Commutative Algebra 88-813

5772 Semester A

Question Sheet 5

Due 22/12/2011, כ"ו כסלו תשע"ב

(1) Let A be a commutative ring and let R be a commutative A-algebra. Let $r_1, r_2, \ldots, r_n$ be integral (שלם) over A. Prove that $A[r_1, r_2, \ldots, r_n]$ is finitely generated as an A-module.

(2) Let A be a Dedekind domain. Prove that it is a UFD (unique factorization domain) if and only if it is a PID (principal ideal domain).

(3) Consider the field $K = \mathbb{Q}(\sqrt{d})$, where d is a square-free integer. Prove that

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\sqrt{d}] & : d \equiv 2, 3 \bmod 4 \\ \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] & : d \equiv 1 \bmod 4. \end{cases}$$

(4) Let $R = \mathbb{Q}[x, y]$, and let $I \subset R$ be the ideal $I = (x^2 - y^3)$. Show that $A = R/I$ is an integral domain, but that A is not integrally closed.

Hint: Find a subring of the polynomial ring in one variable $\mathbb{Q}[t]$ that is isomorphic to A.

(5) Prove that every UFD is integrally closed.