Commutative Algebra 88-813

5769 Semester A

Question Sheet 3, due 11/1/2009

Please feel free to e-mail me at `mschein@math.biu.ac.il` with any questions.

(1) Let F be an algebraically closed field. Prove that an algebraic set $Z \subset F^n$ consists of finitely many points if and only if the corresponding ideal $\mathcal{I}(Z) \subset F[x_1, \dots, x_n]$ has finite codimension. ($\mathcal{I}(Z)$ is said to have finite codimension if the quotient $F[x_1, \dots, x_n]/\mathcal{I}(Z)$ has finite rank as an F-module).

(2) Suppose that R is a commutative ring that has only finitely many maximal ideals. Prove that $R/(\bigcap_{i=1}^n M_i)$ is a direct product of fields, where $M_1, \dots, M_n$ are the maximal ideals.

(3) Prove that $\mathbb{Z}$ is a Noetherian ring. Show that (0) is a radical ideal in $\mathbb{Z}$ but that $(0) \neq p_1\mathbb{Z} \cap p_2\mathbb{Z} \cap \cdots \cap p_n\mathbb{Z}$ for every collection $p_1, \dots, p_n$ of prime numbers.

Is this a counterexample to the theorem that every radical ideal in a Noetherian ring is an intersection of finitely many prime ideals?

(4) Consider the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. Is it Artinian?

(5) Give an example of an integral domain with exactly two maximal ideals.

(6) Let p be a prime number. Let $S = \{p^k : k \in \mathbb{N}\}$ and let $M = S^{-1}\mathbb{Z}/\mathbb{Z}$. In other words,

$$M = \left\{ \frac{m}{n} + \mathbb{Z} : n = p^k, k \in \mathbb{N} \right\}.$$

Show that the $\mathbb{Z}$-module M is Artinian but not Noetherian.