Algebraic Number Theory (88-798)

5779 Semester A

Question Sheet 3

(1) Prove that the rings of integers of the fields $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-3})$, and $\mathbb{Q}(\sqrt{-7})$ are all principal ideal domains.

(2) Find the class number of the field $\mathbb{Q}(\sqrt{6})$.

(3) Prove that if $K = \mathbb{Q}(\sqrt{-5})$, then $h_K = 2$.

(4) Prove that there are no integers x, y such that $x^2 + 5 = y^3$.

Hint: Suppose that such $x, y \in \mathbb{Z}$ do exist. Prove that they must be relatively prime, that x is even, and that y is odd. Consider the ideals $I, J \subset \mathbb{Z}[\sqrt{-5}]$ given by $I = (x+\sqrt{-5})$ and $J = (x - \sqrt{-5})$. Note that $IJ = (y)^3$. Prove that I and J must be relatively prime. It follows that $I = (I')^3$ for some ideal $I' \subset \mathcal{O}_K$. Now use the result of the previous exercise.

(5) Let p be a prime number such that $p \equiv 2 \bmod 3$, and let $K = \mathbb{Q}(\sqrt{-p})$. Suppose that $p > 3^m$. Prove that $h_K > m$.

Hint: Let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime ideal dividing $3\mathcal{O}_K$. Consider its class in Cl_K.

(6) Let $K = \mathbb{Q}(\sqrt{65})$. The purpose of this and the three following exercises is to determine the class number h_K, illustrating a technique that can be used quite generally. Define the set

$$S = \{x \in \mathcal{O}_K : N_{K/\mathbb{Q}}(x) = \pm 2\}.$$

Find a unit $u \in \mathcal{O}_K^*$ that is not a root of unity and show that multiplication by u and u^{-1} preserves S.

(7) Let $\sigma_1, \sigma_2 : K \hookrightarrow \mathbb{R}$ be the two embeddings of K into $\mathbb{R}$, and consider the map

$$\begin{aligned} \lambda : \mathcal{O}_K \backslash \{0\} &\to \mathbb{R} \times \mathbb{R} \\ x &\mapsto (\log|\sigma_1(x)|, \log|\sigma_2(x)|). \end{aligned}$$

Consider the action of u and u^{-1} on S by multiplication and find a number $c > 0$ such that if $S \neq \varnothing$, then there exists $x \in S$ such that

$$\lambda(x) \in \{(z_1, z_2) \in \mathbb{R} \times \mathbb{R} : |z_1| < c, |z_2| < c\}.$$

(8) Find an integer N such that if $x = a + b\sqrt{65} \in S$ is the element found in the previous exercise, then $|a| < N$ and $|b| < N$.

Since $x \in \mathcal{O}_K$ and therefore $a, b \in \frac{1}{2}\mathbb{Z}$, there are only finitely many elements $x = a + b\sqrt{65} \in \mathcal{O}_K$ satisfying the condition that $|a|, |b| < N$. Write a simple computer program to check that none of these elements have norm ± 2. Conclude that $S = \varnothing$.

(9) Use the fact that $S = \varnothing$ to prove that $h_K = 2$.

Hint: It may be helpful to prove that $3\mathcal{O}_K$ is prime and that $2\mathcal{O}_K$ decomposes into a product of two distinct prime ideals.