Commutative Algebra 88-813

5772 Semester A

Question Sheet 1

Due 15/11/2011, יח בחשון תשע"ב

(1) Prove the Five Lemma: Let R be a ring and consider the following commutative diagram of R-modules:

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{g_1} & A_2 & \xrightarrow{g_2} & A_3 & \xrightarrow{g_3} & A_4 & \xrightarrow{g_4} & A_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
B_1 & \xrightarrow[h_1]{} & B_2 & \xrightarrow[h_2]{} & B_3 & \xrightarrow[h_3]{} & B_4 & \xrightarrow[h_4]{} & B_5.
\end{array}$$

Suppose that the two rows are exact and that f_1, f_2, f_4, and f_5 are isomorphisms. Prove that f_3 is also an isomorphism.

(2) Let M be a finitely generated R-module. Prove that M has a maximal submodule.

(3) Recall that $l(M)$ is the composition length of a module M. If $M = M_1 \oplus M_2$, prove that $l(M) = l(M_1) + l(M_2)$.

(4) Let R be a commutative ring and let M be an R-module generated by m elements. Suppose that there is a surjective map $\varphi : M \to R^{(n)}$. Prove that $m \geq n$.

(5) Let R be a commutative ring. An element $s \in R$ is called *regular* if $sr \neq 0$ for all $0 \neq r \in R$. Let M be an R-module, and define $\text{tor}(M) = \{m \in M : sm = 0 \text{ for some regular } s \in R\}$. Prove the following:
   (a) $\text{tor}(M)$ is a submodule of M, and $\text{tor}(R) = 0$.
   (b) $\text{tor}(M_1 \oplus M_2) \simeq \text{tor}(M_1) \oplus \text{tor}(M_2)$.
   (c) If $A = M/\text{tor}(M)$ is a free R-module, then $M \simeq A \oplus \text{tor}(M)$.

(6) Let N_1, N_2, and K be submodules of M such that $N_1 \supset N_2$ and $K \cap N_1 = 0$. Then $(K + N_1)/(K + N_2) \simeq N_1/N_2$.

(7) Let $0 \to K \to M \to N \to 0$ be an exact sequence of R-modules. If K is free of rank k and N is free of rank n, prove that M is free of rank $k + n$.