Commutative Algebra 88-813

5769 Semester A

Question Sheet 1, due 23/11/2008

Please feel free to e-mail me at `mschein@math.biu.ac.il` with any questions of translation or otherwise.

(1) Prove the Five Lemma: Let R be a ring and consider the following commutative diagram of R-modules:

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{g_1} & A_2 & \xrightarrow{g_2} & A_3 & \xrightarrow{g_3} & A_4 & \xrightarrow{g_4} & A_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
B_1 & \xrightarrow[h_1]{} & B_2 & \xrightarrow[h_2]{} & B_3 & \xrightarrow[h_3]{} & B_4 & \xrightarrow[h_4]{} & B_5.
\end{array}$$

Suppose that the two rows are exact and that f_1, f_2, f_4, and f_5 are isomorphisms. Then f_3 is also an isomorphism.

(2) Let M be a finitely generated R-module. Prove that M has a maximal submodule.

(3) Recall that $l(M)$ is the composition length of a module M. If $M = M_1 \oplus M_2$, prove that $l(M) = l(M_1) + l(M_2)$.

(4) Let R be a commutative ring and let M be an R-module generated by m elements. Suppose that there is a surjective map $\varphi : M \to R^{(n)}$. Prove that $m \geq n$.

(5) Let R be a commutative ring. An element $s \in R$ is called *regular* if $sr \neq 0$ for all $0 \neq r \in R$. Let M be an R-module, and define $\mathrm{tor}(M) = \{m \in M : sm = 0 \text{ for some regular } s \in R\}$. Prove the following:

   (a) $\mathrm{tor}(M)$ is a submodule of M, and $\mathrm{tor}(R) = 0$.

   (b) $\mathrm{tor}(M_1 \oplus M_2) \simeq \mathrm{tor}(M_1) \oplus \mathrm{tor}(M_2)$.

   (c) If $A = M/\mathrm{tor}(M)$ is a free R-module, then $M \simeq A \oplus \mathrm{tor}(M)$.

(6) Let R be a commutative ring and $r, s \in R$. Prove that a homomorphism $R/Rr \to R/Rs$ exists if and only if $Rr \subseteq Rs$. In particular, $R/Rr \simeq R/Rs$ if and only if $Rr = Rs$.

(7) Let N_1, N_2, and K be submodules of M such that $N_1 \supset N_2$ and $K \cap N_1 = 0$. Then $(K + N_1)/(K + N_2) \simeq N_1/N_2$.

(8) Complete this alternative proof of the Schreier-Jordan-Hölder Theorem. Suppose that M has a composition series $M = M_0 \supset M_1 \supset \cdots \supset M_t = 0$, denoted $\mathcal{C}$. Let $M = N_0 \supset N_1 \cdots \supset N_k \supset 0$ be an arbitrary chain of submodules, denoted $\mathcal{D}$. We wish to prove that $\mathcal{D}$ can be refined (אפשר לעדן) to a composition series equivalent to $\mathcal{C}$.

Consider the quotient module $\overline{M} = M/M_{t-1}$. For a submodule $N \subset M$, we write $\overline{N}$ for its image in $\overline{M}$. Prove that $\overline{M} = \overline{M_0} \supset \overline{M_1} \supset \cdots \supset \overline{M_{t-1}} = 0$ is a composition series for

$\overline{M}$. Show that $\overline{N_i} = (N_i + M_{t-1})/M_{t-1}$. Consider the chain $\overline{M} = \overline{N_0} \supseteq \overline{N_1} \supseteq \cdots \supseteq \overline{N_k}$, which we call $\overline{\mathcal{D}}$.

Let j be the largest integer such that $N_j \supseteq M_{t-1}$. Show that the desired claim is obvious if $j \geq k$. So assume $j < k$. For all $i > j$, prove that $N_i/N_{i+1} \simeq (N_i + M_{t-1})/(N_{i+1} + M_{t-1}) \simeq \overline{N_i}/\overline{N_{i+1}}$. Deduce that, for $i > j$, $\overline{N_i} \supset \overline{N_{i+1}}$ is a strict inclusion.

Use induction on t, applied to $\overline{M}$, to show that $k - 1 \leq l(\overline{\mathcal{D}}) \leq t - 1$, and deduce that $\mathcal{D}$ can be refined to a composition series equivalent to $\mathcal{C}$.

To show that all composition series for M are equivalent, suppose that $\mathcal{D}$ was a composition series. Now show that $k = t$ and deduce that $\overline{N_j} = \overline{N_{j+1}}$, so that $N_j = N_{j+1} + M_{t-1}$. Use induction to show that $N_j/N_{j+1} \simeq M_{t-1}$ and complete the proof.