Commutative Algebra 88-813
5769 Semester A
Question Sheet 2, due 14/12/2008

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(1) Let F be a field and let R be an affine F-algebra. Prove that if M is a simple R-module (מודול פשוט), then M has finite rank over F.

If you also assume that F is algebraically closed, then what can you say about the rank of M over F?

(2) Let F be a field. The polynomial F-algebra $F[X_1, X_2]$ is clearly affine. But prove that the subalgebra generated by 1, X_1X_2, $X_1X_2^2$, $X_1X_2^3, \ldots$ is not affine. Hence a subalgebra of an affine algebra is not necessarily affine.

(3) Prove that any affine F-algebra has countable dimension as an F-vector space.

(4) Let I be an ideal of a commutative ring R. Prove that the following conditions are equivalent:
  (a) I is a prime ideal (for any elements $x, y \in R$, we have $xy \in I$ if and only if $x \in I$ or $y \in I$).
  (b) The quotient ring R/I is an integral domain.
  (c) If J_1, J_2 are ideals of R with $J_1J_2 \subseteq I$, then $J_1 \subseteq I$ or $J_2 \subseteq I$.
  (d) If $J_1, J_2, \ldots, J_n$ are ideal of R such that $J_1J_2 \cdots J_n \subseteq I$, then $J_i \subseteq I$ for some $1 \leq i \leq n$.
  (e) The complement $R \backslash I$ is a multiplicative submonoid (תת-מונויד כפלי) of R.

(5) Suppose that B and B' are transcendence bases (בסיסי נעלות) of R. Prove that they have the same cardinality. (We did this in class for B and B' finite).

Hint: Each element of B is algebraically dependent on a finite number of elements of B', and the union of these finite subsets is all of B'.

(6) Prove the Noether Normalization Theorem in general. (In class we assumed that the field F was infinite).

Hint: To do this, recall that $R = F[a_1, \ldots, a_n]$ and let f be the polynomial that appeared in our proof in class, and write

$$f = \sum \gamma_{i_1,\ldots,i_n} X_1^{i_1} \cdots X_n^{i_n}.$$

Now let u_j be the highest degree of X_j that appears in any monomial of f, and define $u = 1 + max\{u_1, \ldots, u_n\}$. Now set

$$\hat{f} = f(X_1 + X_n^{u^{n-1}}, X_2 + X_n^{u^{n-2}}, \ldots, X_{n-1} + X_n^u, X_n)$$

and define $c_i = a_i - a_n^{u^{n-1}}$ for $1 \leq i \leq n-1$. Then $\hat{f}(c_1, \ldots, c_{n-1}, a_n) = 0$. Set $R' = F[c_1, \ldots, c_{n-1}]$ and define $h \in R'[X_n]$ by $h(X_n) = \hat{f}(c_1, \ldots, c_{n-1}, X_n)$. Now show that h has an invertible leading coefficient.

(7) Show that the ring $R = \mathbb{Z}[\sqrt{-1}]$ is integral (שלם) over $\mathbb{Z}$, and that it has two different maximal ideals lying over $5\mathbb{Z}$. In general, prove that for any odd prime $p \in \mathbb{Z}$, there are two different prime ideals lying over $p\mathbb{Z}$ if p can be written in the form $p = m^2 + n^2$ for $m, n \in Z$ and one prime ideal lying over $p\mathbb{Z}$ otherwise. (In fact, $p = m^2 + n^2$ if and only if $p \equiv 1 \mod 4$).

(8) Find the Krull dimension of $R = F[X_1, X_2, X_3, X_4]/(X_1^2 + X_2^2 + X_3^2, X_2^3 + X_3^3 + X_4^3)$.

Hint: First show that R is an integral domain, then apply our main theorem.