Commutative Algebra 88-813

5772 Semester A

Question Sheet 3

Due 8/12/2011, י"ב כסלו תשע"ב

(1) Let R be a commutative ring. Prove that an R-module M is Noetherian if and only if for every sequence $f_1, f_2, \dots$ of elements of M there exists $N \in \mathbb{N}$ such that for every $n \geq 1$ it is possible to write f_n in the form $f_n = \sum_{i=1}^{N} r_{in} f_i$, where $r_{1n}, r_{2n}, \dots r_{Nn} \in R$.

(2) Let p be a prime number. Let $S = \{p^k : k \in \mathbb{N}\}$ and let $M = S^{-1}\mathbb{Z}/\mathbb{Z}$. In other words,

$$M = \left\{ \frac{m}{n} + \mathbb{Z} : n = p^k, k \in \mathbb{N} \right\}.$$

Show that the $\mathbb{Z}$-module M is Artinian but not Noetherian.

(3) Let I be an ideal of a commutative ring R. Prove that the following conditions are equivalent:
   (a) I is a prime ideal (for any elements $x, y \in R$, we have $xy \in I$ if and only if $x \in I$ or $y \in I$).
   (b) The quotient ring R/I is an integral domain.
   (c) If J_1, J_2 are ideals of R with $J_1 J_2 \subseteq I$, then $J_1 \subseteq I$ or $J_2 \subseteq I$.
   (d) If $J_1, J_2, \dots, J_n$ are ideal of R such that $J_1 J_2 \cdots J_n \subseteq I$, then $J_i \subseteq I$ for some $1 \leq i \leq n$.
   (e) The complement $R \backslash I$ is closed under multiplication.

(4) Let R be a commutative ring, and let $I \subset R$ be an ideal. Given $x \in R$, consider the ideal $J_x = \{r \in R : rx \in I\}$. Suppose that J_x and $Rx + I$ are both finitely generated ideals of R. Prove that I is finitely generated.

(5) Let R be a commutative ring, and let $I \subset R$ be an ideal. Suppose that I is not finitely generated, but that all ideals of R that strictly contain I are finitely generated. Prove that I is a prime ideal.

(6) Let R be a commutative ring. Prove that R is Noetherian if and only if all prime ideals are finitely generated.

(7) Consider the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. Is it Noetherian? Is it Artinian?

(8) Let R be a ring. An element $x \in R$ is called nilpotent if there exists $n \geq 1$ such that $x^n = 0$. Let N be the set of all nilpotent elements of R. If R is commutative, prove that N is an ideal. If R is commutative and Noetherian, prove that there exists a natural number m such that $N^m = 0$.

(9) Let R be a commutative ring. An ideal $I \subset R$ is called a nilpotent ideal if there exists $n \geq 1$ such that $I^n = 0$. Give an example of a commutative ring such that the ideal of all nilpotent elements is not a nilpotent ideal.