Commutative Algebra 88-813

5772 Semester A

Question Sheet 6

Due 29/12/2011, ג' טבת תשע"ב

(1) Let K be a number field (i.e. K is a finite extension of $\mathbb{Q}$) and let $\mathcal{O}_K$ be the integral closure (הסגור השלם) of $\mathbb{Z}$ in K. Prove that $\text{Frac}\mathcal{O}_K = K$.

Hint: Prove the following more precise statement. Let $y \in K$. Then $c_n y^n + c_{n-1}y^{n-1} + \cdots + c_0 = 0$ for suitable n and suitable $c_0, \ldots, c_n \in \mathbb{Z}$. Let $b = \text{lcm}(c_0, \ldots, c_n)$. Show that $y = a/b$ for some $a \in \mathcal{O}_K$.

(2) An integral basis of K is a collection of elements $\beta_1, \ldots, \beta_n \in \mathcal{O}_K$ such that:
   (a) $K = \beta_1\mathbb{Q} + \beta_2\mathbb{Q} + \cdots + \beta_n\mathbb{Q}$.
   (b) $\mathcal{O}_K = \beta_1\mathbb{Z} + \beta_2\mathbb{Z} + \cdots + \beta_n\mathbb{Z}$.

Every number field K has an integral basis, and you may assume this. Prove that $\mathcal{O}_K$ is a Dedekind domain.

Hint: Let $P \subset \mathcal{O}_K$ be a non-zero prime ideal. Show that $P \cap \mathbb{Z} = p\mathbb{Z}$ for some prime p. The integral basis may help you prove that $\mathcal{O}_K/P$ is a field.

(3) Suppose that B and B' are transcendence bases (בסיסי נעלות) of R. Prove that they have the same cardinality. (We did this in class for B and B' finite).

Hint: Each element of B is algebraically dependent on a finite number of elements of B', and the union of these finite subsets is all of B'.

(4) Prove the Noether Normalization Lemma for arbitrary fields F.

Hint: This can be done using the following variation of the proof we saw in class for infinite fields F. Recall that $R = F[a_1, \ldots, a_n]$, and if $\text{tr.deg}_F(R) < n$, let $f \in F[X_1, \ldots, X_n]$ be a polynomial such that $f(a_1, \ldots, a_n) = 0$. Write

$$f = \sum \gamma_{i_1,\ldots,i_n} X_1^{i_1} \cdots X_n^{i_n}.$$

Now let u_j be the highest degree of X_j that appears in any monomial of f, and define $u = 1 + \max\{u_1, \ldots, u_n\}$. Now set

$$\hat{f} = f(X_1 + X_n^{u^{n-1}}, X_2 + X_n^{u^{n-2}}, \ldots, X_{n-1} + X_n^{u}, X_n)$$

and define $c_i = a_i - a_n^{u^{n-1}}$ for $1 \leq i \leq n-1$. Then $\hat{f}(c_1, \ldots, c_{n-1}, a_n) = 0$. Set $R' = F[c_1, \ldots, c_{n-1}]$ and define $h \in R'[X_n]$ by $h(X_n) = \hat{f}(c_1, \ldots, c_{n-1}, X_n)$. Now show that h has an invertible leading coefficient.

!חנוכה שמח