Algebraic Number Theory (88-798)

5773 Semester A

Question Sheet 4

Due 29/11/2012, ט"ו בכסלו תשע"ג

(1) Let $V = \mathbb{R}^n$, and let $\Gamma = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n \subset V$ be a complete lattice. Let A be the base change matrix from the standard basis $\{e_1, \ldots, e_n\}$ to $\{v_1, \ldots, v_n\}$. Define the volume of Γ to be $\mathrm{vol}(\Gamma) = |\det A|$. Prove that this is independent of the choice of $\{v_1, \ldots, v_n\}$ and that $\mathrm{vol}(\Gamma)$ is equal to the volume of the fundamental domain

$$\Phi = \{a_1 v_1 + \cdots + a_n v_n : 0 \le a_i < 1\}.$$

(2) Let $K = \mathbb{Q}(\sqrt{65})$. The purpose of this and the three following exercises is to determine the class number h_K, illustrating a technique that can be used quite generally. Define the set

$$S = \{x \in \mathcal{O}_K : N_{K/\mathbb{Q}}(x) = \pm 2\}.$$

Find a unit $u \in \mathcal{O}_K^*$ that is not a root of unity and show that multiplication by u and u^{-1} preserves S.

(3) Let $\sigma_1, \sigma_2 : K \hookrightarrow \mathbb{R}$ be the two embeddings of K into $\mathbb{R}$, and consider the map

$$\begin{array}{rcl} \lambda : \mathcal{O}_K \backslash \{0\} & \to & \mathbb{R} \times \mathbb{R} \\ x & \mapsto & (\log|\sigma_1(x)|, \log|\sigma_2(x)|). \end{array}$$

Consider the action of u and u^{-1} on S by multiplication and find a number $c > 0$ such that if $S \neq \varnothing$, then there exists $x \in S$ such that

$$\lambda(x) \in \{(z_1, z_2) \in \mathbb{R} \times \mathbb{R} : |z_1| < c, |z_2| < c\}.$$

(4) Find an integer N such that if $x = a + b\sqrt{65} \in S$ is the element found in the previous exercise, then $|a| < N$ and $|b| < N$.

Since $x \in \mathcal{O}_K$ and therefore $a, b \in \frac{1}{2}\mathbb{Z}$, there are only finitely many elements $x = a + b\sqrt{65} \in \mathcal{O}_K$ satisfying the condition that $|a|, |b| < N$. Write a simple computer program to check that none of these elements have norm ± 2. (If you don't know how to program, don't bother.) Conclude that $S = \varnothing$.

(5) Use the fact that $S = \varnothing$ to prove that $h_K = 2$.

Hint: It may be helpful to prove that $3\mathcal{O}_K$ is prime and that $2\mathcal{O}_K$ decomposes into a product of two distinct prime ideals.

(6) Let p be a prime number such that $p \equiv 5 \bmod 12$, and let $K = \mathbb{Q}(\sqrt{-p})$. Suppose that $p > 3^m$. Prove that $h_K \ge m$.

Hint: Let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime ideal dividing $3\mathcal{O}_K$. Prove that the corresponding class in Cl_K has order at least m.