Algebraic Number Theory (88-798)
5777 Semester B
Question Sheet 2
Due 7/5/2017, י"א באייר תשע"ז

(1) Let A be a Dedekind domain. Let $I, J \subset A$ be ideals. We say that $J|I$ if there exists an ideal $J' \subset A$ such that $JJ' = I$. Prove that $J|I$ if and only if $I \subseteq J$.

(2) Let A be a Dedekind domain. Prove that A is a PID (principal ideal domain) if and only if it is a UFD (unique factorization domain).

(3) Let D be a square-free integer, and let $K = \mathbb{Q}(\sqrt{D})$. Prove that

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\sqrt{D}] & : D \equiv 2, 3 \bmod 4 \\ \mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right] & : D \equiv 1 \bmod 4. \end{cases}$$

Find the discriminant of K.

(4) Prove that every UFD is integrally closed.

(5) Let K be a number field with $n = [K : \mathbb{Q}]$, and let p be a prime number. Prove that there are at most n prime ideals $P \subset \mathcal{O}_K$ such that $p\mathcal{O}_K \subseteq P$.

(6) Let K be a number field and let $x \in \mathcal{O}_K$. Prove that $N(x\mathcal{O}_K) = |N_{K/\mathbb{Q}}(x)|$.

(7) Let $r, s \geq 0$ be integers such that $r + 2s = n$. Let $t \in \mathbb{R}$, and let $X_t \subset \mathbb{R}^n$ be the set

$$X_t = \left\{ (x_1, \ldots, x_r, y_1, z_1, \ldots, y_s, z_s) : |x_1| + \cdots + |x_r| + 2\sqrt{y_1^2 + z_1^2} + \cdots + 2\sqrt{y_s^2 + z_s^2} < t \right\}.$$

Prove that X_t is bounded, convex, and symmetric. Recall that the volume of X_t is $\mathrm{vol}(X_t) = \int_{X_t} 1 \cdot dx_1 \cdots dz_s$. Prove that

$$\mathrm{vol}(X_t) = \frac{2^{r-s}\pi^s t^n}{n!}.$$

Hint: Change to polar coordinates and use induction on r.

(8) Let $I \subset \mathcal{O}_K$ be a non-zero ideal, and let $\mathcal{C} \in \mathrm{Cl}_K$ be any class in the class group. Prove that there exists an ideal $J \subset \mathcal{O}_K$ such that $J \in \mathcal{C}$ and J is co-prime to I.

(9) Let K be a number field. Prove that if $|d_K| \leq 1$, then $K = \mathbb{Q}$.

(10) Prove that the rings of integers of the fields $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-3})$, and $\mathbb{Q}(\sqrt{-7})$ are all PID's.

(11) Prove that if $K = \mathbb{Q}(\sqrt{-5})$, then $h_K = 2$.

(12) Prove that there are no integers x, y such that $x^2 + 5 = y^3$.

Hint: Suppose that such $x, y \in \mathbb{Z}$ do exist. Prove that they must be relatively prime, that x is even, and that y is odd. Consider the ideals $I, J \subset \mathbb{Z}[\sqrt{-5}]$ given by $I = (x + \sqrt{-5})$ and $J = (x - \sqrt{-5})$. Note that $IJ = (y)^3$. Prove that I and J must be relatively prime. It follows that $I = (I')^3$ for some ideal $I' \subset \mathcal{O}_K$. Now use the result of the previous exercise.