ORBITS OF A GROUP ACTION AS OPTIMAL DESIGNS

MICHAEL M. SCHEIN

ABSTRACT. To determine an unknown function belonging to a known n-dimensional space, it suffices to evaluate the function at n generic points. We apply the character theory of finite groups towards finding optimal designs of such points.

Let $Z \subset \mathbb{R}^m$ and suppose that $\mathcal{X} = \{x_1, \ldots, x_s\} \subset Z$ is a set of s distinct points of Z. Let $\mathcal{B}$ be an s-dimensional space of complex-valued functions on Z, and let $\{B_1(x), \ldots, B_s(x)\}$ be a basis of $\mathcal{B}$. If the matrix $(B_i(x_j))$ is non-singular, then there exist Lagrange fundamental functions $L_1(x), \ldots, L_s(x) \in \mathcal{B}$ such that $L_i(x_j) = \delta_{ij}$. Define the variance

$$V_{\mathcal{X},\mathcal{B}} = \sup_{x \in Z} \sum_{i=1}^{s} |L_i(x)|^2.$$

Clearly $V_{\mathcal{X},\mathcal{B}} \geq 1$. We call $\mathcal{X}$ an *optimal design* for $\mathcal{B}$ if $V_{\mathcal{X},\mathcal{B}} = 1$. More generally, if $C \geq 1$, then we say that $\mathcal{X}$ is *C-optimal for $\mathcal{B}$* if $V_{\mathcal{X},\mathcal{B}} \leq C$.

The motivation for this definition is as follows. Suppose we know that f is a function defined on Z and that it lies in an s-dimensional space $\mathcal{B}$. For instance, $f(x)$ might be the activation energy of a certain chemical reaction in the presence of a concentration x of a catalyst, and we may want to approximate it by a polynomial of degree at most $s-1$. Generically, evaluating f at s points of Z will provide enough information to determine f. If $\mathcal{X} = \{x_1, \ldots, x_s\}$ is a set of s points as above, let Y_i be a random variable measuring $f(x_i)$. Then for all $x \in Z$ we have $f(x) = \sum_{i=1}^{s} L_i(x) Y_i$. Hence if the random variables Y_i are independent and normalized so that $\mathrm{Var}(Y_i) = 1$ for all $1 \leq i \leq s$, then $\mathrm{Var}(f(x)) = \sum_{i=1}^{s} |L_i(x)|^2$. By choosing $\mathcal{X}$ to be an optimal design for $\mathcal{B}$ we minimize these variances and thereby can determine the entire function f to a given level of confidence with a minimal number of observations.

This note uses basic results from the representation theory of finite groups to study optimal designs when $\mathcal{X}$ and $\mathcal{B}$ carry some symmetries. More precisely, let $G = \{g_1, \ldots, g_s\}$ be a finite abelian group, where we assume that $g_1 \in G$ is the identity element. Suppose that Z is endowed with a (left) G-action. We will consider designs $\mathcal{X}$ that are orbits of this action: $x_j = g_j x_1$ for all $1 \leq j \leq s$. There is a natural left action of G on the functions on Z: for any function f and any $g \in G$ and $x \in Z$ we have $(gf)(x) = f(g^{-1}x)$. Suppose that $\mathcal{B}$ is stable under this G-action.

Our main result is a theorem establishing some sufficient conditions for $\mathcal{X}$ to be C-optimal for $\mathcal{B}$. As applications, we recover a result of D. Lee about optimal designs in the case where $\mathcal{X}$ consists of equally spaced points on an interval, as well as a multi-dimensional analogue of

---

Key words and phrases. Optimal design, character theory.

This work was supported by German-Israeli Foundation for Scientific Research and Development grant 2264/2010 and by a grant from the Pollack Family Foundation.

2010 MSC classification: 41A65, 41A15.

Lee's result. For some other sets $\mathcal{X}$, we can find spaces $\mathcal{B}$ for which $V_{\mathcal{X},\mathcal{B}}$ is arbitrarily close to 1. It is natural to seek to extend our method to cases where an arbitrary finite group, not necessarily abelian, acts on the spaces $\mathcal{X}$ and $\mathcal{B}$. We remark upon this at the end of the second section. To the author's knowledge this is the first application of algebra to problems in the theory of designs.

The author is grateful to Prof. Boris Levit for introducing him to this problem and for useful discussions.

1. Characters of finite groups

In this section we establish some notation and recall the facts that we will need about characters of finite groups. A more complete exposition may be found in [2], among many other references.

Let $G = \{g_1, \dots, g_s\}$ be an arbitrary finite group of order $|G| = s$, where g_1 is the identity element. Suppose that G has r distinct conjugacy classes $C_1, \dots, C_r$. Then G has r irreducible representations (over $\mathbb{C}$) up to isomorphism. Let $\chi_1, \dots, \chi_r$ be their characters. Then ([2], Corollary 2.7):

$$|G| = \sum_{i=1}^{r} \chi_i(g_1)^2. \tag{1}$$

Furthermore, the first orthogonality relation ([2], Corollary 2.14) states that

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \delta_{ij}. \tag{2}$$

Here δ_{ij} is the Kronecker delta function. Let I_s be the $s \times s$ identity matrix. The character table of G is the $r \times s$ matrix H with entries given by $H_{ij} = \chi_i(g_j)$ for $1 \leq i \leq r$ and $1 \leq j \leq s$. Using it, we can restate (2) as

$$\frac{1}{|G|} HH^* = I_r. \tag{3}$$

For the remainder of the paper (except for Remark 2.5) we assume that G is an abelian group. Then all conjugacy classes of G are singletons, and $r = s$. Thus H is a square matrix, and (3) implies $H^*H = |G| I_s$. It follows from (1) that all the entries in the first column of H are ones. Considering the first column of the product H^*H, we obtain

$$\frac{1}{|G|} \sum_{i=1}^{s} \chi_i(g_j) = \begin{cases} 1 & : j = 1 \\ 0 & : j \neq 1. \end{cases} \tag{4}$$

Since all the irreducible representations of G are one-dimensional, the characters $\chi_i : G \to \mathbb{C}^*$ are group homomorphisms. Moreover, any homomorphism $\chi : G \to \mathbb{C}^* = \mathrm{GL}_1(\mathbb{C})$ must be one of $\chi_1, \dots, \chi_s$.

Remark. Note that (4) also follows from the fact ([2], Lemma 2.11) that, for any finite group G, the regular representation of G is isomorphic to $\sum_{i=1}^{r} (\dim V_i)\rho_i$, where $(\rho_1, V_1), \dots, (\rho_r, V_r)$ are the irreducible representations of G up to isomorphism.

2. The main theorem

Let $\mathcal{F}$ be a $\mathbb{C}$-vector space of complex-valued functions on Z, and suppose that $\mathcal{F}$ is preserved by the action of G. Note that we allow $\mathcal{F}$ to be infinite-dimensional. Since G is finite abelian, it follows from [2], Theorem 2.13, that $\mathcal{F} = \bigoplus_{i=1}^{s} \mathcal{F}_i$, where $\mathcal{F}_i$ is the subspace consisting of all $f \in \mathcal{F}$ such that $f(g^{-1}z) = \chi_i(g)f(z)$ for all $g \in G$ and all $z \in Z$.

In particular, $\mathcal{F}$ has a basis $\{f_\alpha\}_{\alpha\in A}$ consisting of eigenfunctions for the G-action. For each $1 \le i \le s$, set $A_i = \{\alpha \in A : f_\alpha \in \mathcal{F}_i\}$. We assume that each character χ_i appears in the G-module $\mathcal{F}$, or equivalently that $A_i \neq \varnothing$ for all $1 \le i \le s$. If $\alpha, \beta \in A$, we say that $\alpha \equiv \beta$ if f_α and f_β lie in the same isotypical component $\mathcal{F}_i$ of $\mathcal{F}$. Then the A_i are just the equivalence classes for the relation $\equiv$. Finally, define $A_i^- = A_j$, where j is such that $\chi_j = \chi_i^{-1}$.

Recall that the design $\mathcal{X} = \{x_1, \dots, x_s\}$ is assumed to satisfy $x_i = g_i x_1$ for each $1 \le i \le s$. Suppose that $\mathcal{B} \subset \mathcal{F}$ is an s-dimensional space with a basis $\{b_1(x), \dots, b_s(x)\}$ of conjugates of a single function: $b_i(x) = g_i b_1(x)$ for all $1 \le i \le s$. In particular, $\mathcal{B}$ is stable under the action of G. Suppose that $b_1(x)$ may be written as a linear combination of the f_α as follows:

$$b_1(x) = \sum_{\alpha\in A} \hat{B}_\alpha f_\alpha(x).$$

As we will see, this expression may be viewed as a generalized Fourier decomposition.

Lemma 2.1. *Suppose that $f_\alpha(x_1) = 1$ for all $\alpha \in A$ and that $\sum_{\alpha\in A_i} \hat{B}_\alpha \neq 0$ for all $1 \le i \le s$. Then the function $L(x) = \sum_{\alpha\in A} \hat{L_\alpha} f_\alpha(x)$, where*

$$\hat{L_\alpha} = \frac{\hat{B}_\alpha}{|G| \sum_{\substack{\beta\in A \\ \beta\equiv\alpha}} \hat{B}_\beta}.$$

satisfies $L(x_1) = 1$ and $L(x_j) = 0$ for $2 \le j \le s$.

Proof. Let $1 \le j \le s$. For all $\alpha \in A$, observe that $f_\alpha(x_j) = f_\alpha(g_j x_1) = \chi_\alpha(g_j)^{-1} f_\alpha(x_1) = \chi_\alpha(g_j)^{-1}$. Then

$$\sum_{\alpha\in A} \hat{L_\alpha} f_\alpha(x_j) = \sum_{\alpha\in A} \hat{L_\alpha} \chi_\alpha(g_j)^{-1} = \sum_{i=1}^{s} \sum_{\alpha\in A_i} \hat{L_\alpha} \chi_i(g_j)^{-1} =$$
$$\sum_{i=1}^{s} \frac{1}{|G| \sum_{\alpha\in A_i} \hat{B}_\alpha} \sum_{\alpha\in A_i} \hat{B}_\alpha \chi_i(g_j)^{-1} = \sum_{i=1}^{s} \frac{1}{|G|} \chi_i(g_j)^{-1} = \begin{cases} 1 & : j = 1 \\ 0 & : j \neq 1. \end{cases}$$

Here the final equality comes from (4). ☐

Lemma 2.2. *Maintain the hypotheses of Lemma 2.1. Then the function $L(x)$ defined in the statement of Lemma 2.1 is contained in $\mathcal{B}$.*

Proof. For each $1 \le k \le s$, let $b^{(k)}(x) = \sum_{\alpha\in A_k} \hat{B}_\alpha f_\alpha(x)$. Then $b_1(x) = \sum_{k=1}^{s} b^{(k)}(x)$, and for all $1 \le i \le s$ we have

$$b_i(x) = g_i b_1(x) = \sum_{k=1}^{s} \chi_k(g_i) b^{(k)}(x).$$

Since the matrix H is non-singular by (3), it follows that $b^{(k)}(x) \in \mathcal{B}$ for all $1 \le k \le s$. Finally observe that $L(x)$ is a linear combination of the $b^{(k)}(x)$. ☐

From the previous lemma it follows that the Lagrange fundamental functions for $\mathcal{X}$ in $\mathcal{B}$ are given by $L_i(x) = g_i L(x)$ for all $1 \leq i \leq s$.

Now we define an interpolant S as follows. Let $\mathbf{L}(x) = (L_1(x), \ldots, L_s(x))$ be a column vector, and define $\mathbf{S}(x) = (S_1(x), \ldots, S_s(x))$ by $\mathbf{S} = H\mathbf{L}$. Then by (3) we see that $\mathbf{L} = |G|^{-1}H^*\mathbf{S}$.

Lemma 2.3. *Let $C \geq 1$. If $|S_i(x)| \leq C$ for all $1 \leq i \leq s$ and all $x \in Z$, then $\mathcal{X}$ is a C^2-optimal design for $\mathcal{B}$.*

Proof. Suppose that $\sup_{x \in Z} |S_i(x)| \leq C$ for all i. Observe that

$$\sum_{i=1}^{s} |L_i(x)|^2 = \mathbf{L}^*\mathbf{L} = \frac{1}{|G|^2}\mathbf{S}^* H H^* \mathbf{S} = |G|^{-1}\mathbf{S}^*\mathbf{S} = |G|^{-1}\sum_{i=1}^{s}|S_i(x)|^2 \leq C^2.$$

□

Theorem 2.4. *Maintain the notation defined at the beginning of this section. Suppose that there exists a constant $C \geq 1$ such that $\sup_{x \in Z} |f_\alpha(x)| \leq C$ for all $\alpha \in A$ satisfying $\hat{B_\alpha} \neq 0$. Suppose also that $f_\alpha(x_1) = 1$ for all $\alpha \in A$ and that $\sum_{\alpha \in A_i} \hat{B_\alpha} \neq 0$ for all $1 \leq i \leq s$. If $\hat{B_\alpha}$ is a non-negative real number for every $\alpha \in A$, then $\mathcal{X}$ is a C^2-optimal design for $\mathcal{B}$.*

Proof. Observe that, by definition, $S_i(x) = \sum_{g \in G} \chi_i(g) L(g^{-1}x)$. It follows from Lemma 2.1 that the "Fourier coefficients" of $S_i(x) = \sum_{\alpha \in A} \hat{S^i_\alpha} f_\alpha(x)$ are:

$$\hat{S^i_\alpha} = \sum_{g \in G} \chi_i(g) \frac{\hat{B_\alpha}\chi_\alpha(g)}{|G|\sum_{\beta \equiv \alpha}\hat{B_\beta}} = \begin{cases} \frac{\hat{B_\alpha}}{\sum_{\beta\equiv\alpha}\hat{B_\beta}} & : \chi_\alpha = \chi_i^{-1} \\ 0 & : \chi_\alpha \neq \chi_i^{-1}. \end{cases} \tag{5}$$

From this we conclude that

$$|S_i(x)|^2 = \sum_{\alpha,\beta \in A_i^-} \hat{S^i_\alpha}\overline{\hat{S^i_\beta}} f_\alpha(x)\overline{f_\beta(x)} \leq \frac{C^2}{|\sum_{\alpha \in A_i^-}\hat{B_\alpha}|^2}\sum_{\alpha,\beta\in A_i^-}|\hat{B_\alpha}\overline{\hat{B_\beta}}|.$$

If all the $\hat{B_\alpha}$ are non-negative real, then we obtain that $|S_i(x)|^2 \leq C^2$. Since $1 \leq i \leq s$ was arbitrary, Lemma 2.3 implies that $\mathcal{X}$ is a C^2-optimal design for $\mathcal{B}$. □

Remark 2.5. It is not too difficult to obtain analogues of Lemma 2.1 and Lemma 2.2 in special cases when G is not abelian, such as for dihedral groups. We also note that, for an arbitrary group G of order s, the question of finding the Lagrange fundamental functions $L_i(x)$ is considerably simpler computationally than that of inverting a general $s \times s$ matrix. To see this, let $B_{ij} = b_i(x_j)$, where $b_i(x) = g_i b_1(x)$ and $x_i = g_i x_1$ as above. Then $B_{ij} = b_1(g_i^{-1}g_j x_1)$, so that the matrix B is just a weighted sum of permutation matrices. Indeed, the Cayley embedding of G into the symmetric group S_s sends each g_i to the permutation $\sigma_i \in S_s$ such that $g_{\sigma_i(j)} = g_j g_i^{-1}$ for all $1 \leq j \leq s$. The regular representation $\rho_{\mathrm{reg}} : G \to \mathrm{GL}_s(\mathbb{C})$ is the composition of the Cayley embedding with the standard representation of S_s; this corresponds to the left action of G on itself, where $g \in G$ acts by right multiplication by g^{-1}. Observe that $B = \sum_{k=1}^{s} b_1(x_k)\rho_{\mathrm{reg}}(g_k)$.

By assumption B is non-singular, so that the $L_i(x)$ are well-defined. Observe that

$$L_i(x) = \sum_{j=1}^{s}(B^{-1})_{ij} b_j(x), \tag{6}$$

for all $1 \leq i \leq s$. We compute that

$$L_i(x) = \quad g_i L_1(x) = \quad \sum_{j=1}^{s} (B^{-1})_{1j} g_i b_j(x) = \sum_{j=1}^{s} (B^{-1})_{1j} g_i g_j b_1(x) = \tag{7}$$
$$\sum_{k=1}^{s} (B^{-1})_{1,\sigma_k(i)} g_k b_1(x) = \sum_{k=1}^{s} (B^{-1})_{1,\sigma_k(i)} b_k(x).$$

Comparing (6) and (7) we see that $(B^{-1})_{ij} = (B^{-1})_{1,\sigma_j(i)}$. This means that we need only compute s cofactors to find the inverse of B, and not s^2 cofactors as for a general $s \times s$ matrix.

3. Examples

3.1. **Optimal designs.** As a special case of Theorem 2.4 we recover the following result of D. Lee [3], which was originally proved by direct computations with Fourier coefficients relying on work of de Boor [1] on splines. Let $N > 1$ be an integer and let $Z = \mathbb{R}/N\mathbb{Z}$, on which the group $G = \mathbb{Z}/N\mathbb{Z}$ acts by translations: $gx = x + g$ for $x \in Z$ and $g \in G$. Here $\mathcal{F}$ is the set of functions on Z (i.e. N-periodic functions on $\mathbb{R}$) which have Fourier decompositions, and its chosen basis is $\{f_\alpha(x) = e^{2\pi i \alpha x/N} : \alpha \in A\}$, indexed by the set $A = \mathbb{Z}$. Note that the distinct characters of $G = \mathbb{Z}/N\mathbb{Z}$ are precisely

$$\chi_j(g) = e^{2\pi i g j/N}, 1 \leq j \leq N.$$

It is easy to see that $\chi_\alpha = \chi_j$, where $1 \leq j \leq N$ is such that $-\alpha \equiv j \mod N$. In this case the equivalence relation $\equiv$ on A is just congruence modulo N. Let $\mathcal{X} = \{0, 1, \ldots, N-1\} = \{g \cdot 0 : g \in G\}$. Since for all $\alpha \in \mathbb{Z}$ we have $|f_\alpha(x)| = 1$ for all $x \in Z$ and $f_\alpha(0) = 1$, the following is immediate from Theorem 2.4.

Corollary 3.1 (Lee)**.** *Let $N > 1$ be an integer and let $B(x)$ be an N-periodic function on $\mathbb{R}$ with Fourier decomposition $B(x) = \sum_{\alpha=-\infty}^{\infty} \hat{B}_\alpha e^{2\pi i \alpha x/N}$. Suppose that the set $\{\alpha \in \mathbb{Z} : \hat{B}_\alpha \neq 0\}$ includes at least one element from each congruence class modulo N and that all the Fourier coefficients $\hat{B}_\alpha$ are non-negative real numbers. Then $\mathcal{X} = \{0, 1, \ldots, N-1\}$ is an optimal design for the space $\mathcal{B}$ spanned by the functions $B(x), B(x-1), \ldots, B(x-(N-1))$.*

We may obtain a multi-variable analogue of the previous result at no extra cost. Let $m \geq 1$ be an integer, and let $N_1, \ldots, N_m$ be integers greater than 1. The group $G = \mathbb{Z}/N_1\mathbb{Z} \times \mathbb{Z}/N_2\mathbb{Z} \times \cdots \times \mathbb{Z}/N_m\mathbb{Z}$ acts on $Z = \mathbb{R}/N_1\mathbb{Z} \times \cdots \times \mathbb{R}/N_m\mathbb{Z}$ by translations in the obvious way. Then Theorem 2.4 implies:

Corollary 3.2. *Let $m \geq 1$, let $(N_1, \ldots, N_m) \in \mathbb{Z}^m_{>1}$, and let $\mathcal{X} = \{(y_1, \ldots, y_m) \in \mathbb{Z}^m : 0 \leq y_k < N_k\}$. Let $B(x_1, \ldots, x_m)$ be a function on $\mathbb{R}^m$ that is N_k-periodic in the variable x_k for each $1 \leq k \leq m$. Suppose that its multi-variable Fourier decomposition is:*

$$B(x_1, \ldots, x_m) = \sum_{(\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m} \hat{B}_{\alpha_1, \ldots, \alpha_m} e^{2\pi i(\alpha_1 x_1/N_1 + \cdots + \alpha_m x_m/N_m)}.$$

Suppose that for every m-tuple $(\mathcal{C}_1, \ldots, \mathcal{C}_m)$, where $\mathcal{C}_k$ is a congruence class modulo N_k for each $1 \leq k \leq m$, there exists $(\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$ such that $\alpha_k \in \mathcal{C}_k$ for each k and $\hat{B}_{\alpha_1, \ldots, \alpha_m} \neq 0$.

If all the Fourier coefficients $\hat{B}_{\alpha_1,\ldots,\alpha_m}$ are non-negative real numbers, then $\mathcal{X}$ is an optimal design for the $N_1N_2\cdots N_m$-dimensional space spanned by the functions

$$\{B(x_1 - D_1, \ldots, x_m - D_m) : (D_1, \ldots, D_m) \in \mathbb{Z}^m, 0 \leq D_k < N_k\}.$$

3.2. **Almost optimal designs.** The next simplest automorphisms of an interval, after the translations considered in the previous section, are reflections. In this section, let $Z = \mathbb{R}/2\pi\mathbb{Z}$. Let $t : Z \to Z$ be translation by π, so that $t(x + 2\pi\mathbb{Z}) = x + \pi + 2\pi\mathbb{Z}$. Let $r : Z \to Z$ be reflection about zero: $r(x + 2\pi\mathbb{Z}) = -x + 2\pi\mathbb{Z}$. The group $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generated by these two operators acts on Z.

The four characters of G are determined by:

$$\begin{array}{rr} \chi_1(r) = -1 & \chi_1(t) = -1 \\ \chi_2(r) = -1 & \chi_2(t) = 1 \\ \chi_3(r) = 1 & \chi_3(t) = -1 \\ \chi_4(r) = 1 & \chi_4(t) = 1 \end{array}$$

Let $a \in Z$ be any point that is not a rational multiple of π. We will consider the design $\mathcal{X} = \{a, -a, a + \pi, -a + \pi\}$, with a in the role of x_1. Consider the space $\mathcal{F}$ of functions spanned by $f_\alpha : \alpha \in \mathbb{Z}$, where

$$f_\alpha(x) = \begin{cases} (\cos \alpha x)/(\cos \alpha a) & : \alpha \geq 0 \\ (\sin \alpha x)/(\sin \alpha a) & : \alpha < 0. \end{cases}$$

Then $\mathcal{F}$ is the space of 2π-periodic functions on $\mathbb{R}$ with Fourier decompositions. Note that $f_\alpha(a) = 1$ for all $\alpha \in A$ and that

$$\begin{array}{ll} A_1 = \{\alpha \in \mathbb{Z} : \alpha < 0, \alpha \text{ odd}\} & A_3 = \{\alpha \in \mathbb{Z} : \alpha \geq 0, \alpha \text{ odd}\} \\ A_2 = \{\alpha \in \mathbb{Z} : \alpha < 0, \alpha \text{ even}\} & A_4 = \{\alpha \in \mathbb{Z} : \alpha \geq 0, \alpha \text{ even}\}. \end{array}$$

In other words,

$$\begin{array}{rcl} \text{span}\{f_\alpha : \alpha \in A_1\} & = & \text{span}\{\sin mx : m \geq 1 \text{ odd}\} \\ \text{span}\{f_\alpha : \alpha \in A_2\} & = & \text{span}\{\sin mx : m \geq 1 \text{ even}\} \\ \text{span}\{f_\alpha : \alpha \in A_3\} & = & \text{span}\{\cos mx : m \geq 1 \text{ odd}\} \\ \text{span}\{f_\alpha : \alpha \in A_4\} & = & \text{span}\{\cos mx : m \geq 0 \text{ even}\} \end{array}$$

In our situation, when translated into a more usual basis for Fourier decompositions, Theorem 2.4 says the following:

Corollary 3.3. *Consider the 2π-periodic function*

$$B(x) = c_0 + \sum_{m=1}^{\infty} (c_m \cos mx + d_m \sin mx).$$

Suppose that $c_m \neq 0$ for at least one even and one odd index m, and that $d_m \neq 0$ for at least one even and one odd index m. Let $a \in [-\pi, \pi]$ be a number which is not a rational multiple of π. Define

$$C = \sup\left(\{\frac{1}{|\cos ma|} : c_m \neq 0\} \cup \{\frac{1}{|\sin ma|} : d_m \neq 0\}\right).$$

If $(\operatorname{sgn}(\cos ma))c_m$ *and* $(\operatorname{sgn}(\sin ma))d_m$ *are non-negative real numbers for all* $m \geq 0$*, then* $\mathcal{X} = \{a, -a, a+\pi, -a+\pi\}$ *is a* C^2*-optimal design for the four-dimensional space* $\mathcal{B}$ *spanned by the functions* $B(x)$*,* $B(-x)$*,* $B(x+\pi)$*, and* $B(-x+\pi)$*.*

Observe that $\mathbb{Z}a$ is dense in $\mathbb{R}/2\pi\mathbb{Z}$ if a is not a rational multiple of π. Therefore, given any such a and any $\varepsilon > 0$ we can find a function $B(x)$ such that $\mathcal{X}$ is $(1+\varepsilon)$-optimal for $\mathcal{B}$ by Corollary 3.3. For instance, let $a = 1$, so that $\mathcal{X} = \{\pm 1, \pm(\pi - 1)\}$. Observe that $|\sin 11|^{-1}$, $|\sin 366|^{-1}$, $|\cos 0|^{-1}$, and $|\cos 355|^{-1}$ are all less than $\sqrt{1.00002}$. Therefore, if we set

$$B(x) = c_0 - c_{355}\cos 355x - d_{11}\sin 11x + d_{366}\sin 366x,$$

where c_0, c_{355}, d_{11}, and d_{366} are any positive real numbers, then the space $\mathcal{B}$ spanned by the orbit of $B(x)$ under the action of $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ satisfies $1 \leq V_{\mathcal{X},\mathcal{B}} < 1.00002$.

At the expense of making the design $\mathcal{X}$ somewhat less optimal, we can take smaller harmonics in $B(x)$. For instance, if

$$B(x) = c_0 - c_3\cos 3x - d_{11}\sin 11x + d_{14}\sin 14x,$$

where c_0, c_3, d_{11}, and d_{14} are positive, then the space $\mathcal{B}$ spanned by the orbit of $B(x)$ satisfies $V_{\mathcal{X},\mathcal{B}} < 1.0204$.

On the other hand, if we want our function to have the form $B(x) = c_0 + c_1\cos x + d_1\sin x + d_2\sin 2x$, then observe that the quantity

$$C = \max\{\frac{1}{\cos a}, \frac{1}{\sin a}, \frac{1}{\sin 2a}\}$$

is minimized when $a = \pi/4$, in which case $C = \sqrt{2}$. Therefore, the best optimality result we can extract from Theorem 2.4 for functions of this form is the following: if

$$B(x) = c_0 + c_1\cos x + d_1\sin x + d_2\sin 2x,$$

where the constants c_0, c_1, d_1, d_2 are all positive real, if $\mathcal{B}$ is the space spanned by the G-orbit of $B(x)$, and if $\mathcal{X} = \{\frac{\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, -\frac{3\pi}{4}\}$, then $V_{\mathcal{X},\mathcal{B}} \leq 2$.

References

[1] Carl de Boor. On the cardinal spline interpolant to e^{iut}. *SIAM J. Math. Anal.*, 7:930–941, 1976.

[2] I. Martin Isaacs. *Character theory of finite groups*. AMS Chelsea Publishing, Providence, RI, 2006. Corrected reprint of the 1976 original [Academic Press, New York].

[3] Daniel Lee. On a minimal property of cardinal and periodic Lagrange splines. *J. Approx. Theory*, 70:335–338, 1992.

Department of Mathematics, Bar-Ilan University, Ramat Gan 52900, Israel
E-mail address: mschein@math.biu.ac.il