Commutative Algebra 88-813
5772 Semester A
Question Sheet 11
Due 9/2/2012, ט"ז שבט תשע"ב

(1) Let F be a field. If $V \subset F^n$ is an irreducible algebraic variety of dimension $n-1$, prove that it is defined by a single polynomial, i.e. that there exists $f \in F[x_1, \dots, x_n]$ such that $V = \{(a_1, \dots, a_n) \in F^n : f(a_1, \dots, a_n) = 0\}$.

(2) Let F be an algebraically closed field, and let $Z \subset F^n$ be any algebraic set. Prove that there exists a finite number of irreducible algebraic sets $V_1, \dots, V_r$ such that $Z = V_1 \cup V_2 \cup \cdots V_r$.

Recall that an algebraic set Z is called irreducible if for any two algebraic sets Z_1, Z_2 such that $Z = Z_1 \cup Z_2$, one always has either $Z_1 \subseteq Z_2$ or $Z_2 \subseteq Z_1$.

(3) Let F be a field which is not algebraically closed. Prove that the Nullstellensatz fails for F, in other words that there exists a radical ideal $I \subset F[x_1, \dots, x_n]$ for which there is so algebraic set $Z \subset F^n$ such that $I = \mathcal{I}(Z)$.

(4) Let R be a Noetherian domain. Prove that R is a principal ideal domain (PID) if and only if every maximal ideal is principal.

(5) Let $C \subset \mathbb{C}^3$ be the curve defined parametrically as follows:

$$C = \left\{ (t^3, t^4, t^5) : t \in \mathbb{C} \right\}.$$

Prove that C is an algebraic set. Let $P \subset \mathbb{C}[x, y, z]$ be the ideal $\mathcal{I}(C)$, and justify our choice of notation by proving that P is prime.

(6) Let $C \subset \mathbb{C}^3$ and the ideal $P \subset \mathbb{C}[x, y, z]$ be as in the previous question. Prove that $\mathrm{ht}_{\mathbb{C}[x,y,z]}(P) = 2$ but that P cannot be generated by any two elements. This shows that the inequality in the Hauptidealsatz is sometimes a strict inequality.