Algebraic Number Theory (88-798)
5779 Semester A
Question Sheet 2

(1) Let A be a Dedekind domain. Let $I, J \subset A$ be ideals. We say that $J|I$ if there exists an ideal $J' \subset A$ such that $JJ' = I$. Prove that $J|I$ if and only if $I \subseteq J$.

(2) Let A be a Dedekind domain. Prove that A is a PID (principal ideal domain) if and only if it is a UFD (unique factorization domain).

(3) Let K be a number field with $n = [K : \mathbb{Q}]$, and let p be a prime number. Prove that there are at most n prime ideals $P \subset \mathcal{O}_K$ such that $p\mathcal{O}_K \subseteq P$.

(4) Let K be a number field and let $x \in \mathcal{O}_K$. Prove that $N(x\mathcal{O}_K) = |N_{K/\mathbb{Q}}(x)|$.

(5) Let $r, s \geq 0$ be integers such that $r + 2s = n$. Let $t \in \mathbb{R}$, and let $X_t \subset \mathbb{R}^n$ be the set

$$X_t = \left\{(x_1, \ldots, x_r, y_1, z_1, \ldots, y_s, z_s) : |x_1| + \cdots + |x_r| + 2\sqrt{y_1^2 + z_1^2} + \cdots + 2\sqrt{y_s^2 + z_s^2} < t\right\}.$$

Prove that X_t is bounded, convex, and symmetric. Recall that the volume of X_t is $\mathrm{vol}(X_t) = \int_{X_t} 1 \cdot dx_1 \cdots dz_s$. Prove that

$$\mathrm{vol}(X_t) = \frac{2^{r-s}\pi^s t^n}{n!}.$$

Hint: Change to polar coordinates and use induction on r.

(6) Let $I \subset \mathcal{O}_K$ be a non-zero ideal, and let $\mathcal{C} \in \mathrm{Cl}_K$ be any class in the class group. Prove that there exists an ideal $J \subset \mathcal{O}_K$ such that $J \in \mathcal{C}$ and J is co-prime to I.

(7) Let K be a number field. Prove that if $|d_K| \leq 1$, then $K = \mathbb{Q}$.

(8) Let K be a number field. In this exercise we will give a proof of the finiteness of Cl_K that avoids the geometry of numbers.

(a) Let $x_1, \ldots, x_n$ be an integral basis of K, and let $a = c_1x_1 + \cdots c_na_n \in K$. Consider the map

$$\begin{aligned} \varphi : \mathbb{Z} &\to [0,1]^n \\ t &\mapsto (\{tc_1\}, \ldots, \{tc_n\}), \end{aligned}$$

where $\{y\}$ denotes the fractional part of $y \in \mathbb{Q}$. Subdivide each edge of the n-cube $[0,1]^n$ into L equal segments (for some $L \in \mathbb{N}$ to be chosen judiciously later). Let $\sigma_1, \ldots, \sigma_n$ be the embeddings $K \hookrightarrow \mathbb{C}$. Using the pigeonhole principle, show that there exists an integer $1 \leq t \leq L^n$ and $b \in \mathcal{O}_K$ such that

$$|N_{K/\mathbb{Q}}(ta - b)| \leq \frac{1}{L^n} \sum_{i_1=1}^{n} \cdots \sum_{i_n=1}^{n} \prod_{j=1}^{n} |\sigma_j(x_{i_j})|.$$

(b) Fix some $H_K > \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n \prod_{j=1}^n |\sigma_j(x_{i_j})|$ and observe that this condition is independent of a. Show that for any $a \in K$ there exists $b \in \mathcal{O}_K$ and an integer $1 \leq t \leq H_K$

such that $|N_{K/\mathbb{Q}}(ta-b)|<1$. This statement should be viewed as a basic Diophantine approximation theorem: every $a \in K$ is close to a fraction with bounded denominator.

(c) Let $I \subset \mathcal{O}_K$ be a non-zero ideal, and fix $0 \neq y \in I$ such that $|N_{K/\mathbb{Q}}(y)|$ is minimal. Let $a \in I$ be arbitrary. Using the previous part of the question, show that there exists $w \in \mathcal{O}_K$ and an integer t such that $|t| \leq H_K$ and $ta = wy$.

(d) Conclude that every class in Cl_K contains an (integral) ideal J satisfying $N(J) \leq (H_K!)^n$. As we discussed in the lecture, this implies that Cl_K is finite.