Commutative Algebra 88-813

5772 Semester A

Question Sheet 10

Due 26/1/2012, ב' שבט תשע"ב

(1) Let R be a commutative ring. Let $I \subset R$ be an ideal, and suppose that $I \subset \bigcup_{i=1}^{r} P_i$, where $P_1, P_2, \ldots, P_r \subset R$ are prime ideals. Prove that there exists some $1 \leq j \leq r$ such that $I \subset P_j$.

(2) Let R be a Noetherian UFD. Let $P \subset R$ be a prime ideal such that $\mathrm{ht}_R(P) = 1$. Prove that P is principal.

(3) Suppose that R is a Noetherian domain. Prove that every non-zero element can be factored as a product of irreducible elements. (Recall that $x \in R$ is irreducible if for all pairs $y, z \in R$ such that $x = yz$, either y or z is a unit.)

(4) Let R be a Noetherian domain. Prove that R is a UFD if and only if every prime ideal $P \subset R$ such that $\mathrm{ht}_R(P) = 1$ is principal.

(5) Let R be a Noetherian ring. Let $P \subset R$ be a prime ideal such that $\mathrm{ht}_R(P) = r$. Prove that there exist $a_1, \ldots, a_r \in P$ such that P is minimal over the ideal $I = Ra_1 + \cdots Ra_r$ and I cannot be generated by fewer than r elements.

(6) Here is an example, due to Nagata, of a Noetherian ring with infinite Krull dimension. Let F be a field and let $R = F[x_1, x_2, \ldots]$ be the commutative polynomial ring in (countably) infinitely many variables. Consider the ideal

$$P_i = \langle x_{i(i-1)/2+1}, \ldots, x_{i(i+1)/2-1}, x_{i(i+1)/2} \rangle.$$

Show that P_i is a prime ideal with i generators. Let $S = R \backslash (\bigcup_{i=1}^{\infty} P_i)$. Show that the ring $S^{-1}R$ is Noetherian and that if $M \subset S^{-1}R$ is a maximal ideal, then $M = S^{-1}P_i$ for some i. Show that $\mathrm{ht}_{S^{-1}R}(S^{-1}P_i) = i$. Conclude that $\mathrm{Kdim}(R) = \infty$.