Commutative Algebra 88-813

5770 Semester A

Question Sheet 2

(1) Prove the Noether Normalization Theorem in general. (In class we assumed that the field F was infinite).

Hint: To do this, recall that $R = F[a_1, \dots, a_n]$ and let f be the polynomial that appeared in our proof in class, and write

$$f = \sum \gamma_{i_1,\dots,i_n} X_1^{i_1} \cdots X_n^{i_n}.$$

Now let u_j be the highest degree of X_j that appears in any monomial of f, and define $u = 1 + max\{u_1, \dots, u_n\}$. Now set

$$\hat{f} = f(X_1 + X_n^{u^{n-1}}, X_2 + X_n^{u^{n-2}}, \dots, X_{n-1} + X_n^u, X_n)$$

and define $c_i = a_i - a_n^{u^{n-1}}$ for $1 \leq i \leq n-1$. Then $\hat{f}(c_1, \dots, c_{n-1}, a_n) = 0$. Set $R' = F[c_1, \dots, c_{n-1}]$ and define $h \in R'[X_n]$ by $h(X_n) = \hat{f}(c_1, \dots, c_{n-1}, X_n)$. Now show that h has an invertible leading coefficient.

(2) Prove that any ring R contains a prime ideal P such that $\mathrm{ht}(P) = 0$.

(3) Let F be a field and let $R = F[x_1, x_2, \dots]$ be the commutative polynomial ring in infinitely many variables. Consider the ideal

$$P_i = \langle x_{i(i-1)/2+1}, \dots, x_{i(i+1)/2-1}, x_{i(i+1)/2} \rangle.$$

Show that P_i is a prime ideal with i generators. Let $S = R \backslash (\bigcup_{i=1}^{\infty} P_i)$. Show that the ring $S^{-1}R$ is Noetherian and that if $M \subset S^{-1}R$ is a maximal ideal, then $M = S^{-1}P_i$ for some i. Show that $\mathrm{ht}_{S^{-1}R}(S^{-1}P_i) = i$. Conclude that every prime ideal of $S^{-1}R$ has finite height but that $\mathrm{Kdim}(S^{-1}R) = \infty$.

(4) Let R be a local ring with maximal ideal J. Let $F = R/J$.
   (a) Prove that J is not generated by fewer than $\mathrm{Kdim}(R)$ elements.
   (b) Let M be a finitely generated R-module. Prove that the minimal number of generators needed to generate M over R is equal to the dimension of M/JM as an F-vector space. Conclude that $\mathrm{Kdim}(R) \leq \dim_F(J/J^2)$.

(5) Let R be a local ring and J the maximal ideal. Suppose that J is a principal ideal (אידאל ראשי) and that $\bigcap_{n=1}^{\infty} J^n = 0$.

Prove that R is Noetherian and that if $0 \neq I \subset R$ is a proper ideal (אידאל אמיתי) then $I = J^n$ for some n.

(6) Let F be a field. Prove that any maximal ideal of $F[x_1, \dots, x_n]$ is generated by n elements $f_1, \dots, f_n$, where each f_i involves only the variables $x_1, \dots, x_i$.

Hint: Look at the proof that each maximal ideal of $F[x_1, \dots, x_n]$ has height n.

(7) Consider the ideal $I = (x^2 - y) \subset F[x, y]$. Show that it is prime and find its height.

(8) Let F be an arbitrary field that is not algebraically closed. Find a counterexample showing that the Nullstellensatz is false for F.

(9) Prove the prime exclusion principle: Let R be a commutative ring and let $P_1, \ldots, P_r$ be prime ideals. Then an ideal $I \subset R$ such that $I \subset P_1 \cup \cdots \cup P_r$ must be contained in some P_i.

Hint: Assume not. Without loss of generality, r is minimal such that $I \subset P_1 \cup \cdots \cup P_r$. Then $I \cap P_i \not\subset \cup_{j \neq i} P_j$ for each $1 \leq i \leq r$. Choose a suitable element $a \in I$ and elements $a_i \in (I \cap P_i) \backslash P_r$ for $1 \leq i \leq r-1$. Show that $a + a_1 \cdots a_{r-1}$ is not contained in any of the P_i.

(10) Let R be a noetherian ring, $P \subset R$ a prime ideal such that $\mathrm{ht}_R(P) \geq 2$. Prove that P contains infinitely many prime ideals of height 1.

Hint: Deduce a contradiction to the principal ideal theorem.

(11) Let $P \subset Q$ be prime ideals of a noetherian ring R. Prove that if there is a prime ideal strictly between P and Q, then there are infinitely many such prime ideals.