a sequence of subsets $J_1 \subset \cdots \subset J_r = [g]$ such that $\sum_{i \in J_j \setminus J_{j-1}} f_i = L_j - L_{j-1}$ for all $j \in [r]$. Setting $\mathcal{A}_j = J_j \setminus J_{j-1}$, we obtain precisely the set partitions of $[g]$ that are compatible with w, in the sense of [24, Definition 3.4]. Recall that the set of set partitions compatible with w was denoted $\mathcal{P}_w$ in [24].

We see from Theorem 4.21, applied to $L = \mathfrak{g}_{1,1}(\mathfrak{O}_1) \times \cdots \times \mathfrak{g}_{1,1}(\mathfrak{O}_g)$, that

$$\zeta_L^{\triangleleft \mathfrak{o}}(s) = \frac{\zeta_{\mathfrak{o}^{2c}}(s)}{\prod_{i=1}^g \zeta_{\mathfrak{O}_i}(s)^2} \sum_{w \in \mathcal{D}_{2c}} \sum_{\rho \in \mathcal{M}_{\underline{n},w}} D_{w,\rho}(q,t) = \zeta_{\mathfrak{o}^{2c}}(s) \left(\prod_{i=1}^g (1 - t^{f_i})^2 \right) \sum_{\substack{w \in \mathcal{D}_{2c} \\ \rho \in \mathcal{M}_{\underline{n},w}}} D_{w,\rho}(q,t).$$

Now set $\mathfrak{o} = \mathbb{Z}_p$; in particular, $q = p$. A comparison with [24, eq. (2.20)] and the displayed equation immediately before [24, Theorem 3.6] shows that, to recover the results obtained there, it suffices to prove that if $\rho \in \mathcal{M}_{\underline{n},w}$ is associated to a set partition $\{\mathcal{A}_j\}_{j \in [r]} \in \mathcal{P}_w$, then

$$\left(\prod_{i=1}^g (1 - t^{f_i})^2 \right) D_{w,\rho}(p,t) = \left(\prod_{i=1}^g (1 - t^{2f_i}) \right) D^{\mathbf{f}}_{w,\mathcal{A}}(p,t), \tag{5.2}$$

where $D^{\mathbf{f}}_{w,\mathcal{A}}(p,t)$ is defined by [24, (3.12)].

We read off from Definition 4.18 that, for $\rho \in \mathcal{M}_{\underline{n},w}$,

$$\begin{aligned} D_{w,\rho}(p,t) = \prod_{j=1}^r &\left(\binom{L_j - M_{j-1}}{L_j - M_j}_{p^{-1}} I^{\mathrm{wo}}_{\prod_{k \in \mathcal{A}_j} a_k a_{k+g}}(\mathbf{y}^{(j)}) \right) \cdot \\ &\left(\prod_{j=1}^{r-1} I^{\circ}_{M_j - M_{j-1}}(p^{-1}; x_{M_{j-1}+1}, \ldots, x_{M_j}) \right) I_{M_r - M_{r-1}}(p^{-1}; x_{M_{r-1}+1}, \ldots, x_{M_r}), \end{aligned} \tag{5.3}$$

with the numerical data specified there. Since the parameters q_i^{-1} do not actually appear in the relevant generalized Igusa functions, we have omitted them from the notation (just as in Proposition 3.11). Observe that the numerical data x_k in (5.3) match those in the formula for $D^{\mathbf{f}}_{w,\mathcal{A}}(p,t)$ given in [24, Theorem 3.6]. Moreover, if $r_{\mathcal{J}} = \prod_{k \in \mathcal{J}} a_k a_{k+g}$ is a radical subword of $\prod_{k \in \mathcal{A}_j} a_k a_{k+g}$, then the numerical datum $y^{(j)}_{r_{\mathcal{J}}}$ matches the numerical datum $y^{(j)}_{\mathcal{J}}$ of [24, Theorem 3.6]. In addition, we observe that the numerical data of Definition 4.18 satisfy the hypothesis of Proposition 3.11. Recalling from Example 3.6 how to express the weak order Igusa functions of [24, Definition 2.9] in terms of the generalized Igusa functions of Definition 3.5 above, we find that Proposition 3.11 indeed implies (5.2).

Remark 5.9. Observe that $\mathfrak{h} = \mathfrak{f}_{2,2}$. Thus we can view $L = \mathfrak{f}_{2,2}(\mathfrak{O}_1) \times \cdots \times \mathfrak{f}_{2,2}(\mathfrak{O}_g)$ and obtain an expression for $\zeta_L^{\triangleleft \mathfrak{o}}(s)$ by specializing the analysis of Section 5.2. This expression is not obviously equal to the one obtained above by considering $L = \mathfrak{g}_{1,1}(\mathfrak{O}_1) \times \cdots \times \mathfrak{g}_{1,1}(\mathfrak{O}_g)$ and using the approach of Section 5.3, or to that of [24, Theorem 3.6]. To verify the equality directly, one has to prove identities between generalized Igusa functions that depend on the numerical data, in the style of Proposition 3.11. We leave this as an exercise for the reader.

5.4. **The higher Heisenberg Lie rings.** Let $d \in \mathbb{N}$. The higher Heisenberg Lie ring $\mathfrak{h}_d$ consists of d copies of the Heisenberg Lie ring $\mathfrak{h}$, amalgamated over their centres; in particular $\mathfrak{h}_1 = \mathfrak{h}$. More precisely, $\mathfrak{h}_d$ is spanned over $\mathbb{Z}$ by $2d+1$ elements $x_1, \dots, x_d, y_1, \dots, y_d, z$, with the relations $[x_i, y_i] = z$ for all $i \in [d]$; all other pairs of generators commute. Let

$$L = \mathfrak{h}_{d_1}(\mathfrak{O}_1) \times \cdots \times \mathfrak{h}_{d_g}(\mathfrak{O}_g),$$

where $(d_1, \dots, d_g) \in \mathbb{N}^g$ and each $\mathfrak{O}_i$ is a finite, not necessarily unramified extension of $\mathfrak{o}$. In the case of $d_1 = \cdots = d_g$ and $\mathfrak{o} = \mathfrak{O}_1 = \cdots = \mathfrak{O}_g = \mathbb{Z}_p$, the zeta function $\zeta_L^{\triangleleft \mathfrak{o}}(s)$ was computed by Bauer in his unpublished M.Sc. thesis [1] by adapting the methods of [24]. Observe that

$$\overline{L} \simeq \mathfrak{O}_1^{d_1} \times \mathfrak{O}_1^{d_1} \times \cdots \times \mathfrak{O}_g^{d_g} \times \mathfrak{O}_g^{d_g} = \underbrace{\mathfrak{O}_1 \times \cdots \times \mathfrak{O}_1}_{2d_1 \text{ copies}} \times \cdots \times \underbrace{\mathfrak{O}_g \times \cdots \times \mathfrak{O}_g}_{2d_g \text{ copies}}. \tag{5.4}$$

Set $S_i = \sum_{j=1}^{i} 2d_j$. We have naturally expressed $\overline{L}$ as a product of S_g submodules, giving rise to projections $\pi_1, \dots, \pi_{S_g}$ as in Section 4.3, where $\pi_k : \overline{L} \to \mathfrak{O}_i$ when $S_{i-1} < k \leqslant S_i$. Let $\Lambda \leqslant \overline{L}$ be an $\mathfrak{o}$-sublattice, and let $\boldsymbol{\nu}(\Lambda) = (\nu^{(1)}, \dots, \nu^{(S_g)})$ be the corresponding projection data with respect to (5.4); each of these S_g partitions has only one part. Note that $L' = Z(L)$ has rank $c = \sum_{i=1}^{g} e_i f_i$ as an $\mathfrak{o}$-module.

Lemma 5.10. *Let $\Lambda \leqslant \overline{L}$ be an $\mathfrak{o}$-sublattice. The $\mathfrak{o}$-elementary divisor type $\lambda(\Lambda)$ of the commutator $[\Lambda, \overline{L}] \leqslant L'$ is obtained from the following multiset with c elements:*

$$\coprod_{i=1}^{g} \left\{ \min\{\nu_1^{(S_{i-1}+1)}, \nu_1^{(S_{i-1}+2)}, \dots, \nu_1^{(S_i)}\}_{e_i, f_i} \right\}.$$

Proof. Let $(x_1^{(i)}, \dots, x_{d_i}^{(i)}, y_1^{(i)}, \dots, y_{d_i}^{(i)}, z^{(i)})$ be the natural basis of $\mathfrak{h}_{d_i}(\mathfrak{O}_i)$ as an $\mathfrak{O}_i$-module. Let the decomposition (5.4) be such that, for every $k \in [d_i]$, the images of $\pi_{S_{i-1}+k}$ and $\pi_{S_{i-1}+d_i+k}$ are $\mathfrak{O}_i x_k^{(i)}$ and $\mathfrak{O}_i y_k^{(i)}$, respectively. If $\Pi_i \in \mathfrak{O}_i$ is a uniformizer, then it is clear that, for all $i \in [g]$ and all $k \in [d_i]$,

$$[\Lambda, \mathfrak{O}_i x_k^{(i)}] = \Pi_i^{\nu_1^{(S_{i-1}+g_i+k)}} \mathfrak{O}_i z^{(i)}$$
$$[\Lambda, \mathfrak{O}_i y_k^{(i)}] = \Pi_i^{\nu_1^{(S_{i-1}+k)}} \mathfrak{O}_i z^{(i)}.$$

The claim follows. ☐

It is immediate from the previous lemma that Hypothesis 4.5 is satisfied if all the extensions $\mathfrak{O}_i/\mathfrak{o}$ are unramified. In this case, we set $Z = \sum_{i=1}^{g} f_i$ and take the collection $\widetilde{\mathfrak{S}}_1, \dots, \widetilde{\mathfrak{S}}_Z$ to consist of f_i copies of the pair $([S_{i-1}+1, S_i], (1, 1, \dots, 1))$ for every $i \in [g]$. The following is then given by Lemma 4.14.

Lemma 5.11. *Let $v = \prod_{k=1}^{S_g} a_k^{\alpha_k} \in C_{\underline{n}}$. Then $\ell(v) = \sum_{i=1}^{g} \left(\prod_{k=S_{i-1}+1}^{S_i} \alpha_k \right) f_i$.*

An explicit expression for $\zeta_L^{\triangleleft \mathfrak{o}}(s)$ can now be obtained from Theorem 4.21.

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