Commutative Algebra 88-813

5772 Semester A

Question Sheet 4

Due 15/12/2011, י"ט כסלו תשע"ב

(1) Let F be a field. The polynomial F-algebra $F[X_1, X_2]$ is clearly affine. But prove that the subalgebra generated by 1, X_1X_2, $X_1X_2^2$, $X_1X_2^3, \ldots$ is not affine. This shows that a subalgebra of an affine algebra is not necessarily affine.

(2) Prove that any affine F-algebra has countable dimension as an F-vector space.

(3) Let F be an algebraically closed field and let R be an affine F-algebra. Prove that there is only one non-zero simple R-module up to isomorphism.

(4) Let R be a commutative ring. For $n \geq 1$, consider the set $M_n(R)$ of $n \times n$ matrices with entries in R. It is a ring under the usual addition and multiplication of matrices. Show that every two-sided ideal $I \subset M_n(R)$ is of the form $I = M_n(J)$, where J is an ideal of R.

Hint: Take J to be the set of all elements of R that appear as the entry in the top left corner of an element of I.

(5) Let F be a field and consider the affine algebra $R = F[x]$. For all $n \geq 1$, show that the subalgebra $F[x^{2n-1}, x^{2n} + x, x^{2n+1}] \subset R$ is equal to R.

(6) Prove that $S = F[x^3 + x, x^2]$ is a proper subalgebra of $F[x]$ by showing that $x \notin S$.

Hint: Show that every element of S may be written as $f + g(x^3 + x)$, where f and g are polynomials that involve only even powers of x.