Commutative Algebra 88-813

5772 Semester A

Question Sheet 8

Due 12/1/2012, י"ז טבת תשע"ב

In your solution to any question you may assume the statements of previous questions to be true even if you did not prove them.

(1) Let F be a field, and suppose the integral domain R is an F-algebra. If $r \in R$ is algebraic over F, prove that $r^{-1} \in \operatorname{Frac} R$ is also algebraic over F.

(2) Let $A \subset R$ be integral domains, and suppose that R is integral over A. Prove that R is a field if and only if A is a field.

(3) Let $F \subset K$ be an extension of fields such that K is algebraic over F. Let $\alpha \in K$. Then $I_\alpha = \{f \in F[x] : f(\alpha) = 0\}$ is a non-zero ideal in $F[x]$. Since $F[x]$ is a PID, I_α is principal. Therefore there is a uniquely defined *monic* polynomial (פולינום מתוקן) f_α such that $I_\alpha = (f_\alpha)$. This f_α is called the minimal polynomial of α.

Suppose that A is an integrally closed domain such that $\operatorname{Frac} A = F$. Prove that $\alpha \in K$ is integral over A if and only if $f_\alpha \in A[x]$.

Hint: One direction is trivial. To prove the other, let $K \subset E$ be the splitting field of f_α. Then in $E[x]$ we have $f_\alpha(x) = \prod_{i=1}^{n}(x - \beta_i)$, where $\beta_1 = \alpha$. Show that every β_i is integral over A and express the coefficients of f_α in terms of the elements β_i.

(4) The aim of the remaining exercises is to prove the Going-Down Theorem: Let $A \subset R$ be an extension of integral domains, where A is an integrally closed domain and R is integral over A. Then $A \subset R$ has the property GD.

Recall that this means that, given prime ideals $P_0 \subset P_1 \subset A$ and a prime ideal $Q_1 \in \operatorname{Spec} R$ lying over P_1, there exists $Q_0 \in \operatorname{Spec} R$ such that $Q_0 \subset Q_1$ and Q_0 lies over P_0.

Let $S_0 = A - P_0$, and define

$$S = \{ar : a \in S_0, r \in R - Q_1\} \subset R.$$

Show that S is a monoid under multiplication.

(5) Let $\langle P_0 \rangle$ be the ideal of R generated by the set P_0 (note that P_0 is an ideal of A, but not necessarily of R). Suppose that there exists an element $s \in S \cap \langle P_0 \rangle$. Then we may write $s = ar$, where $a \in S_0$ and $r \in R - Q_1$. Similarly, we can write $s = \sum_{i=1}^{m} p_i r_i$ for a suitable m, where $p_i \in P_0$ and $r_i \in R$.

Show that there exist $h_0, h_1, \ldots, h_{m-1} \in P_0$ such that $g(s) = 0$, where

$$g(x) = x^m + h_{m-1}x^{m-1} + \cdots + h_1 x + h_0.$$

Hint: Let $M = A[r_1, \dots, r_m]$. Prove that M is finitely generated as an A-module. Consider the map $\varphi : M \to M$ given by $\varphi(x) = sx$ (why is this indeed a homomorphism of modules?) and use the proof of Nakayama's lemma.

(6) We use the notation of the previous question. Let $F = \operatorname{Frac} A$, and let $f_s(x) = x^n + d_{n-1}x^{n-1} + \cdots + d_1 x + d_0 \in F[x]$ be the minimal polynomial of s. Show that $d_i \in P_0$ for all $0 \le i \le n-1$.

Hint: The ring $(A/P_0)[x]$ is a UFD.

(7) Let $f_r(x) = x^n + d'_{n-1}x^{n-1} + \cdots + d'_1 x + d'_0 \in F[x]$ be the minimal polynomial of r. Why does f_r indeed have degree n? Prove that $d_i = a^{n-i} d'_i$ for all $0 \le i \le n-1$.

(8) Prove that $d'_i \in P_0$ for all $0 \le i \le n-1$ and deduce from this that $r \in Q_1$. On the other hand, we know that $r \in R - Q_1$. It follows from this contradiction that $S \cap \langle P_0 \rangle = \varnothing$.

(9) Consider the set $\mathcal{S} = \{Q \subset R : S \cap Q = \varnothing, \langle P_0 \rangle \subset Q\}$ of ideals of R. By the result of the previous question, $\mathcal{S}$ is non-empty. Prove that it contains a maximal element and that any maximal element is a prime ideal.

(10) Let Q_0 be a maximal element of $\mathcal{S}$. Prove that $\mathbb{Q}_0 \subset Q_1$ and that $Q_0 \cap A = P_0$. Therefore the extension $A \subset R$ does indeed satisfy the property GD.