LECTURE NOTES FOR COURSE 88-909
COHOMOLOGY OF GROUPS
BAR-ILAN UNIVERSITY, 5776 SEMESTER B

MICHAEL M. SCHEIN

Before starting, we'll say a few words of motivation. We wish to understand the infinite Galois group $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, which is very important in number theory. In particular, we wish to understand G-modules, namely modules over some ring that are endowed with an action of G. Such objects tend to be complicated. Recall that in algebraic topology, we define the homology and cohomology of simplicial complexes. These are collections of invariants that can be used, for instance, to prove that two simplicial complexes are not homeomorphic. One loses information when passing from a complex to its cohomology – two different complexes can have the same cohomology groups in every dimension – but retains enough to do some useful things. On the other hand, the cohomology groups have the advantage of sometimes being computable; for simple complexes they can be computed directly, and for more complicated ones they can often be deduced with tools such as the long exact cohomology sequence, the Mayer-Vietoris sequence, etc. The aim of this course is to develop an analogous theory in the setting of G-modules.

1. Profinite groups

Definition 1.1. Let I be a directed partially ordered set. This means that every pair of elements has a common upper bound: if $i, j \in I$ then there exists $k \in I$ such that $i \leq k$ and $j \leq k$. A *projective system* of groups indexed by I consists of a collection of groups $\{G_i : i \in I\}$, and, for each pair $i \geq j$ of elements of I, a group homomorphism $\varphi_{ij} : G_i \to G_j$ such that φ_{ii} is always the identity. Moreover, we demand that these maps be compatible, in the sense that if $i \geq j \geq k$ then $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$.

Definition 1.2. Let $\{G_i\}_{i \in I}$ be a projective system. Its projective limit $\varprojlim_I G_i$ is the subgroup of $\prod_{i \in I} G_i$ consisting of "compatible" tuples $(a_i)_{i \in I}$, namely those satisfying $\varphi_{ij}(a_i) = a_j$ for all $i > j$.

If the G_i are topological groups and the homomorphisms φ_{ij} are continuous maps, then we can define the *projective limit topology* on $\varprojlim_I G_i$ as the weakest topology such that the maps

$$\begin{aligned} \pi_i : \varprojlim_I G_i &\to G_i \\ (a_j)_j &\mapsto a_i \end{aligned}$$

are continuous for all $i \in I$.

For our purposes, the G_i will usually be finite groups endowed with the discrete topology. In this case, one checks that the projective limit topology is exactly the subset topology induced on $\varprojlim_I G_i$ by the product topology on $\prod_{i \in I} G_i$. We will implicitly use this fact several times below.

As an example, let I be the set of all finite Galois extensions of $\mathbb{Q}$, ordered by inclusion. Then $\{\mathrm{Gal}(K/\mathbb{Q})\}_{K\in I}$ is a projective system, with the obvious homomorphisms. We observe that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \simeq \varprojlim_I \mathrm{Gal}(K/\mathbb{Q})$. Indeed, define a homomorphism $f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \varprojlim_I \mathrm{Gal}(K/\mathbb{Q})$ by $f(\sigma) = (\sigma_{|K})_{K\in I}$. Since every element of $\overline{\mathbb{Q}}$ lies in some finite Galois extension of $\mathbb{Q}$, we see that f is injective. Moreover, given an element $(\sigma_K)_K$ of the projective limit, define $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by $\sigma(\alpha) = \sigma_K(\alpha)$, where $\alpha \in \overline{\mathbb{Q}}$ and K is a finite Galois extension of $\mathbb{Q}$ containing α. The compatibility condition of the projective limit exactly ensures that the σ_K glue and σ is well-defined.

Since we are interested in representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, it is natural to study projective limits of finite groups. It will be convenient to have an alternative characterization of such groups.

Definition 1.3. A topological space X is called *totally disconnected* if its only non-empty connected subsets are single points. In other words, for any pair of distinct points $x, y \in X$ there exist open sets $U, V \subset X$ such that $x \in U$, $y \in V$, $U \cap V = \varnothing$, and $U \cup V = X$.

Note that a totally disconnected topological space is necessarily Hausdorff.

Definition 1.4. A topological group is called *profinite* if it is compact and totally disconnected.

The following basic results will be used frequently throughout the course.

Proposition 1.5. *Let G be a compact group. Any subgroup $H \subset G$ is open if and only if it is closed and has finite index in G.*

Proof. Suppose that H is open. The cosets of H are open, disjoint, and cover G. Since G is compact, there are only finitely many of them. Moreover, the complement of H is a union of cosets and hence open; thus H is closed.

Conversely, if H is closed and of finite index, then its complement is a finite union of closed cosets, thus closed. Hence H is open. □

Proposition 1.6. *Let G be a topological group and $H \leq G$ a subgroup. Then H is open if and only if it contains an open neighborhood of the identity.*

Proof. One direction is obvious. If H contains an open set $e \in U$, then $H = \bigcup_{h\in H} hU$ is open. □

Theorem 1.7. *A topological group G is profinite if and only if $G \simeq \varprojlim_I G_i$, where $\{G_i\}_{i\in I}$ is a projective system of (discrete) finite groups.*

Proof. Suppose that $G = \varprojlim_I G_i$, and consider the natural inclusion $f : G \to \prod_{i\in I} G_i$; recall that f is a topological isomorphism onto its image. We claim that $f(G)$ is closed; since $\prod_{i\in I} G_i$ is compact by Tychonoff's theorem, this will imply the compactness of G. Now, for each pair $i > j$, let

$$X_{ij} = \{(a_k)_k \in \prod_k G_k : \varphi_{ij}(a_i) = a_j\}.$$

Observe that X_{ij} is the preimage of the diagonal under the continuous map (recall that the φ_{ij} are continuous)

$$\begin{aligned} \prod_k G_k &\to G_j \times G_j \\ (a_k)_k &\mapsto (a_j, \varphi_{ij}(a_i)). \end{aligned}$$

Therefore X_{ij} is closed, and thus $G = \bigcap_{i>j} X_{ij}$ is closed and hence compact.

The G_i are discrete and hence totally disconnected, and a product of totally disconnected spaces is clearly totally disconnected. (Indeed, let $a = (a_i)_i$ and $b = (b_i)_i$ be distinct elements of $\prod_{i\in I} X_i$. Let j be such that $a_j \neq b_j$, and let $U_j, V_j \subset X_j$ such that $a_j \in U_j$, $b_j \in V_j$, $U_j \cup V_j = X_j$, $U_j \cap V_j = \varnothing$. Set $U = U_j \times \prod_{i\neq j} X_i$ and $V = V_j \times \prod_{i\neq j} X_i$.) But a subspace of a totally disconnected space is clearly totally disconnected. Thus G is profinite.

Conversely, suppose that G is a profinite group, and let $\{H_q\}_{q\in Q}$ be the family of its open normal subgroups, ordered by inclusion: $q > q'$ if $H_q \subset H_{q'}$. Note that the quotient subgroups G/H_q are all finite (by Proposition 1.5) and form a projective system of groups. We have a natural homomorphism of groups

$$\begin{aligned} f : G &\to \varprojlim_Q G/H_q \\ g &\mapsto (gH_q)_q \end{aligned}$$

We claim that f is a topological isomorphism. The pre-image under the quotient map $G \to G/H_q$ of any subset of G/H_q is a union of cosets of the open subgroup H_q and hence is open. From this it follows that f is continuous. Any closed subset of the compact group G is compact, hence its image under f is compact and hence closed, since $\varprojlim_Q G/H_q$ is Hausdorff. Thus, it suffices to prove that f is an isomorphism of abstract groups.

To show that f is surjective, let $(g_qH_q)_q \in \varprojlim_Q G/H_q$. We claim that $\bigcap_{q\in Q} g_qH_q \neq \varnothing$. If this is true, then it is easy to see that $(g_qH_q)_q = f(g)$ for any g in the above intersection. Assume, by way of contradiction, that the intersection is empty. This means that $\bigcup_{q\in Q}(G \setminus g_qH_q) = G$. Since G is compact and each g_qH_q is closed, there is a finite subcover $G = \bigcup_{i=1}^r (G \setminus g_{q_i}H_{q_i})$. Hence $\bigcap_{i=1}^r g_{q_i}H_{q_i} = \varnothing$. The subgroup $H_{q_1} \cap \cdots \cap H_{q_r}$ is open and normal, so it is H_q for some $q \in Q$. Clearly $g_q \in \bigcap_{i=1}^r g_{q_i}H_{q_i}$, contradicting the emptiness of this intersection.

It remains to show that f is injective, and this will follow immediately from the following claim. □

Proposition 1.8. *Let G be a profinite group, and let $\{H_q\}_{q\in Q}$ be the family of all open normal subgroups of G (note that G is in the family, so it is non-empty). Then $\bigcap_{q\in Q} H_q = \{e\}$.*

Proof. Let $x \in G$ be a non-trivial element. We need to construct an open normal subgroup $H \trianglelefteq G$ such that $x \notin H$. By total disconnectedness of G, there exist open and closed sets U and V that separate e and x. Suppose that $e \in U$. Then $x \notin U$. We will construct an open normal subgroup H that is contained in U and hence does not contain x.

For any subset $A \subset G$, we denote by A^n the set of all products of n elements of G and by A^{-1} the set of all inverses of elements of A. Note that U^2 is the image of the compact set $U \times U$ (closed subset of compact $G \times G$) under the continuous multiplication map $\mu : G \times G \to G$, we see that U^2 is compact, and thus closed, since G is Hausdorff. By induction, U^n is compact for all $n \in \mathbb{N}$.

Set $W = (G \setminus U) \cap U^2$; then W consists of all elements of U^2 that do not lie in U. This is a closed subset of G, hence compact. Note that $U \subset G \setminus W$. We claim that for any $u \in U$, there exist open neighborhoods $u \in X_u$ and $e \in Y_u$, contained in U, such that $X_uY_u \subset (G \setminus W) \cap U^2 \subset U$ (the last inclusion holds since the middle set consists of all elements of U^2 that do lie in U). Indeed, $u \in G \setminus W$. Thus $\mu^{-1}(G \setminus W) \subset G \times G$ is open and contains (u, e). In particular, it contains an open box of the form $X'_u \times Y'_u$, where $u \in X'_u$

and $e \in Y'_u$ are open. Setting $X_u = X'_u \cap U$ and $Y_u = Y'_u \cap U$ (recall that $e \in U$), we get the desired neighborhoods.

Now $\{X_u\}_{u\in U}$ is an open cover of U, and since U is compact there exists a finite subcover $\{X_{u_1}, \dots, X_{u_r}\}$. Set $Y = Y_{u_1} \cap \dots \cap Y_{u_r}$. This is an open set containing e. Set $Z = Y \cap Y^{-1}$. This is still an open set containing e. Moreover, $Z \subset U$ because $Y \subset U$.

Observe that $UZ = \cup_{i=1}^{r} X_{u_i} Z \subset U$, since $Z \subset Y_{u_i}$ for each i. By induction, $UZ^n \subset U$ for all $n \geq 1$, and thus $Z^n \subset U$ for all $n \geq 1$ (since $Z \subset U$). Thus, $\langle Y \rangle = \bigcup_{n=1}^{\infty} Z^n$ is a subgroup of G that is contained in U. Since it contains the open set $e \in Z$, it is open. Call it K.

Since K has finite index in G, it has only finitely many conjugate subgroups. Thus, the intersection $H = \bigcap_{g\in G} gKg^{-1}$ is a finite intersection, so it is open. Clearly, H is a normal subgroup of G. Finally, $H \subset K \subset U$, so we are done. □

Corollary 1.9. *Let G be a profinite group. Any open neighborhood of e contains an open normal subgroup.*

Proof. Let U be an open neighborhood of e. Observe that if U is open and closed, then the claim follows from the proof of the previous proposition.

Let $\{H_q\}_{q\in Q}$ be the family of open normal subgroups of G. From the previous proposition it follows that $\bigcup_{q\in Q}(G \setminus H_q) = G \setminus \{e\}$, hence $U \cup \bigcup_{q\in Q}(G \setminus H_q) = G$. By compactness of G, there is a finite subcover $G = U \cup (G \setminus H_{q_1}) \cup \dots \cup (G \setminus H_{q_r})$. Hence $(G \setminus U) \cap H_{q_1} \cap \dots \cap H_{q_r} = \varnothing$.

Set $H = H_{q_1} \cap \dots \cap H_{q_r}$. This is an open normal subgroup of G, and $H \subset U$. □

Corollary 1.10. *Let G be a profinite group and $H \subset G$ a subgroup. Then H is closed (normal) if and only if it is an intersection of open (normal) subgroups.*

Proof. Since open subgroups are closed, one direction is trivial. Conversely, let H be a closed subgroup. Let $\{K_q\}_{q\in Q}$ be the family of open normal subgroups of G. Observe that for every $q \in Q$,

$$HK_q = \{hk : h \in H,\, k \in K_q\},$$

since the K_q are normal, and that HK_q is normal if H is normal as well. We claim that

$$\bigcap_{q\in Q} HK_q = H\left(\bigcap_{q\in Q} K_q\right)$$

for any closed H. Since the right-hand side is equal to H by Proposition 1.8, this implies our claim. It is obvious that the right-hand side is contained in the left-hand side for any subgroup H. To prove the opposite containment, let $g \in \bigcap_{q\in Q} HK_q$. We want to show that g is contained in the right-hand side, or, equivalently, that $Hg \cap \left(\bigcap_{q\in Q} K_q\right) \neq \varnothing$.

Indeed, if this intersection were empty, then, by the argument that is by now standard for us, $(G \setminus Hg) \cup \bigcup_{q\in Q}(G \setminus K_q)$ would be an open cover of G, hence would have a finite subcover, hence $Hg \cap (K_{q_1} \cap \dots \cap K_{q_r}) = \varnothing$. But $K = K_{q_1} \cap \dots \cap K_{q_r}$ is itself an open normal subgroup, hence by assumption $g \in HK$. Thus we have arrived at a contradiction. □

Proposition 1.11. *Let G be a profinite group and $H \subset G$ a closed subgroup. Then H is profinite. Moreover, if H is normal, then G/H (with the quotient topology) is profinite.*

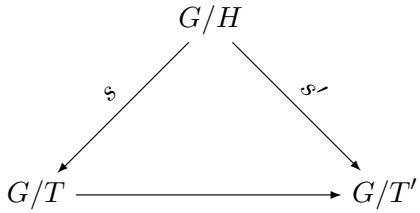
Proof. Easy to see that H and G/H are compact and totally disconnected. Alternatively, if $G \simeq \varprojlim_I G_i$ and $H_i = \pi_i(H) \subset G_i$ for each $i \in I$, then one can show that $H \simeq \varprojlim_I H_i$ and $G/H \simeq \varprojlim_I G_i/H_i$. □

Proposition 1.12. *Suppose that G is profinite and $K \subset H \subset G$ are closed subgroups. There exists a continuous section $s : G/H \to G/K$ of the natural surjection $G/K \to G/H$.*

Proof. First consider the case $[H : K] < \infty$. Then K is open in H, so there exists an open set $V \subset G$ such that $K = V \cap H$. Let $U \trianglelefteq G$ be an open normal subgroup contained in V (this exists by Corollary 1.9), hence that $H \cap U \subset K$.

Decompose G into double cosets $G = \sqcup_{i=1}^{r} U g_i H$; there are only finitely many double cosets because U has finite index in G. On each piece, define $s : Ug_iH/H \to Ug_iH/K$ by $s(ug_ihH) = ug_iK$. This is well-defined, since if $ug_ih = u'g_ih'$, then $h(h')^{-1} = g_i^{-1}u^{-1}u'g_i \in U \cap H \subset K$, since U is normal in G. Thus $ug_iK = u'g_iK$, so s is indeed well-defined. It is obvious that s is continuous.

We treat the general case by a Zorn's Lemma argument. Consider the set $\mathcal{S}$ of pairs (T, s), where $H \supseteq T \supseteq K$ is a closed subgroup of G and s is a continuous section of the surjection $G/T \to G/H$. We define an order on $\mathcal{S}$ by setting $(T, s) \geq (T', s')$ if $T \subseteq T'$ and the obvious diagram commutes:



If $(T_1, s_1) \leq (T_2, s_2) \leq \cdots$ is a chain, we can set $T_\infty = \bigcap_{i=1}^{\infty} T_i$ (this is a closed subgroup containing K). It is easy to see that $G/T_\infty = \varprojlim G/T_i$, and by the universal property of the projective limit (which we haven't stated – can also just say that the s_i glue) we get a section $s_\infty : G/H \to G/T_\infty$ that is compatible with all the other data. Thus the chain has an upper bound, and by Zorn's Lemma the set $\mathcal{S}$ contains a maximal element (T, s).

We need to prove that $T = K$. Since K is the intersection of all open subgroups containing K by Corollary 1.10, it suffices to show that $T \subset U$ for any open subgroup $U \supset K$. However, for any such U, we have that $V = T \cap U$ is a closed subgroup such that $T \supseteq V \supseteq K$ and $[T : V] < \infty$.

By the first case we treated in this proof, there is a continuous section $G/T \to G/V$. Composing it with s gives a continuous section $G/H \to G/V$. By maximality of (T, s), this implies that $V = T$ and hence $T \subset U$. □

2. G-modules

Definition 2.1. Let G be a profinite group. A *(discrete) G-module* is an abelian group M endowed with an action of G, such that every $g \in G$ acts by group homomorphisms: for any $g \in G$ and $m_1, m_2 \in M$, we have $g(m_1 + m_2) = gm_1 + gm_2$. Moreover, we assume that for any $m \in M$ the stabilizer $\mathrm{stab}_G(m) = \{g \in G : gm = m\}$ is an open subgroup of G.

Remark 2.2. Observe that the condition above is equivalent to the continuity of the action $G \times M \to M$, where M is given the discrete topology. Also, we clearly have $M = \bigcup_U M^U$, where U runs over open subgroups of G, and $M^U = \{m \in M : \forall u \in U, um = m\}$ is the space of U-invariants.

Example 2.3. Let K be a number field and $G = \mathrm{Gal}(\overline{K}/K)$. Then G acts on $\overline{K}$ in the obvious way. Moreover, for any $\alpha \in \overline{K}$, the field $K(\alpha)$ has finite degree over K, so that

$\mathrm{stab}_G(\alpha) = \mathrm{Gal}(\overline{K}/K(\alpha))$ is open in G. (If $K(\alpha)/K$ is Galois, then $\mathrm{Gal}(\overline{K}/K(\alpha))$ is the kernel of the continuous map $G \to \mathrm{Gal}(K(\alpha)/K)$ and hence open. Otherwise, it contains the absolute Galois group of the Galois closure of $K(\alpha)$, which is open.) Thus $\overline{K}$ is a G-module.

Example 2.4. Let $G = \{e\}$. (Note that finite groups are profinite). A G-module is just an abelian group, with no extra structure.

We write Mod_G for the category of discrete G-modules. The morphisms are, of course, group homomorphisms $f : M \to N$ that respect the G-action, i.e. such that $f(gm) = gf(m)$ for all $g \in G$ and $m \in M$. Such maps are often called *G-equivariant*. One checks that Mod_G is an abelian category.

Definition 2.5. Let G be a profinite group, $H \subset G$ a closed subgroup, and M a discrete H-module. The induced module $\mathrm{Ind}_H^G M$ is the space of functions $f : G \to M$ such that

(1) For all $h \in H$ and $g \in G$, we have $f(hg) = hf(g)$.
(2) f is locally constant, i.e. continuous (recall that M has the discrete topology).

The group G acts on this space by right translation: $(gf)(x) = f(xg)$ for any $x, g \in G$ and any $f \in \mathrm{Ind}_H^G M$.

Proposition 2.6. *The G-action defined above gives $\mathrm{Ind}_H^G M$ the structure of a discrete G-module.*

Proof. Let $f \in \mathrm{Ind}_H^G M$. Then for every $g \in G$ there exists an open set $g \in V_g$ such that f is constant on V_g. Since a base of the topology on G is given by the cosets of open normal subgroups, there exists an open subgroup U_g such that $g \in gU_g \subset V_g$. Then $G = \bigcup_{g \in G} gU_g$, and by compactness of G there is a finite subcover $G = g_1U_{g_1} \cup \cdots \cup g_rU_{g_r}$ such that f is constant on each $g_iU_{g_i}$. Let $U = U_{g_1} \cap \cdots \cap U_{g_r}$. This is an open subgroup of G that is contained in $\mathrm{stab}_G(f)$. Indeed, for any $u \in U$ and $x \in G$, if $x \in g_iU_{g_i}$, then $xu \in g_iU_{g_i}$ as well, and thus $(uf)(x) = f(xu) = f(x)$. □

Proposition 2.7. *Let G be a profinite group and $H \subset G$ a closed subgroup.*

(1) $\mathrm{Ind}_H^G : \mathrm{Mod}_H \to \mathrm{Mod}_G$ *is a functor.*
(2) *If $K \subset H \subset G$ are closed subgroups, then $\mathrm{Ind}_K^G \simeq \mathrm{Ind}_H^G \mathrm{Ind}_K^H$.*
(3) *The functor Ind_H^G is exact. In other words, if $0 \to M \to N \to P \to 0$ is an exact sequence of H-modules, then $0 \to \mathrm{Ind}_H^G M \to \mathrm{Ind}_H^G N \to \mathrm{Ind}_H^G P \to 0$ is also exact.*

Proposition 2.8 (Frobenius Reciprocity)**.** *Let G be a profinite group and $H \subset G$ a closed subgroup. Let $M \in \mathrm{Mod}_G$ and $N \in \mathrm{Mod}_H$. Then*

$$\mathrm{Hom}_G(M, \mathrm{Ind}_H^G N) \simeq \mathrm{Hom}_H(M, N).$$

Proof. We can write down an explicit bijection. Indeed, if $\varphi \in \mathrm{Hom}_G(M, \mathrm{Ind}_H^G N)$, define $A(\varphi) \in \mathrm{Hom}_H(M, N)$ by $A(\varphi)(m) = (\varphi(m))(e) \in N$ for any $m \in M$.

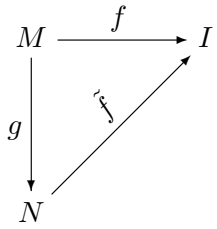
In the other direction, if $\psi \in \mathrm{Hom}_H(M, N)$, we define $B(\psi) \in \mathrm{Hom}_G(M, \mathrm{Ind}_H^G N)$ by $(B(\psi)(m))(g) = \psi(gm)$.

It is simple to check that $B(A(\varphi)) = \varphi$ and $A(B(\psi)) = \psi$. For instance, for all $m \in M$ and $g \in G$ and $\varphi \in \mathrm{Hom}_G(M, \mathrm{Ind}_H^G N)$, we have

$$B(A(\varphi))(m)(g) = A(\varphi)(gm) = \varphi(gm)(e) = g(\varphi(m))(e) = \varphi(m)(g).$$

Similarly, $A(B(\psi))(m) = (B(\psi)(m))(e) = \psi(em) = \psi(m)$. □

Definition 2.9. A G-module I is called *injective* if for any G-module morphism $f : M \to I$ and any injective $g : M \hookrightarrow N$, there exists $\tilde{f} : N \to I$ that completes the diagram:



Proposition 2.10. *Any divisible abelian group is an injective $\{e\}$-module. In particular, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\{e\}$-modules.*

Proof. Recall that a group G, written additively, is called divisible if for any $g \in G$ and any $n \in N$ there exists $h \in G$ such that $nh = g$.

Let G be a divisible abelian group, and f and g as above. View M as a submodule of N via g, and consider the set of all pairs (M', h), where $M \subset M' \subset N$ and h extends f. Define a partial order on this set by extension: $(M', h') \leq (M'', h'')$ if $M' \subset M''$ and $h''_{|M'} = h'$. Any ascending chain glues, so has an upper bound, so by Zorn's Lemma there exists a maximal pair (M', h). We need to show that $M' = N$. Suppose not. Then there exists $n \in N \setminus M'$. Let s be the order of n in N/M'. Then the map $\tilde{h} : M' + \langle n \rangle \to G$ given by

$$\tilde{h}(m' + an) = \begin{cases} h(m') & : s = \infty \\ h(m') + at & : s < \infty \end{cases}$$

extends h, contradicting the maximality of (M', h). Here $a \in \mathbb{Z}$ is arbitrary and $t \in G$ is an element satisfying $st = h(sn)$. □

The converse of Proposition 2.10 is also true, but we will not use this fact.

Corollary 2.11. *Let G be a profinite group and $H \subset G$ a closed subgroup. If I is an injective H-module, then $\mathrm{Ind}_H^G I$ is an injective G-module.*

Proof. Suppose we have $f : M \to \mathrm{Ind}_H^G I$ and $\varepsilon : M \hookrightarrow N$ is an embedding of G-modules. By Frobenius reciprocity, $f \in \mathrm{Hom}_G(M, \mathrm{Ind}_H^G I)$ corresponds to an H-module morphism $A(f) \in \mathrm{Hom}_H(M, I)$. By injectivity of I, we get an H-module map $\tilde{f} : N \to I$ satisfying $\tilde{f} \circ \varepsilon = A(f)$, hence, by a second application of Frobenius reciprocity, a map $B(\tilde{f}) : N \to \mathrm{Ind}_H^G I$ of G-modules.

We claim that $B(\tilde{f})$ is the map we are looking for, namely that it satisfies $B(\tilde{f}) \circ \varepsilon = f$. Indeed, for any $m \in M$ and any $g \in G$, since ε is a map of G-modules we have

$$\begin{aligned} (B(\tilde{f}) \circ \varepsilon)(m)(g) &= \tilde{f}(g\varepsilon(m)) = \tilde{f}(\varepsilon(gm)) = A(f)(gm) = f(gm)(e) = \\ &\quad g \cdot (f(m))(e) = f(m)(eg) = f(m)(g). \end{aligned}$$

Thus $(B(\tilde{f}) \circ \varepsilon)(m) = f(m) \in \mathrm{Ind}_H^G I$ for each $m \in M$, which proves our claim. □

Corollary 2.12. *If G is a profinite group and M is any G-module, there exists a G-module I into which M embeds (i.e. the category* Mod_G has enough injectives*).*

Proof. The claim is true for $\{e\}$-modules, i.e. abelian groups. Indeed, let M be any abelian group and $m \in M$ a non-trivial element. If $\langle m \rangle$ is a torsion group, then it embeds in $I_m = \mathbb{Q}/\mathbb{Z}$, and otherwise it embeds in $I_m = \mathbb{Q}$. Since $\langle m \rangle \subset M$, we can extend this

embedding to a map $f_m : M \to I_m$ satisfying $f_m(m) \neq 0$. We can collect all of these into a map $M \to \prod_{m \in M} I_m$ which clearly has trivial kernel. (Observe that a direct product of injective modules is injective.)

Now let M be any G-module, and let I be an injective group such that there exists an embedding $\psi : M \to I$ of abelian groups. By Frobenius reciprocity, it corresponds to a G-module map $B(\psi) : M \to \mathrm{Ind}_{\{e\}}^G I$. Note that $\mathrm{Ind}_{\{e\}}^G I$ is an injective G-module by Corollary 2.11, so it remains only to show that $B(\psi)$ is injective. But for any $m \in M$, we have that $B(\psi)(m)$ is the map $g \mapsto \psi(gm)$. Since ψ is an embedding, this can be the zero map only when $m = 0$. □

3. Cohomology of G-modules

3.1. **Definition of cohomology.** For any profinite group G, there is a left exact functor $\mathrm{Mod}_G \to \mathrm{Ab}$ sending a G-module M to the space of invariants $M^G = \{m \in M : \forall g \in G, gm = m\}$. The cohomology of M will be defined as the right derived functors of this functor. To make the exposition more elegant, we will define some notions (first introduced by Grothendieck in the famous Tohoku paper) that axiomatize the properties of derived functors.

Definition 3.1. A *(cohomological) δ-functor* $\mathrm{Mod}_G \to \mathrm{Ab}$ (the same definition can be made for any two abelian categories) consists of the following data:

- A series of covariant functors $H^i : \mathrm{Mod}_G \to \mathrm{Ab}$, for $i \in \mathbb{N}_0$.
- For each exact sequence of G-modules $0 \to M \to N \to P \to 0$ and each $i \geq 0$, a G-module homomorphism $\delta^i : H^i(P) \to H^{i+1}(M)$ with the following properties:
  (1) For $0 \to M \to N \to P \to 0$ exact as above, there is a long exact sequence of cohomology groups

$$0 \to H^0(M) \to H^0(N) \to H^0(P) \xrightarrow{\delta^0} H^1(M) \to H^1(N) \to H^1(P) \xrightarrow{\delta^1} H^2(M) \to \cdots$$

  (2) Given a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & P' & \longrightarrow & 0 \end{array}$$

  (where the diagram is commutative with exact rows), for each $i \geq 0$ the corresponding diagram commutes:

$$\begin{array}{ccc} H^i(P) & \xrightarrow{\delta^i} & H^{i+1}(M) \\ \downarrow & & \downarrow \\ H^i(P') & \xrightarrow{\delta^i} & H^{i+1}(M') \end{array}$$

Definition 3.2. A δ-functor (H^i, δ^i) is called *universal* if, for any δ-functor (G^i, ε_i) and any natural transformation $f_0 : H^0 \to G^0$, there exists a unique sequence of natural transformations $f_i : H^i \to G^i$ that commutes with the δ^i maps.

(Recall that a natural transformation of functors $f : \mathcal{F} \to \mathcal{G}$ provides, for each $M \in \mathrm{Ob}(\mathcal{F})$, a map $f_M : \mathcal{F}(M) \to \mathcal{G}(M)$, such that for any arrow $M \to N$ the obvious square below commutes.)

$$\begin{array}{ccc} \mathcal{F}(M) & \xrightarrow{f_M} & \mathcal{G}(M) \\ \downarrow & & \downarrow \\ \mathcal{F}(N) & \xrightarrow{f_N} & \mathcal{G}(N). \end{array}$$

Proposition 3.3. *Let G be a profinite group. There exists a δ-functor $(H^i(G,-),\delta^i)$ from Mod_G to* Ab *satisfying the following two properties:*

1. *For every $M \in \mathrm{Mod}_G$, we have $H^0(G,M) = M^G$.*
2. *If $I \in \mathrm{Mod}_G$ is injective, then $H^i(G,I) = 0$ for every $i > 0$.*

Proof. Suppose that such a δ-functor exists. What can we say about it? Any $M \in \mathrm{Mod}_G$ embeds into an injective G-module I^0; let N be the cokernel, so that we have a short exact sequence $0 \to M \xrightarrow{\iota} I^0 \xrightarrow{\pi} N \to 0$. By (2) and the long exact sequence, we would have an exact sequence

$$0 \to M^G \xrightarrow{\iota^G} (I^0)^G \xrightarrow{\pi^G} N^G \to H^1(G,M) \to H^1(G,I^0) \to \cdots,$$

whence $H^1(G,M) = \operatorname{coker} \pi^G$ and we must have $H^{i+1}(M) \simeq H^i(N)$ for all $i \geq 1$. Since we know that $H^0(G,M) = M^G$ for any $M \in \mathrm{Mod}_G$, the observations above provide a recursive algorithm for computing $H^i(G,M)$.

We can streamline this process by fixing an injective resolution of M, namely an exact sequence

$$0 \to M \to I^0 \xrightarrow{\alpha_0} I^1 \xrightarrow{\alpha_1} I^2 \xrightarrow{\alpha_2} \cdots.$$

This is possible because the category Mod_G has enough injectives by Corollary 2.12. Indeed, M embeds in an injective module I^0 as above. The quotient $I^0/M = N$ embeds into an injective module I^1, and this lifts to a map $\alpha_0 : I^0 \to I^1$. Then $I^1/\alpha_0(I^0)$ embeds into I^2, and this lifts to a map $\alpha_1 : I^1 \to I^2$. Continue forever.

Now apply the G-invariants functor to the injective resolution above, to obtain a complex

$$0 \to M^G \to (I^0)^G \xrightarrow{d_0} (I^1)^G \xrightarrow{d_1} (I^2)^G \xrightarrow{d_2} \cdots \tag{1}$$

This complex need not be exact, since the G-invariants functor is only left exact. However, it obviously inherits the property that $d_{i+1} \circ d_i = 0$ for all $i \geq 0$, and hence $\operatorname{im} d_{i-1} \subseteq \ker d_i$ for all $i \geq 1$.

We claim that $H^1(G,M) \simeq \operatorname{coker} \pi^G \simeq (\ker d_1)/(\operatorname{im} d_0)$. Indeed, since $\alpha_0 : I^0 \to I^1$ is the composition of π with an embedding $\varepsilon : N \hookrightarrow I^1$, we see that $\operatorname{im} d_0 = \varepsilon(\operatorname{im} \pi^G)$. On the other hand, $\alpha_1 : I^1 \to I^2$ is the composition $I^1 \twoheadrightarrow I^1/\alpha_0(I^0) = I^1/\varepsilon(N) \hookrightarrow I^2$, hence we see that $\ker \alpha_1 = \varepsilon(N)$ and $\ker d_0 = \varepsilon(N^G)$. Thus, the embedding ε induces an isomorphism $(\ker d_1)/(\operatorname{im} d_0) \simeq N^G/(\operatorname{im} \pi^G) = \operatorname{coker} \pi^G$.

Observe that $0 \to N \to I^1 \to I^2 \to \cdots$ is an injective resolution of N. Thus, by the argument of the previous paragraph, $H^2(G,M) \simeq H^1(G,N) \simeq (\ker d_2)/(\operatorname{im} d_1)$. By induction, for all $i \geq 1$, we get

$$H^i(G,M) = (\ker d_i)/(\operatorname{im} d_{i-1}).$$

In fact, this construction works. However, it is not at all obvious that the $H^i(G,-)$ are well-defined functors and that there exist maps δ^i that turn all this data into a δ-functor. We will soon sketch a proof of these facts, but for the time being we will assume them. □

The construction of the cohomology groups $H^i(G,M)$ that we just gave does not, in practice, provide a useful way of computing them. Indeed, the injective modules constructed in Corollaries 2.11 and 2.12 are huge and unwieldy. Moreover, the proof of the existence of an embedding $M \hookrightarrow I$ ultimately relies on the argument with Zorn's Lemma in the proof of Proposition 2.10, and thus is completely non-constructive. In practice, when one works with injective resolutions one often loses an explicit understanding of the situation. Thus, our first priority is to find a more explicit way to compute cohomology.

Definition 3.4. A G-module J is called *acyclic* if $H^i(G,J) = 0$ for all $i > 0$.

Clearly, all injective G-modules are acyclic. Note that in the construction of the previous proof, all we used about injective modules was their acyclicity and the existence of injective resolutions. The injectivity will be used more intensively in the proof that the above construction actually gives a δ-functor. In the meantime, we can conclude the following, which is useful for computing cohomology in practice.

Proposition 3.5. *Let M be a G-module, and consider an acyclic resolution*

$$0 \to M \to J^0 \to J^1 \to J^2 \to \cdots,$$

namely an exact sequence where the J^i are acyclic for all $i \geq 0$. Taking G-invariants gives rise to a complex

$$0 \to M^G \to (J^0)^G \xrightarrow{d_0} (J^1)^G \xrightarrow{d_1} (J^2)^G \cdots$$

Then for all $i > 0$ we have $H^i(G,M) \simeq (\ker d_i)/(\operatorname{im} d_{i-1})$.

Definition 3.6. Let G be a profinite group and M a G-module. For each $i \geq 0$, define $\mathcal{C}^i(G,M)$ to be the space of continuous (i.e. locally constant) functions $\varphi : G^{i+1} \to M$. Let G act on this space by

$$(g\varphi)(g_0,\ldots,g_i) = g \cdot (\varphi(g_0 g,\ldots,g_i g))$$

for $\varphi \in \mathcal{C}^i(G,M)$ and $g, g_0, \ldots, g_i \in G$.

We have an obvious injective map $f : M \to \mathcal{C}^0(G,M)$ that sends each $m \in M$ to the constant function $\varphi_m : G \to M$ satisfying $\varphi_m(g) = m$ for all $g \in G$. Observe that f is G-equivariant, since $(x\varphi_m)(g) = x\varphi_m(gx) = xm = \varphi_{xm}(g)$ for all $x, g \in G$.

Moreover, for all $i \geq 0$ we have a map $f_i : \mathcal{C}^i(G,M) \to \mathcal{C}^{i+1}(G,M)$ given by

$$(f_i\varphi)(g_0,\ldots,g_{i+1}) = \sum_{j=0}^{i+1} (-1)^j \varphi(g_0,\ldots,\hat{g}_j,\ldots,g_{i+1}), \tag{2}$$

where $\hat{g}_j$ denotes omission of the variable g_j, so that each summand on the right-hand side provides $i+1$ arguments for the function φ. It is clear that f_i is G-equivariant.

Lemma 3.7. *The complex*

$$0 \to M \xrightarrow{f} \mathcal{C}^0(G,M) \xrightarrow{f_0} \mathcal{C}^1(G,M) \xrightarrow{f_1} \mathcal{C}^2(G,M) \xrightarrow{f_2} \cdots \tag{3}$$

is exact.

Proof. It is easy to check that this is indeed a complex, namely that the composition of any two consecutive arrows is the zero map. Observe that f is obviously injective. For any $\varphi \in \mathcal{C}^0(G, M)$ we have $(f_0\varphi)(g_0, g_1) = \varphi(g_1) - \varphi(g_0)$. Hence, if $\varphi \in \ker f_0$ then φ is a constant function and thus contained in the image of f.

Now let $i > 0$. Suppose that $\varphi \in \mathcal{C}^i(G, M)$ lies in the kernel of f_i, and define a locally constant map $\psi : G^i \to M$ by $\psi(g_1, \dots, g_i) = \varphi(e, g_1, \dots, g_i)$. Then one checks that

$$\varphi(g_1, \dots, g_{i+1}) = \sum_{j=1}^{i+1} (-1)^{j-1} \psi(g_1, \dots, \hat{g}_j, \dots, g_{i+1}),$$

so that $\varphi = f_{i-1}(\psi)$. This gives exactness at $\mathcal{C}^i(G, M)$. □

3.2. **Universal δ-functors and Shapiro's Lemma.** If we knew that the complex (3) were an acyclic resolution of M, we would be able to apply Proposition 3.5 to compute cohomology. To prove the acyclicity of $\mathcal{C}^i(G, M)$ we will need a few more tools at our disposal.

Definition 3.8. A functor $\mathcal{F} : \mathrm{Mod}_G \to \mathrm{Ab}$ is called *effaceable*[1] if for any $M \in \mathrm{Mod}_G$ there exists an injection $\iota : M \hookrightarrow N$ such that $\mathcal{F}(\iota) : \mathcal{F}(M) \to \mathcal{F}(N)$ is the zero map.

Lemma 3.9 (Grothendieck)**.** *If (H^i, δ^i) is a δ-functor such that the functors H^i are effaceable for all $i > 0$, then (H^i, δ^i) is a universal δ-functor.*

Proof. Let (G^i, ε^i) be another δ-functor, and let $f_0 : H^0 \to G^0$ be a natural transformation. The first step is to construct the unique natural transformations $f_i : H^i \to G^i$ for all $i > 0$. Assume, by induction, that f_j has been constructed and shown to be unique for all $j < i$. Let $M \in \mathrm{Mod}_G$ and let $\iota : M \to J$ be an embedding, where J is a G-module such that $H^i(J) = 0$; this exists because (H^i, δ^i) is effaceable. Let C be the cokernel. Then the short exact sequence $0 \to M \xrightarrow{\iota} J \to C \to 0$ gives rise to the diagram

$$\begin{array}{ccccccccc}
H^{i-1}(M) & \longrightarrow & H^{i-1}(J) & \longrightarrow & H^{i-1}(C) & \xrightarrow{\delta^{i-1}} & H^i(M) & \longrightarrow & H^i(J) = 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle f_{i-1,C}} & & \vdots & & \\
G^{i-1}(M) & \longrightarrow & G^{i-1}(J) & \longrightarrow & G^{i-1}(C) & \xrightarrow{\varepsilon^{i-1}} & G^i(M) & \longrightarrow & G^i(J).
\end{array}$$

The dotted arrow $f_{i,M}$ is constructed by a simple diagram chase. Indeed, let $c \in H^i(M)$. Since δ^{i-1} is surjective, there exists $b \in H^{i-1}(C)$ such that $\delta^{i-1}(c)$. Then we define $f_{i,M}(c) = \varepsilon^{i-1} f_{i-1,C}(b)$. It is easy to check that this is independent of the choice of b. It remains to show that $f_{i,M} : H^i(M) \to G^i(M)$ is independent of the choice of ι and that f_i is a natural transformation. We will omit these details here. □

Corollary 3.10. *For any profinite group G, the δ-functor $(H^i(G, -), \delta^i)$ constructed in Proposition 3.3 is universal.*

Proof. Since every G-module M injects into an injective module, the functors $H^i(G, -)$ are effaceable for all $i > 0$. □

[1] "Effaceable" is a French word taken directly from Grothendieck's Tohoku paper. S. Lang argues forcefully that authors writing in English should employ its English translation "erasable." Nevertheless, here we follow the standard usage.

Proposition 3.11 (Shapiro's Lemma)**.** *Let G be a profinite group, $H \subset G$ a closed subgroup, and M an H-module. Then there is a natural isomorphism $H^i(H, M) \simeq H^i(G, \mathrm{Ind}_H^G M)$ for all $i \geq 0$.*

By "natural" we mean that these isomorphisms arise from a natural transformation of functors $H^i(H, -) \to H^i(G, -) \circ \mathrm{Ind}_H^G$.

Proof. The strategy of the proof is to show that both functors are universal δ-functors extending the same H^0. Indeed, observe that $H^0(G, \mathrm{Ind}_H^G M) = (\mathrm{Ind}_H^G M)^G$ consists of (continuous) functions $f : G \to M$ such that $f(hg) = h \cdot f(g)$ for all $h \in H, g \in G$ and $f(gx) = f(g)$ for all $g, x \in G$. The second condition forces f to be a constant function $f_m : g \mapsto m$ for some $m \in M$, while the first condition imposes $hm = m$ for all $h \in H$, in other words that $m \in M^H$. Thus we have $M^H \xrightarrow{\sim} (\mathrm{Ind}_H^G M)^G$ for all $M \in \mathrm{Mod}_H$, and this is clearly a natural isomorphism of functors.

We already know by Corollary 3.10 that $(H^i(H, -), \delta^i)$ is a universal δ-functor. Since $(H^i(G, -), \delta^i)$ is a δ-functor and Ind_H^G is exact, it is not hard to see that $(H^i(G, \mathrm{Ind}_H^G -))$ is a δ-functor. For any H-module M, let $\iota : M \to I$ be an embedding of M into an injective H-module. The induced map $H^i(G, \mathrm{Ind}_H^G M) \to H^i(G, \mathrm{Ind}_H^G I) = 0$ is the zero map when $i > 0$, since $\mathrm{Ind}_H^G I$ is an injective G-module by Corollary 2.11. Thus the functors $H^i(G, \mathrm{Ind}_H^G -)$ are effaceable for $i > 0$, and universality follows from Lemma 3.9. □

Lemma 3.12. *All $\{e\}$-modules are acyclic. If G is any profinite group and M any $\{e\}$-module, then $\mathrm{Ind}_{\{e\}}^G M$ is an acyclic G-module.*

Proof. Since the functor taking $\{e\}$-invariants is just the identity functor, the complex (1) is exact for any $\{e\}$-module M. Thus M is acyclic. The second part of the claim now follows from Shapiro's Lemma. □

Before returning to our explicit computation of the cohomology $H^i(G, M)$, we point out a simple consequence of Lemma 3.12.

Corollary 3.13. *Let L/K be a finite Galois extension of fields. Then $H^i(\mathrm{Gal}(L/K), L) = 0$ for all $i > 0$.*

Proof. Here $G = \mathrm{Gal}(L/K)$ acts on L in the obvious way. Since any finite separable extension is simple, there exists an element $\alpha \in L$ such that $L = K(\alpha)$. Moreover, it is a standard fact from Galois theory that $\{\sigma(\alpha) : \sigma \in G\}$ is a K-basis of L. Thus any $\beta \in L$ can be written in the form $\beta = \sum_{\sigma \in G} \beta_\sigma \sigma(\alpha)$, where $\beta_\sigma \in K$.

We can obtain a G-module isomorphism $L \to \mathrm{Ind}_{\{e\}}^G K$ by sending $\beta \in L$ to the function $f_\beta : G \to K$ given by $f_\beta(\sigma) = \beta_{\sigma^{-1}}$. Indeed, for any $\tau \in G$ we have $\tau(\beta) = \sum_{\sigma \in G} \beta_\sigma \tau\sigma(\alpha) = \sum_{\sigma \in G} \beta_{\tau^{-1}\sigma} \sigma(\alpha)$. On the other hand, we have

$$(\tau f_\beta)(\sigma) = f_\beta(\sigma\tau) = \beta_{\tau^{-1}\sigma^{-1}} = f_{\tau(\beta)}(\sigma)$$

for all $\sigma, \tau \in G$. The claim now follows from Lemma 3.12. □

3.3. **Explicit computation of cohomology.** We are now ready to return to the G-modules $\mathcal{C}^i(G, M)$ that were defined earlier.

Lemma 3.14. *For every $i \geq 0$, let $\mathcal{LC}^i(G, M)$ be the abelian group of locally constant functions $f : G^i \to M$, viewed as a $\{e\}$-module. There is a G-module isomorphism $\theta : \mathcal{C}^i(G, M) \xrightarrow{\sim} \mathrm{Ind}_{\{e\}}^G \mathcal{LC}^i(G, M)$.*

Proof. Given $\varphi \in \mathcal{C}^i(G,M)$, define $\theta_\varphi \in \mathrm{Ind}_{\{e\}}^G \mathcal{LC}^i(G,M)$ by

$$\theta_\varphi(g)(g_1,\dots,g_i) = g\varphi(g,g_1g,\dots,g_ig).$$

Clearly θ is a group homomorphism. We note that θ is G-equivariant. Indeed, if $x \in G$, then

$$\theta_{x\varphi}(g)(g_1,\dots,g_i) = g(x\varphi)(g,g_1g,\dots,g_ig) = gx\varphi(gx,g_1gx,\dots,g_igx) = \theta_\varphi(gx)(g_1,\dots,g_i).$$

Thus $\theta_{x\varphi}(g) = \theta_\varphi(gx) = (x\theta_\varphi)(g)$ for all $g \in G$.

Note that if $\theta_\varphi(g)$ is the zero element of $\mathcal{LC}^i(G,M)$ for all $g \in G$, then $\varphi(g,g_1g,\dots,g_ig) = 0$ for all $g,g_1,\dots,g_i \in G$, whence $\varphi = 0$. Thus θ is injective. Moreover, given $\psi \in \mathcal{LC}^i(G,M)$ we can define $\eta \in \mathcal{C}^i(G,M)$ by $\eta(g_0,\dots,g_i) = g_0^{-1}\psi(g_0)(g_1g_0^{-1},\dots,g_ig_0^{-1})$. It is easy to see that $\psi = \theta_\eta$, so that θ is also surjective. □

Corollary 3.15. *Let G be a profinite group and M a G-module. The G-modules $\mathcal{C}^i(G,M)$ are acyclic for all $i \geq 0$.*

Proof. This is immediate from Lemma 3.14 and Lemma 3.12. □

Therefore, the complex (3) is an acyclic resolution of M, and by Proposition 3.5 it may be used to compute the cohomology of M. We will now do this and observe some consequences.

First of all, it will be useful to find a convenient parametrization of the spaces $\mathcal{C}^i(G,M)^G$ of G-invariants. Recall that these spaces consist of locally constant functions $\varphi : G^{i+1} \to M$ such that $g \cdot \varphi(g_0g,\dots,g_ig) = \varphi(g_0,\dots,g_i)$ for all $g,g_0,\dots,g_i \in G$.

If $i > 0$, let $C^i(G,M)$ be the space of locally constant functions $\psi : G^i \to M$, with no G-module structure. There is a group homomorphism $C^i(G,M) \to \mathcal{C}^i(G,M)^G$ sending $\psi \in C^i(G,M)$ to the map

$$\varphi_\psi(g_0,\dots,g_i) = g_0^{-1}\psi(g_0g_1^{-1},g_1g_2^{-1},\dots,g_{i-1}g_i^{-1}).$$

We wish to show that this is an isomorphism of groups by constructing an inverse map. Given $\varphi \in \mathcal{C}^i(G,M)^G$, we want to find $\psi_\varphi \in C^i(G,M)$ such that $g_0^{-1}\psi_\varphi(g_0g_1^{-1},g_1g_2^{-1},\dots,g_{i-1}g_i^{-1}) = \varphi(g_0,\dots,g_i) = g_0^{-1}\varphi(e,g_1g_0^{-1},\dots,g_ig_0^{-1})$. The map

$$\psi_\varphi(x_1,\dots,x_i) = \varphi(e,x_1^{-1},(x_1x_2)^{-1},\dots,(x_1x_2\cdots x_i)^{-1}) \tag{4}$$

works. Note that all of this holds for $i = 0$ as well, where $C^0(G,M)$ is the space of constant functions. The next step is to determine the map $d_i : C^i(G,M) \to C^{i+1}(G,M)$ induced by f_{i+1}. Given $\psi \in C^i(G,M)$, a straightforward computation shows that

$$\begin{aligned}(f_{i+1}\varphi_\psi)(g_0,\dots,g_{i+1}) &= g_1^{-1}\psi(g_0g_1^{-1},\dots,g_{i-1}g_i^{-1}) + \\ &\quad \sum_{j=1}^{i}(-1)^j g_0^{-1}\psi(g_0g_1^{-1},\dots,g_{j-1}g_{j+1}^{-1},\dots,g_ig_{i+1}^{-1}) + \\ &\quad (-1)^{i+1}g_0^{-1}\psi(g_0g_1^{-1},\dots,g_{i-1}g_i^{-1}).\end{aligned}$$

It follows that $d_i\psi = \psi_{f_{i+1}(\varphi_\psi)}$ is given by the following formula:

$$\begin{aligned}d_i(\psi)(x_1,\dots,x_{i+1}) &= x_1\psi(x_2,\dots,x_{i+1}) - \psi(x_1x_2,x_3,\dots,x_{i+1}) + \\ &\quad \psi(x_1,x_2x_3,x_4,\dots,x_{i+1}) + \cdots + (-1)^i\psi(x_1,x_2,\dots,x_ix_{i+1}) + \\ &\quad (-1)^{i+1}\psi(x_1,\dots,x_i).\end{aligned} \tag{5}$$

Finally, note that if $\varphi \in \mathcal{C}^0(G,M)^G$, then $\varphi(g_1) = g\varphi(g_1 g)$ for all $g, g_1 \in G$. Taking $g_1 = e$, we see that $m = g\varphi(g)$ is independent of g. Moreover, $\psi_{f_0(\varphi)}(g) = f_0\varphi(e, g^{-1}) = \varphi(g^{-1}) - \varphi(e)$. Thus $\operatorname{im} d_0$ consists of functions $\psi : G \to M$ of the form $\psi(g) = gm - m$ for a fixed $m \in M$.

For each $i > 0$, let $Z^i(G,M) = \ker d_i$ and $B^i(G,M) = \operatorname{im} d_{i-1}$ denote the cocycles and coboundaries, respectively. The Proposition 3.5 tells us that:

Proposition 3.16. *Let G be a profinite group and M a G-module. Then for all $i > 0$, we have $H^i(G,M) \simeq Z^i(G,M)/B^i(G,M)$.*

In particular, if $\psi \in C^1(G,M)$, then $d_1(\psi)(g_1, g_2) = g_1\psi(g_2) - \psi(g_1 g_2) + \psi(g_1)$. Thus the 1-cocycles are maps $\psi : G \to M$ satisfying $\psi(g_1 g_2) = \psi(g_1) + g_1\psi(g_2)$. These are called *crossed homomorphisms*.

Example 3.17. If G is a finite cyclic group and M is a G-module, then $Z^1(G,M)$ is isomorphic to the abelian group $N(M) = \{m \in M : \sum_{g\in G} gm = 0\}$.

Proof. Let σ be a generator of G and let $\varphi \in Z^1(G,M)$. Observe that $\varphi(e) = \varphi(e) + e\varphi(e)$, whence $\varphi(e) = 0$. Similarly, for all $j \geq 1$ we have $\varphi(\sigma^j) = \varphi(\sigma) + \sigma(\varphi(\sigma^{j-1}))$. It follows by induction that $\varphi(\sigma^j) = \sum_{k=0}^{j-1} \sigma^k(\varphi(\sigma))$ for all $j \geq 1$. Taking $j = |G|$, we get

$$0 = \varphi(e) = \varphi(\sigma^j) = \sum_{k=0}^{j-1} \sigma^k(\varphi(\sigma)) = \sum_{\tau\in G} \tau(\varphi(\sigma)).$$

Thus $\varphi(\sigma) \in N(M)$, and $\varphi \mapsto \varphi(\sigma)$ is our candidate for an isomorphism between $Z^1(G,M)$ and $N(M)$. Clearly it is injective, since φ is determined by $\varphi(\sigma)$ by the recursive formula above. To show surjectivity, let $m \in N(M)$ and define $\varphi_m : G \to M$ by $\varphi_m(\sigma^j) = \sum_{k=0}^{j-1} \sigma^k(m)$. It is easy to check that this is indeed a 1-cocycle, and clearly $\varphi_m(\sigma) = m$. □

The following important result is a generalization, by Emmy Noether, of Theorem 90 in Hilbert's *Zahlbericht*, which itself originated with Kummer.

Theorem 3.18 (Hilbert 90)**.** *If L/K is a finite Galois extension, then $H^1(\mathrm{Gal}(L/K), L^\times) = 0$.*

Proof. We write the abelian group $L^\times$ multiplicatively and denote $G = \mathrm{Gal}(L/K)$. For every $\alpha \in L^\times$ consider the "Poincaré series"

$$b(\alpha) = \sum_{\sigma\in G} \varphi(\sigma)\sigma(\alpha) \in L.$$

By Dedekind's Lemma the automorphisms σ are linearly independent over L, so there must exist some $\alpha \in L^\times$ such that $b = b(\alpha) \neq 0$. Let $\varphi \in Z^1(G, L^\times)$. This means that for all $\sigma, \tau \in G$ we have $\varphi(\sigma\tau) = \varphi(\sigma)\sigma(\varphi(\tau))$. Now for every $\sigma \in G$ we have

$$\sigma(b) = \sum_{\tau\in G} \sigma(\varphi(\tau))\sigma\tau(\alpha) = \sum_{\tau\in G} \varphi(\sigma\tau)\varphi(\sigma)^{-1}\sigma\tau(\alpha) = (\varphi(\sigma))^{-1} \sum_{\tau\in G} \varphi(\sigma\tau)\sigma\tau(\alpha) = (\varphi(\sigma))^{-1} b.$$

This means that $\varphi(\sigma) = \sigma(b^{-1}) \cdot b = \sigma(m)m^{-1}$, where $m = b^{-1}$. Thus $\varphi \in B^1(G, L^\times)$. □

We now derive the original Hilbert 90 as a corollary.

Corollary 3.19. *Let L/K be a cyclic Galois extension and let σ be a generator of $G = \mathrm{Gal}(L/K)$. Let $\alpha \in L$ be an element with $\mathrm{N}_{L/K}(\alpha) = 1$. Then there exists $\beta \in L^\times$ such that $\alpha = \sigma(\beta)\beta^{-1}$.*

Proof. In the notation of Example 3.17, we have that $N(L^\times) = \{\gamma \in L^\times : \mathrm{N}_{L/K}(\gamma) = 1\}$. By that example, there exists $\varphi \in Z^1(G, L^\times)$ such that $\varphi(\sigma) = \alpha$. By Hilbert 90, we see that $Z^1(G, L^\times) = B^1(G, L^\times)$, hence there exists $\beta \in L^\times$ such that $\varphi(\tau) = \tau(\beta)\beta^{-1}$ for all $\tau \in G$. □

3.4. **Some homological algebra.** At this point we finally go back to the construction in Proposition 3.3 and show why it indeed gives a well-defined δ-functor. Recall that a complex $\mathbf{C}$ consists of the data

$$C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2 \xrightarrow{\partial^2} C^3 \cdots$$

where $\partial^{i+1} \circ \partial^i = 0$ for all $i \geq 0$. The cohomology of the complex is given by $H^i(\mathbf{C}) = (\ker \partial^{i+1})/(\operatorname{im} \partial^i)$ for all $i \geq 0$.

Definition 3.20. Let $\mathbf{C}$ and $\mathbf{D}$ be two complexes. A *cochain map* $\varphi : \mathbf{C} \to \mathbf{D}$ is a collection of morphisms $\varphi^i : C^i \to D^i$ that commute with the coboundary maps of the complexes. In other words, the diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial^{i-1}} & C^i & \xrightarrow{\partial^i} & C^{i+1} & \xrightarrow{\partial^{i+1}} & \cdots \\
 & & \downarrow{\scriptstyle \varphi^i} & & \downarrow{\scriptstyle \varphi^{i+1}} & & \\
\cdots & \xrightarrow{\partial^{i-1}} & D^i & \xrightarrow{\partial^i} & D^{i+1} & \xrightarrow{\partial^{i+1}} & \cdots
\end{array}$$

commutes for each i.

Observe that a cochain map $\varphi : \mathbf{C} \to \mathbf{D}$ induces maps $H^i(\varphi) : H^i(\mathbf{C}) \to H^i(\mathbf{D})$ in a natural way. We will sometimes write ∂^i_C and ∂^i_D to distinguish the coboundary maps coming from our two complexes. The main tool for proving that two complexes have the same cohomology is the notion of homotopy.

Definition 3.21. Let $\mathbf{C}$ and $\mathbf{D}$ be two complexes, and let $\varphi, \psi : \mathbf{C} \to \mathbf{D}$ be two cochain maps. A *homotopy* $\Sigma : \varphi \to \psi$ is a family of morphisms $\Sigma^i : C^i \to D^{i-1}$ such that $\psi - \varphi = \partial\Sigma + \Sigma\partial$, i.e. $\psi^i - \varphi^i = \partial^{i-1} \circ \Sigma^i + \Sigma^{i+1} \circ \partial^i$ for all i.

Lemma 3.22. *Let $\mathbf{C}$ and $\mathbf{D}$ be two complexes, and let $\varphi, \psi : \mathbf{C} \to \mathbf{D}$ be two cochain maps. Suppose that there exists a homotopy $\Sigma : \varphi \to \psi$. Then $H^i(\varphi) = H^i(\psi)$ for all i.*

Proof. Suppose that $x \in C^i$ lies in the kernel of ∂^i. We need to show that the two elements $\varphi^i(x)$ and $\psi^i(x)$ of D^i, which clearly lie in the kernel of ∂^i, differ by an element of the image of ∂^{i-1}. However,

$$\psi^i(x) - \varphi^i(x) = \partial^{i-1}(\Sigma^i(x)) + \Sigma^{i+1}(\partial^i(x)) = \partial^{i-1}(\Sigma^i(x))$$

by the definition of homotopy, and this is exactly what we need. □

We say that φ and ψ are *homotopic* if there exists a homotopy $\Sigma : \varphi \to \psi$ and leave it to the reader to verify that this is an equivalence relation on the set of cochain maps.

Proposition 3.23. *Let $\mathbf{C}$ and $\mathbf{D}$ be two complexes. Suppose that $\mathbf{C}$ is acyclic, i.e. that $H^i(\mathbf{C}) = 0$ for all $i \geq 1$, whereas $\mathbf{D}$ is injective, namely that all the D^i are injective objects. Let $\eta : H^0(\mathbf{C}) \to H^0(\mathbf{D})$ be a homomorphism. Then there exists a cochain map $\varphi : \mathbf{C} \to \mathbf{D}$ inducing η. Moreover, any two cochain maps inducing η are homotopic.*

Proof. Observe that $H^0(\mathbf{C}) = \ker \partial_C^0 \subset C^0$, and similarly $H^0(\mathbf{D}) \subset D^0$. Thus η gives a homomorphism $H^0(\mathbf{C}) \to D^0$, which extends to a homomorphism $\varphi^0 : C^0 \to D^0$ by injectivity of D^0. Now assume by induction that φ^{i-1} has been constructed. If $y \in \operatorname{im} \partial_C^{i-1}$, then choose $x \in C^{i-1}$ such that $\partial_C^{i-1}(x) = y$ and define $\varphi^i(y) = \partial_D^{i-1}(\varphi^{i-1}(x))$. This is well-defined; indeed, if x and x' are two preimages of x, then $z = x' - x \in \ker \partial_C^{i-1}$. If $i = 1$, then $\varphi^0(z) = \eta(z) \in \ker \partial_D^0$. If $i > 1$, then $z \in \operatorname{im} \partial_C^{i-2}$, and a very simple diagram chase again shows that $\varphi^{i-1}(z) \in \ker \partial_D^{i-1}$. Hence we get a map $\varphi^i : \operatorname{im} \partial_C^{i-1} \to D^i$. It extends to a homomorphism $\varphi^i : C^i \to D^i$ by the injectivity of i, and our definition ensures that the following square commutes:

$$\begin{array}{ccc} C^{i-1} & \xrightarrow{\partial_C^{i-1}} & C^i \\ \downarrow{\scriptstyle \varphi^{i-1}} & & \downarrow{\scriptstyle \varphi^i} \\ D^{i-1} & \xrightarrow{\partial_D^{i-1}} & D^i \end{array}$$

Now let φ and ψ be two cochain maps inducing $H^0(\varphi)$; we will construct a homotopy Σ between them. If $y \in \operatorname{im} \partial_C^0$, then define $\Sigma^1(y) = \psi^0(x) - \varphi^0(x)$, where $\partial^0(x) = y$; observe that this is well-defined. We can extend this to a homomorphism $\Sigma^1 : C^1 \to D^0$ by injectivity of D^0.

Suppose now that Σ^i has been constructed. If $y \in \operatorname{im} \partial_C^i$, then define $\Sigma^{i+1}(y) = \psi^i(x) - \varphi^i(x) - \partial_D^{i-1}(\Sigma^i(x))$, where $x \in C^i$ satisfies $\partial_C^i(x) = y$. If this is well-defined, it will extend to a homomorphism $\Sigma^{i+1} : C^{i+1} \to D^i$ by the injectivity of D^i. It is indeed well-defined, because if x' is another pre-image of y, then $z = x' - x \in \ker \partial_C^i = \operatorname{im} \partial_C^{i-1}$ by acyclicity of $\mathbf{C}$. Let $w \in C^{i-1}$ satisfy $\partial_C^{i-1}(w) = z$. Then

$$\begin{aligned} (\psi^i - \varphi^i)(z) = \quad & (\psi^i - \varphi^i)(\partial_C^{i-1}(w)) = \\ & \partial_D^{i-1}(\psi^{i-1} - \varphi^{i-1})(w) = \partial_D^{i-1}(\partial_D^{i-2}\Sigma^{i-1}w + \Sigma^i \partial_C^{i-1} w) = \partial_D^{i-1}\Sigma^i(z). \quad \square \end{aligned}$$

Corollary 3.24. *Let G be a profinite group and M a G-module. The cohomology groups $H^i(G, M)$, as constructed in the proof of Proposition 3.3, do not depend on the choice of injective resolution. Moreover, $H^i(G, -) : \mathrm{Mod}_G \to \mathrm{Ab}$ is a functor for all $i \geq 0$.*

Proof. Let $0 \to M \to I^0 \overset{\partial_I^0}{\to} I^1 \overset{\partial_I^1}{\to} \cdots$ and $0 \to M \to J^0 \overset{\partial_J^0}{\to} J^1 \overset{\partial_J^1}{\to} \cdots$ be two injective resolutions of M. Note that $H^0(\mathbf{I}) = H^0(\mathbf{J}) = M$. By the previous proposition, the identity maps in either direction extend to cochain maps $\varphi : \mathbf{I} \to \mathbf{J}$ and $\psi : \mathbf{J} \to \mathbf{I}$. Moreover, the cochain map $\psi \circ \varphi : \mathbf{I} \to \mathbf{I}$ induces the identity on $H^0(\mathbf{I})$ and is thus homologous to the identity cochain map $\mathbf{1}_\mathbf{I} : \mathbf{I} \to \mathbf{I}$. Let $\Sigma : \mathbf{1}_\mathbf{I} \to \psi \circ \varphi$ be a homotopy.

Now apply the G-invariants functor to everything. The resulting complexes $\mathbf{I}^G$ and $\mathbf{J}^G$ are no longer necessarily acyclic or injective, so we can't apply the previous proposition directly. However, the relation $\psi^i \circ \varphi^i - 1 = \partial_I^{i-1} \circ \Sigma^i + \Sigma^{i+1} \circ \partial_I^i$ clearly survives the application of the G-invariants functor. Since $\mathbf{1}_{\mathbf{I}^G}$ obviously induces the identity on cohomology, we find that the composition $H^i(\mathbf{I}^G) \overset{\mathrm{H}^i(\varphi)}{\to} H^i(\mathbf{J}^G) \overset{H^i(\psi)}{\to} H^i(\mathbf{I}^G)$ is the identity by Lemma 3.22. Hence we get isomorphic results when we compute $H^i(G, M)$ via the injective resolutions $\mathbf{I}$ and $\mathbf{J}$.

Finally if $f : M \to N$ is any map of G-modules, then we can choose injective resolutions $0 \to M \to \mathbf{I}$ and $0 \to N \to \mathbf{J}$. Then $H^0(\mathbf{I}) \simeq M$ and $H^0(\mathbf{J}) \simeq N$, so the homomorphism

$f : H^0(\mathbf{I}) \to H^0(\mathbf{J})$ is induced by a cochain map φ. By the previous proposition, all possible choices of φ are homologous and thus induce the same maps on cohomology. Applying the G-invariants functor as above, we find that $H^i(f) : H^i(G, M) \to H^i(G, N)$ is well-defined and functorial. □

Remark 3.25. A careful inspection of our arguments shows that the only property of the G-invariants functor that figures in the proof that $H^i(G, -)$ are indeed functors is that $M \mapsto M^G$ is left exact. Thus we can make the same construction for any left exact functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $\mathcal{A}$ has enough injectives. Indeed, if M is an object of $\mathcal{A}$, we can find a resolution by injective objects:

$$0 \to M \to I^0 \stackrel{\partial^0}{\to} I^1 \stackrel{\partial^1}{\to} I^2 \cdots$$

Applying $\mathcal{F}$, we get a complex in $\mathcal{B}$ that need no longer be exact:

$$\mathcal{F}(I^0) \stackrel{\mathcal{F}(\partial^0)}{\to} \mathcal{F}(I^1) \stackrel{\mathcal{F}(\partial^1)}{\to} \mathcal{F}(I^2) \cdots$$

Note that $\ker \mathcal{F}(\partial^0) = \mathcal{F}(M)$. By the arguments above, the cohomology of this complex is independent of the choice of injective resolution. We obtain the series of *right derived functors* of $\mathcal{F}$, given by

$$R^i\mathcal{F}(M) = (\ker \mathcal{F}(\partial^i))/(\operatorname{im} \mathcal{F}(\partial^{i-1})).$$

Observe that $R^0\mathcal{F} = \mathcal{F}$.

4. Restriction, Corestriction, and Inflation

Let G be a profinite group and $H \subset G$ a closed subgroup. The "restriction of scalars" functor $\mathrm{Res}^G_H : \mathrm{Mod}_G \to \mathrm{Mod}_H$ is clearly an exact functor, and hence $(H^i(H, \mathrm{Res}^G_H -), \delta^i)$ is a δ-functor.

Definition 4.1. For any G-module M we have an embedding $M^G \hookrightarrow M^H$. This gives a natural transformation of functors $H^0(G, -) \to H^0(H, \mathrm{Res}^G_H -)$. By the universality of $(H^i(G, -))$, we get natural transformations $\mathrm{res} : H^i(G, -) \to H^i(H, \mathrm{Res}^G_H -)$ for all $i > 0$ as well. This is called *restriction*.

Lemma 4.2. *Let $H \subset G$ be an open subgroup. For each $i > 0$, the functor $H^i(H, \mathrm{Res}^G_H -)$ is effaceable.*

Proof. Let M be a G-module. As we saw in the proof of Corollary 2.12, as an abelian group M injects into a divisible group I. We concluded, using Frobenius reciprocity, that M injects, as a G-module, into the injective G-module $\mathrm{Ind}^G_{\{e\}} I$. To establish our claim, it suffices to show that $\mathrm{Res}^G_H \mathrm{Ind}^G_{\{e\}} I$ is an injective H-module.

Observe that $\mathrm{Res}^G_H \mathrm{Ind}^G_{\{e\}} I = \bigoplus_{gH \in G/H} \Phi_{gH}$, where Φ_{gH} is the abelian group of locally constant functions $gH \to I$; note that each Φ_{gH} is stable under the H-action on $\mathrm{Res}^G_H \mathrm{Ind}^G_{\{e\}} I$. For each $g \in G$, there is an H-module isomorphism

$$\begin{aligned} \Phi_{gH} &\to \mathrm{Ind}^H_{\{e\}} I \\ f &\mapsto (h \mapsto f(gh)). \end{aligned}$$

Since $\mathrm{Ind}^H_{\{e\}} I$ is an injective H-module by Corollary 2.11, and a direct sum of injective H-modules is injective, we are done. □

Definition 4.3. Let $H \subset G$ be an open subgroup and M a G-module. If $m \in M^H$ and $g \in G$, then gm depends only on the left coset gH. Therefore, since H has finite index in G, we have a map

$$\begin{aligned} M^H &\to M^G \\ m &\mapsto \sum_{gH \in G/H} gm. \end{aligned}$$

It is simple to check that the right-hand side is indeed G-invariant. This is a natural transformation $\mathrm{tr}_{G/H} : H^0(H, \mathrm{Res}^G_H -) \to H^0(G, -)$. Since $(H^i(H, \mathrm{Res}^G_H -), \delta^i)$ is a universal δ-functor by Lemma 3.9 and the previous lemma, we obtain a natural transformation $\mathrm{cor} : H^i(H, \mathrm{Res}^G_H -) \to H^i(G, -)$ for all $i \geq 0$. This is called *corestriction*.

Lemma 4.4. *Let $H \subset G$ be an open subgroup and M a G-module. For any $i \geq 0$, the composition $\mathrm{cor} \circ \mathrm{res} : H^i(G, M) \to H^i(G, M)$ is multiplication by the index $[G : H]$.*

Proof. If $m \in M^G$, then $\mathrm{tr}_{G/H}(m) = \sum_{gH \in G/H} gm = \sum_{gH \in G/H} m = [G : H]m$, so the claim is true if $i = 0$. It follows for $i > 0$ by universality. □

For any open subgroup $H \subset G$ and any G-module M, consider the G-module map $i : M \to \mathrm{Ind}^G_H M$ given by $i(m) = (g \mapsto gm)$; note that $i(m)$ is locally constant because M is a discrete G-module. Similarly, consider the trace map $\mathrm{tr} : \mathrm{Ind}^G_H M \to M$ given by $\mathrm{tr}(f) = \sum_{Hg \in H\backslash G} g^{-1} f(g)$; one quickly checks that this is well-defined.

Lemma 4.5. *Let $H \subset G$ be an open subgroup and M a G-module. Then for each $i \geq 0$ there is a commutative diagram*

$$\begin{array}{ccc} H^i(G, M) & \xrightarrow{i} & H^i(G, \mathrm{Ind}^G_H M) \\ \Big\downarrow{\scriptstyle \mathrm{res}} & \nearrow & \Big\downarrow{\scriptstyle \mathrm{tr}} \\ H^i(H, M) & \xrightarrow[\mathrm{cor}]{} & H^i(G, M) \end{array}$$

where the diagonal isomorphism comes from Shapiro's Lemma.

Proof. Since all the cohomology functors here are universal, it suffices to show that in the case $i = 0$ our maps arise from a commutative diagram of natural transformations. This is easily checked using the definitions above and the proof of Shapiro's Lemma. The only point that is not completely obvious is the bottom triangle: here one uses the observation that if $\{g_1, \ldots, g_r\}$ is a system of coset representatives of G/H, then $\{g_1^{-1}, \ldots, g_r^{-1}\}$ are coset representatives of $H\backslash G$. □

The construction of the restriction map can be generalized. Let $f : G' \to G$ be a continuous group homomorphism, and let M be a G-module. Then M can be given the structure of a G'-module by $g' \cdot m = f(g') \cdot m$ for all $g' \in G'$ and $m \in M$. We denote this G'-module (why is it discrete?) by f^*M. Observe that if $H \subset G$ is a subgroup and $f : H \to G$ is the inclusion map, then $f^*M = \mathrm{Res}^G_H M$. Note that $M \mapsto f^*M$ is an exact functor, and clearly $M^G \subset (f^*M)^{G'}$, so by universality this inclusion gives rise to functors $H^i(G, M) \to H^i(G', f^*M)$ for all $i \geq 0$. Moreover, if $h : f^*M \to M'$ is a G'-module homomorphism, then we can consider the composition

$$(f, h)^* : H^i(G, M) \to H^i(G', f^*M) \to H^i(G', M').$$

Definition 4.6. Let G be a profinite group and $H \trianglelefteq G$ a closed normal subgroup. Consider the natural projection $f : G \to G/H$, which is continuous by definition of the quotient topology. Let M be a G-module. Then M^H is a G-submodule (since for any $g \in G$, $h \in H$, and $m \in M^H$ we have $h(gm) = g(g^{-1}hg)m = gm$). It has an obvious G/H-module structure. The inclusion $h : f^*M^H \to M$ is G-equivariant, and the map $(f, h)^*$ is called *inflation*:

$$\inf : H^i(G/H, M^H) \to H^i(G, M).$$

Sometimes we will need to compute explicitly with cocyles. Given a cocyle $\psi \in Z^i(G, M)$, we denote its cohomology class by $[\psi]$. The following description of the action of the boundary map follows from (5) and the proof of the Zigzag Lemma.

Lemma 4.7. *Let $0 \to M \to N \to P \to 0$ be a short exact sequence of G-modules. Let $\psi \in Z^i(G, P)$. The connecting map $\delta^i : H^i(G, P) \to H^{i+1}(G, M)$ sends $[\psi]$ to the cohomology class of $\eta \in Z^{i+1}(G, M)$, where $\eta : G^{i+1} \to M$ is given by*

$$\begin{aligned}\eta(g_1, \dots, g_{i+1}) \;=\; & g_1\psi(\widetilde{g_2, \dots, g_{i+1}}) - \psi(\widetilde{g_1g_2, g_3, \dots, g_{i+1}}) + \psi(\widetilde{g_1, g_2g_3, g_4, \dots, g_{i+1}}) + \cdots + \\ & (-1)^i\psi(\widetilde{g_1, g_2, \dots, g_ig_{i+1}}) + (-1)^{i+1}\psi(\widetilde{g_1, \dots, g_i}).\end{aligned}$$

Here, for any $p \in P$, we denote by $\tilde{p}$ an arbitrary lift of p to N.

Note that the element of N in the right-hand side of the equality above indeed lies in M, since its image in P vanishes. For the purpose of future computations, it will be useful to know how restriction and inflation act on cochains.

Lemma 4.8. *Let G be a profinite group, and let $H \subset G$ be a closed subgroup. Let M be a G-module.*

1. *Let $\psi \in Z^i(G, M)$. Then $\mathrm{res}^G_H([\psi]) = [\psi_{|H^i}]$.*
2. *Suppose that H is normal. Let $\psi \in Z^i(G/H, M^H)$. Then $\inf([\psi]) = [\tilde{\psi}]$, where $\tilde{\psi} : G^i \to M$ is the composition $G^i \twoheadrightarrow (G/H)^i \xrightarrow{\psi} M$.*

Proof. The claim holds trivially for $i = 0$. Since the formulas in our claim are compatible with the boundary maps by Lemma 4.7, we obtain the claim in general by the universal property used to define the restriction and inflation maps. □

Definition 4.9. Let G be a profinite group and let $\{G_i\}_{i \in I}$ be a projective system of finite groups, with connecting homomorphisms $\varphi_{ij} : G_i \to G_j$ for $i \geq j$, such that $G = \varprojlim G_i$. Suppose that for each $i \in I$ we have a G_i-module M_i. Moreover, suppose that whenever $i \geq j$, we have a G_i-module homomorphism $h_{ij} : \varphi_{ij}^* M_j \to M_i$. Then we can define a G-module structure on the direct limit $M = \varinjlim M_i$: if $g = (g_i) \in G$ and $m = (m_i) \in M$, we define $gm = (g_im_i) \in M$. (Why is this a discrete G-module?)

Proposition 4.10. *Let $G = \varprojlim G_i$ be a profinite group, and let $\{M_i\}$ be a system of G_i-modules as above. Then*

$$H^k(G, M) \simeq \varinjlim H^k(G_i, M_i)$$

for all $k \geq 0$. The connecting homomorphisms on the right-hand side are the maps $(\varphi_{ij}, h_{ij})^*$.

Proof. If $\pi_i : G \to G_i$ are the natural projections, then the maps $(\pi_i : M_i \to M)^* : H^k(G_i, M_i) \to H^k(G, M)$ are clearly compatible with the $(\varphi_{ij}, h_{ij})^*$, so by the universal property of direct limits we get a map $\varinjlim H^k(G_i, M_i) \to H^k(G, M)$. We claim that it is an isomorphism. This can be established by checking on cocycles: it is easy to show that $C^k(G, M) \simeq \varinjlim C^k(G_i, M_i)$. □

Remark 4.11. The previous proposition does not assume that the G_i are finite groups. Indeed, if $G_i = G$ for all i, and all the homomorphisms φ_{ij} are identities, then we clearly have $G = \varprojlim G_i$ and the proposition states that cohomology commutes with direct limits:

$$H^k(G, \varinjlim M_i) = \varinjlim H^k(G, M_i).$$

Corollary 4.12. *Let K be a field, and let its absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ act on $\overline{K}^\times$ in the natural way. Then $H^1(G_K, \overline{K}^\times) = 0$.*

Proof. This is immediate from Proposition 4.10 and Hilbert 90. □

Corollary 4.13. *Let G be a profinite group, and let M be a $\mathbb{Q}$-vector space with a G-module structure. Then $H^i(G, M) = 0$ for all $i > 0$.*

Proof. Let $\{H_j\}_{j\in J}$ be the family of open normal subgroups of G, ordered by reverse inclusion. Since $M = \bigcup_{j\in J} M^{H_j}$, it is easy to see that $M = \varinjlim M^{H_j}$, where the connecting homomorphisms are the inclusions $M^{H_j} \subset M^{H_k}$ for $j \le k$, i.e. $H_k \subseteq H_j$. Thus $H^i(G, M) = \varinjlim H^i(G/H_j, M^{H_j})$ by Proposition 4.10. Since the M^{H_j} are all $\mathbb{Q}$-vector spaces, it suffices to prove our claim in the case where G is a finite group.

So let G be finite. In this case, $\{e\} \subset G$ is an open subgroup, so by Lemma 4.4 the composition

$$H^i(G, M) \overset{\mathrm{res}}{\to} H^i(\{e\}, \mathrm{Res}^G_{\{e\}} M) \overset{\mathrm{cor}}{\to} H^i(G, M)$$

is multiplication by $|G|$. On the other hand, this composition is the zero map, since we have $H^i(\{e\}, \mathrm{Res}^G_{\{e\}} M) = 0$ for $i > 0$ by Lemma 3.12. Since $H^i(G, M)$ is naturally a $\mathbb{Q}$-vector space (the spaces of cochains are, and the boundary maps commute with the $\mathbb{Q}$-vector space structure), it follows that $H^i(G, M) = 0$. □

Definition 4.14. A G-module M is said to be a *torsion module* if every element of m is annihilated by some integer.

Lemma 4.15. *Let G be a profinite group. Suppose there exists $i \ge 1$ such that $H^i(G, M) = 0$ for all G-modules M of finite cardinality. Then $H^j(G, M) = 0$ for all $j \ge i$ and for all torsion G-modules M.*

Proof. First we prove the claim in the case $j = i$. Observe that every element m of a torsion G-module M is contained in a finite G-module. Indeed, since $\mathrm{stab}_G(m)$ has finite index in G, the G-orbit of m has only finitely many elements, say $m_1, \dots, m_r$. The subgroup $\langle m_1, \dots, m_r\rangle$ of the abelian group M is thus finite, and it is clearly stable under the G-action. Now let $\{M_k\}_{k\in K}$ be the family of finite G-submodules of M, ordered by inclusion and with the natural inclusions as the connecting homomorphisms. By the considerations above we have $M = \varinjlim M_k$. By Proposition 4.10 (note the remark following it) we have $H^i(G, M) = \varinjlim H^i(G, M_k) = 0$.

Now suppose that the claim is known for $j - 1$. We will prove it for j by a "dimension shifting" argument. For brevity, we will write $\mathrm{Ind}^G_{\{e\}} M$ for $\mathrm{Ind}^G_{\{e\}} \mathrm{Res}^G_{\{e\}} M$. Observe that this is a torsion G-module. Indeed, every element is a locally constant function $f : G \to M$. Since G is compact, the function f only takes on finitely many values, so there is an integer that annihilates all of them. There is a natural injection $\varepsilon : M \to \mathrm{Ind}^G_{\{e\}} M$, where $\varepsilon(m)$ is the function $g \mapsto gm$. Let Q be the cokernel; it is a quotient of a torsion module and thus is a

torsion module itself. The short exact sequence $0 \to M \to \mathrm{Ind}_{\{e\}}^G M \to Q \to 0$ gives rise to the following bit of the long exact sequence:

$$\cdots \to H^{j-1}(G, \mathrm{Ind}_{\{e\}}^G M) \to H^{j-1}(G, Q) \to H^j(G, M) \to H^j(G, \mathrm{Ind}_{\{e\}}^G M) \to \cdots$$

Since $H^{j-1}(G, Q) = 0$ by the inductive hypothesis and $H^j(G, \mathrm{Ind}_{\{e\}}^G M) = 0$ by Shapiro's Lemma, we conclude that $H^j(G, M) = 0$. □

Our next goal is to prove the "inflation-restriction exact sequence": if G is a profinite group, $H \trianglelefteq G$ is a normal subgroup, and M is a G-module, then the following sequence is exact:

$$0 \to H^1(G/H, M^H) \overset{\mathrm{inf}}{\to} H^1(G, M) \overset{\mathrm{res}}{\to} H^1(H, \mathrm{Res}_H^G M)^{G/H} \to H^2(G/H, M^H) \overset{\mathrm{inf}}{\to} H^2(G, M).$$

Here the action of G on $H^1(H, \mathrm{Res}_H^G M)$ is the one arising naturally from the G-action on cocycles, and the unlabelled map has yet to be defined. The desired result could be proven by direct computations on cocycles; while long and unpleasant, such a proof would still be shorter than the one we will give. However, we will take the opportunity to develop the machinery of spectral sequences in the next section. This is a tool for computing cohomology that turns out to be ubiquitous in number theory and algebraic geometry. The inflation-restriction sequence will turn out to be a special case of a very general phenomenon.

Exercises

(1) A triple $(x, y, z) \in \mathbb{Z}^3$ is called a *Pythagorean triple* if $x^2 + y^2 = z^2$. It has been known since ancient times that (x, y, z) is a Pythagorean triple if and only if it is proportional to $(m^2 - n^2, 2mn, m^2 + n^2)$ for some integers m, n. It was observed by N. Elkies that this fact can be deduced straightforwardly from Hilbert 90. Do it.

Hint: Let (x, y, z) be a Pythagorean triple such that $z \neq 0$. Consider the element $\frac{x+y\sqrt{-1}}{z} \in \mathbb{Q}(\sqrt{-1})$.

(2) Generalize your solution to the preceding exercise to obtain a parametrization of all solutions (x, y, z) to the Diophantine equation $x^2 + axy + by^2 = z^2$, where $a, b \in \mathbb{Z}$ are such that $a^2 - 4b$ is not a perfect square.

(3) Let K be a field of characteristic prime to $n \geq 1$. Let $\mu_n \subset \overline{K}^\times$ be the subgroup of n-th roots of unity; clearly it is preserved by the action of the absolute Galois group G_K. Prove that $H^1(G_K, \mu_n) \simeq K^\times/(K^\times)^n$.

5. Spectral sequences

5.1. **Basic definitions.** The goal of this section is to understand the Hochschild-Serre spectral sequence, which is an important tool for computing cohomology. We start off with a general treatment of spectral sequences. The underlying theory is not complicated, once one sees past all the cluttered diagrams. Let R be a ring.

Definition 5.1. (1) A *bigraded R-module* E is a family of R-modules $\{E^{p,q} : p, q \in \mathbb{Z}\}$.

(2) A *differential* $d : E \to E$ of degree (r, s) is a family of R-module homomorphisms $d : E^{p,q} \to E^{p+r,q+s}$ such that $d \circ d = 0$.

Definition 5.2. A *spectral sequence* is a sequence $\{E_r\}_{r \geq t}$, for some integer t, of bigraded R-modules $E_r = (E_r^{p,q})$ equipped with differentials $d_r : E_r \to E_r$ of degree $(r, 1-r)$ such that,

for every $p, q \in \mathbb{Z}$ and every integer $r \geq r_0$, the following holds:

$$E_{r+1}^{p,q} \simeq \ker(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1})/\mathrm{im}(d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}). \tag{6}$$

The spectral sequence is called *positive* if $E_r^{p,q} = 0$ whenever $p < 0$ or $q < 0$. Most of the interesting applications of spectral sequences happen in the case $t = 2$, and we will assume this from now on, except in Example 5.6 below.

From now on, all our spectral sequences will be assumed to be positive. The bigraded modules E_r are called *sheets* of the spectral sequence. For a fixed r, think of the modules $E_r^{p,q}$ as lattice points on the plane. Then the differential maps are slanted arrows that form complexes which run in slanted lines across the plane. The (co)homology of these complexes computes the modules $E_{r+1}^{p,q}$ of the following sheet.

The terms of the form $E_r^{p,0}$ and $E_r^{0,q}$ are called base terms and fiber terms, respectively.

Lemma 5.3. *Let $\{E_r\}$ be a positive spectral sequence. Fix $p, q \in \mathbb{N}$, and let $r_0 = \max\{p, q+1\} + 1$. Then $E_r^{p,q} \simeq E_{r_0}^{p,q}$ for all $r \geq r_0$.*

Proof. If $r \geq r_0$, then $E_r^{p+r,q-r+1}$ and $E_r^{p-r,q+r-1}$ both vanish. The claim is then immediate from (6). □

Definition 5.4. A *filtered R-module* is an R-module A together with a family of submodules

$$A = F^0 A \supseteq F^1 A \supseteq F^2 \supseteq \cdots$$

We will assume throughout that $\bigcap_{i=0}^{\infty} F^i A = 0$. The graded pieces are the quotients $\mathrm{gr}^i A = F^i A / F^{i+1} A$.

Definition 5.5. Let $\{E_r^{p,q}\}$ be a spectral sequence.

(1) For any $p, q \in \mathbb{N}$, we define $E_\infty^{p,q} = E_{r_0}^{p,q}$, in the notation of Lemma 5.3.
(2) We say that the spectral sequence $\{E_r^{p,q}\}$ *converges to* or *abuts to* the family $\{A^n\}_{n\in\mathbb{N}}$ of filtered modules if $E_\infty^{p,q} \simeq \mathrm{gr}^p A^{p+q}$ for all $p, q \in \mathbb{N}$. In this case, one writes $E_r^{p,q} \Rightarrow A^n$ or $E_2^{p,q} \Rightarrow A^n$.

Example 5.6. As our simplest example of a spectral sequence, let $C^0 \stackrel{\partial^0}{\to} C^1 \stackrel{\partial^1}{\to} \cdots$ be a cochain complex. Define $E_1^{p,q} = C^p$ whenever p and q are both non-negative, and set $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ to be ∂^p. Then (6) forces $E_2^{p,q} = H^p(\mathbf{C})$. The only natural choice for the differential is the zero map. Thus we get $E_\infty^{p,q} = E_2^{p,q} = H^p(\mathbf{C})$ for all p, q. If for all n we define $H^n = \bigoplus_{i=0}^{\infty} H^i(\mathbf{C})$, with the grading $F^j H^n = \bigoplus_{i=j}^{\infty} H^i(\mathbf{C})$, then clearly $E_1^{p,q} \Rightarrow H^n$. While this example is silly, it already suggests that spectral sequences could be useful for computing cohomology.

5.2. **The five-term exact sequence.** In this section we derive a general five-term exact sequence associated to any positive spectral sequence. The inflation-restriction sequence will be the special case of this result for the Hochschild-Serre spectral sequence.

Lemma 5.7. *Let $\{E_r^{p,q}\} \Rightarrow A^n$ be a spectral sequence. For each $n \geq 1$ there is a natural injection $e_B : E_\infty^{n,0} \hookrightarrow A^n$ and a natural surjection $e_F : A^n \twoheadrightarrow E_\infty^{0,n}$. Moreover, in the case $n = 1$ the sequence*

$$E_\infty^{1,0} \stackrel{e_B}{\to} A^1 \stackrel{e_F}{\to} E_\infty^{0,1}$$

is exact.

Proof. Observe by the definition of abutment that if $m > n$, then $\mathrm{gr}^m A^n \simeq E_\infty^{m,n-m} = 0$, since our spectral sequences are positive. Since $\bigcap_i F^i A^n = 0$, this implies that $F^{n+1}A^n = 0$, and hence that $\mathrm{gr}^n A^n = F^n A^n$ is a submodule of A^n. Since $E_\infty^{n,0} \simeq \mathrm{gr}^n A^n$, this provides our injection.

Similarly, $E_\infty^{0,n} \simeq \mathrm{gr}^0 A^n = A^n/F^1 A^n$. Since this is a quotient of A^n, we obtain the desired surjection. It follows from the above that $\ker e_F = F^1 A^n$ for all n, whereas $\operatorname{im} e_B = F^n A^n$, and hence we get exactness when $n = 1$. □

Lemma 5.8. *Let $\{E_r^{p,q}\}$ be a spectral sequence. Then for all $n \geq 1$ there is a natural surjection $E_2^{n,0} \twoheadrightarrow E_\infty^{n,0}$ and injection $E_\infty^{0,n} \hookrightarrow E_2^{0,n}$.*

Proof. This is immediate from (6). Indeed, for every $r \geq 2$ we have $E_r^{n+r,-r+1} = 0$ by positivity, whence $E_{r+1}^{n,0} = E_r^{n,0}/\mathrm{im}(d_r : E_r^{n-r,r-1} \to E_r^{n,0})$ is a quotient of $E_r^{n,0}$. By induction, every $E_r^{n,0}$ is thus naturally a quotient of $E_2^{n,0}$.

Similarly, $E_{r+1}^{0,n} = \ker(d_r : E_r^{0,n} \to E_r^{r,n-r+1}) \subseteq E_r^{0,n}$ for every $r \geq 1$, and hence $E_r^{0,n} \subseteq E_2^{0,n}$ for every $r \geq 2$. □

Definition 5.9. The compositions of the maps from Lemmas 5.7 and 5.8 provide maps $E_2^{n,0} \to A^n$ and $A^n \to E_2^{0,n}$, which we abusively call e_B and e_F, respectively. Note that these maps are not necessarily injective or surjective.

Definition 5.10. Let $n \geq 1$. The spectral sequence $\{E_r^{p,q}\}$ is said to *satisfy condition* $(*)_n$ if $E_2^{p,q} = 0$ for all pairs p, q such that $1 \leq q \leq n-1$ and $p+q \in \{n-1, n, n+1\}$. Observe that the condition $(*)_1$ is vacuous and thus is satisfied by all spectral sequences.

Remark 5.11. Observe that if condition $(*)_n$ holds, then for all $2 \leq r \leq n$ we have $E_2^{r,n-r+1} = 0$, and hence $E_r^{r,n-r+1} = 0$ by (6). This implies that the inclusions $E_{n+1}^{0,n} \subseteq E_n^{0,n} \subseteq \cdots \subseteq E_2^{0,n}$ of Lemma 5.8 are isomorphisms. Similarly, we have $E_2^{(n+1)-r,r-1} = 0$ for all $2 \leq r \leq n$, and thus the projections $E_2^{n+1,0} \twoheadrightarrow E_3^{n+1,0} \twoheadrightarrow \cdots \twoheadrightarrow E_{n+1}^{n+1,0}$ are also all isomorphisms. This means that, for all $n \geq 1$, we may consider the composition

$$E_2^{0,n} \simeq E_{n+1}^{0,n} \overset{d_{n+1}}{\to} E_{n+1}^{n+1,0} \simeq E_2^{n+1,0},$$

which is called the transgression map and will be denoted d_{n+1}.

We finally have all the necessary ingredients to establish the "five-term exact sequences" associated to the spectral sequence $\{E_r^{p,q}\}$.

Proposition 5.12. *Let $\{E_r^{p,q} \Rightarrow A^n\}$ be a spectral sequence satisfying condition $(*)_n$ for some $n \geq 1$. Then there is an exact sequence*

$$0 \to E_2^{n,0} \overset{e_B}{\to} A^n \overset{e_F}{\to} E_2^{0,n} \overset{d_{n+1}}{\to} E_2^{n+1,0} \overset{e_B}{\to} A^{n+1}.$$

Proof. We successively verify exactness at each node.

Exactness at $E_2^{n,0}$: We need to show that $e_B : E_2^{n,0} \to A^n$ is injective. Since $E_{n+1}^{n,0} = E_\infty^{n,0} \to A^n$ is injective by Lemma 5.8, it suffices to show that the maps $E_r^{n,0} \twoheadrightarrow E_{r+1}^{n,0}$ are injective, hence isomorphisms, for all $2 \leq r \leq n$. Now by (6) we have $\ker(E_r^{n,0} \to E_{r+1}^{n,0}) = \mathrm{im}(d_r : E_r^{n-r,r-1} \to E_r^{n,0}) = 0$ for each $2 \leq r \leq n$, since $E_r^{n-r,r-1} = 0$ by condition $(*)_n$.

Exactness at A^n: In view of the injectivity of e_B and the end of the proof of Lemma 5.7, we find that $\operatorname{im} e_B = F^n A^n$, whereas $\ker e_F = F^1 A^n$. However, these are equal, since condition $(*)^n$ implies that for all $1 \leq i \leq n-1$ we have $\mathrm{gr}^i A^n = E_\infty^{i,n-i} = 0$.

Exactness at $E_2^{0,n}$: Observe by Lemma 5.3 that $E_\infty^{0,n} = E_{n+2}^{0,n}$ and $A^n \to E_\infty^{0,n}$ is surjective. Thus $\operatorname{im} e_F = \operatorname{im}(E_{n+2}^{0,n} \hookrightarrow E_{n+1}^{0,n}) = \ker(d_{n+1} : E_{n+1}^{0,n} \to E_{n+1}^{n+1,0})$, which implies exactness by the definition of the transgression map.

Exactness at $E_2^{n+1,0}$: Again by Lemma 5.3, we see that $E_{n+2}^{n+1,0} = E_\infty^{n+1,0}$ injects into A^{n+1}, and hence that $\ker e_B = \ker(E_{n+1}^{n+1,0} \twoheadrightarrow E_{n+2}^{n+1,0}) = \operatorname{im}(d_{n+1} : E_{n+1}^{0,n} \to E_{n+1}^{n+1,0})$, which is what we need by the definition of the transgression map. □

Corollary 5.13. *Let* $E_r^{p,q} \Rightarrow A^n$ *be any positive spectral sequence. Then there is an exact sequence*

$$0 \to E_2^{1,0} \xrightarrow{e_B} A^1 \xrightarrow{e_F} E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \xrightarrow{e_B} A^2.$$

Proof. As noted above, the condition $(*)_1$ is vacuous. □

5.3. **Double complexes.** So far the only example we have seen of a spectral sequence is the trivial one in Example 5.6. In this section we will study a general construction that produces many useful examples of spectral sequences.

Definition 5.14. A filtered complex is a complex $C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} \cdots$ of filtered modules whose filtrations are compatible with the boundary maps. In other words, for every i, j, we have $\partial^i(F^jC^i) \subseteq F^jC^{i+1}$.

A filtration of a complex induces a filtration on its cohomology. Indeed, for every $i, j \geq 0$ we may define $F^jH^i(\mathbf{C}) = \operatorname{im}(H^i(F^j\mathbf{C}) \to H^i(\mathbf{C}))$, where the maps whose image we are considering is the one induced by the natural inclusion of complexes $F^j\mathbf{C} \hookrightarrow \mathbf{C}$. The family of filtered modules $\{H^i(\mathbf{C})\}$ will be denoted $H(\mathbf{C})$.

For every $r \in \mathbb{Z}$, we set $Z_r^{p,q} = \{x \in F^pC^{p+q} : \partial^{p+q}(x) \in F^{p+r}C^{p+q+1}\}$. Similarly, set $B_r^{p,q} = \partial^{p+q-1}Z_{r-1}^{p-r+1,q+r-2} = \partial^{p+q-1}F^{p-r+1}C^{p+q-1} \cap F^pC^{p+q}$, the second equality following immediately from the definition. Observe that $B_r^{p,q} \subseteq Z_r^{p,q}$, since the composition of two boundary maps is zero. Similarly, note that $Z_{r-1}^{p+1,q-1} \subseteq Z_r^{p,q}$. Now set

$$E_r^{p,q} = Z_r^{p,q}/(B_r^{p,q} + Z_{r-1}^{p+1,q-1}). \tag{7}$$

Proposition 5.15. *Let* $\mathbf{C}$ *be a filtered complex as above.*

1. *There exists a spectral sequence, with* $E_r^{p,q}$ *defined as in* (7) *and differentials induced by the boundary maps of* $\mathbf{C}$.
2. *If the filtration of each* C^i *is bounded, so that there exists some* j *such that* $F^jC^i = 0$ *(we allow* j *to depend on* i*), then* $E_r^{p,q} \Rightarrow H(\mathbf{C})$.

Proof. The definitions of $Z_r^{p,q}$ and $E_r^{p,q}$ are exactly the ones needed to make this claim work. Since $\partial^{p+q}Z_r^{p,q} \subseteq Z_r^{p+r,q-r+1}$ by definition, the boundary map ∂^{p+q} clearly induces a map $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$, and the composition of two such maps is zero.

Again by definition, $\operatorname{im}(\partial^{p+q-1} : Z_r^{p-r,q+r-1} \to Z_r^{p,q}) = B_{r+1}^{p,q}$. Observing that $B_r^{p,q} \subseteq B_{r+1}^{p,q}$, we conclude that

$$\operatorname{im}(d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}) = (B_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1})/(B_r^{p,q} + Z_{r-1}^{p+1,q-1}).$$

Furthermore, for $x \in Z_r^{p,q}$, we have $\partial^{p+q}(x) \in Z_r^{p+r+1,q-r}$ if and only if $\partial^{p+q}(x) \in F^{p+r+1}C^{p+q+1}$, which is equivalent to $x \in Z_{r+1}^{p,q}$. Also, $B_r^{p+r,q-r+1} = \partial^{p+q}(Z_{r-1}^{p+1,q-1})$. Thus, $\partial^{p+q}(x) \in B_r^{p+r,q-r+1}$ implies $x \in Z_{r-1}^{p+1,q-1} + \ker \partial^{p+q}$.

Hence, $\ker(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}) = (Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1})/(B_r^{p,q} + Z_{r-1}^{p+1,q-1})$.

Therefore, $\ker(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1})/\mathrm{im}\,(d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}) \simeq E_{r+1}^{p,q}$, as claimed.

To prove the second part of the proposition, let p, q be fixed. For sufficiently large r, we have that $F^{p+r}C^{p+q+1} = 0$, and hence $Z_r^{p,q} = \{x \in F^pC^{p+q} : \partial^{p+q}(x) = 0\}$. But for sufficiently large r we have $F^{p-r+1}C^{p+q-1} = C^{p+q-1}$ and hence $B_r^{p,q} = \partial^{p+q-1}C^{p+q-1} \cap F^pC^{p+q}$. It follows that $E_\infty^{p,q}$ is exactly $\mathrm{im}\,(H^{p+q}(F^p\mathbf{C}) \to H^{p+q}(\mathbf{C}))/\mathrm{im}\,(H^{p+q}(F^{p+1}\mathbf{C}) \to H^{p+q}(\mathbf{C}))$. □

Definition 5.16. A double complex is a family $\mathbf{K} = \{K^{p,q}\}$ of R-modules, where the indices p and q run over the natural numbers, equipped with horizontal and vertical differential maps

$$\begin{aligned} \partial' : K^{p,q} &\to K^{p+1,q} \\ \partial'' : K^{p,q} &\to K^{p,q+1} \end{aligned}$$

such that $\partial' \circ \partial' = 0$, $\partial'' \circ \partial'' = 0$, and $\partial' \circ \partial'' + \partial'' \circ \partial' = 0$.

Observe that a double complex is not a commutative diagram, since the squares

$$\begin{array}{ccc} K^{p,q} & \xrightarrow{\partial'} & K^{p+1,q} \\ \downarrow{\scriptstyle \partial''} & & \downarrow{\scriptstyle \partial''} \\ K^{p,q+1} & \xrightarrow{\partial'} & K^{p+1,q+1} \end{array}$$

anti-commute. Moreover, it is conventional to refer to the maps ∂' and ∂'' as horizontal and vertical, respectively, so one visualizes $K^{p,q}$ as lying in the q-th row and p-th column of the double complex. Given a double complex as in Definition 5.16, we define a (usual) cochain complex $\mathbf{C}$ as follows: for each $i \geq 0$ set

$$C^i = \bigoplus_{p+q=i} K^{p,q} \tag{8}$$

and define the boundary maps $\partial^i : C^i \to C^{i+1}$ by $\partial^i = \partial' + \partial''$. It is simple to check that $\partial^{i+1} \circ \partial^i = 0$ for all $i \geq 0$; this is the reason for the condition $\partial' \circ \partial'' + \partial'' \circ \partial' = 0$ in the definition of double complexes. The complex $\mathbf{C}$ is called the *total complex* of $\mathbf{K}$ and is often denoted $\mathrm{Tot}\,\mathbf{K}$ in the literature.

We give $\mathbf{C}$ the structure of a filtered complex in two different ways. Define two filtrations

$$\begin{aligned} {}'F^jC^i &= \bigoplus_{\substack{p+q=i \\ p \geq j}} K^{p,q} \\ {}''F^jC^i &= \bigoplus_{\substack{p+q=i \\ q \geq j}} K^{p,q} \end{aligned}$$

Both filtrations are compatible with the boundary maps and are obviously bounded. By Proposition 5.15 we get two spectral sequences, $\{{}'E_r^{p,q}\}$ and $\{{}''E_r^{p,q}\}$, that both abut to $H(\mathbf{C})$.

We now investigate the terms of these sequences. For any element $x \in A$ of a filtered module A with bounded filtration, let $\deg(x) = \max\{p : x \in F^pA\}$. Observe that if $x \in C^i$, then $\deg(\partial'(x)) = \deg(x) + 1$, whereas $\deg(\partial''(x)) = \deg(x)$. It follows that

$${}'Z_1^{p,q} = \ker(\partial'' : K^{p,q} \to K^{p,q+1}) \oplus F^{p+1}C^{p+q},$$

whereas ${}'B_1^{p,q} = \partial F^p C^{p+q-1} \cap F^p C^{p+q} = \partial F^p C^{p+q-1}$. Now ${}'Z_0^{p+1,q-1} = F^{p+1}C^{p+q}$ by definition, and $\partial'(F^p C^{p+q-1}) \subset F^{p+1}C^{p+q}$, so

$$ {}'B_1^{p,q} + {}'Z_0^{p+1,q-1} = \operatorname{im}(\partial'' : K^{p,q-1} \to K^{p,q}) \oplus F^{p+1}C^{p+q}. $$

We conclude that

$$ {}'E_1^{p,q} = \ker(\partial'' : K^{p,q} \to K^{p,q+1}) / \operatorname{im}(\partial'' : K^{p,q-1} \to K^{p,q}), $$

Thus the spectral sequence $\{{}'E_1^{p,q}\}$ computes the cohomology of the columns of the double complex $\{K^{p,q}\}$. Precisely, ${}'E_1^{p,q} = H^q(\mathbf{K}^{p,\bullet})$, where $K^{p,\bullet}$ is the cochain complex $K^{p,0} \xrightarrow{\partial''} K^{p,1} \xrightarrow{\partial''} K^{p,2} \to \cdots$.

Similarly, ${}''E_1^{p,q} = H^p(\mathbf{K}^{\bullet,q})$ computes the cohomology of the rows of the original double complex.

Such a setup is very useful, since often one can arrange a double complex for which one of these two spectral sequences is of independent interest, whereas the other one is less interesting but has a readily computable limit. The prime example of such a situation is described in the following section.

5.4. **Grothendieck's theorem.** Recall, from Remark 3.25, the right derived functors $R^i\mathcal{F}$ of a left exact functor $\mathcal{F}$.

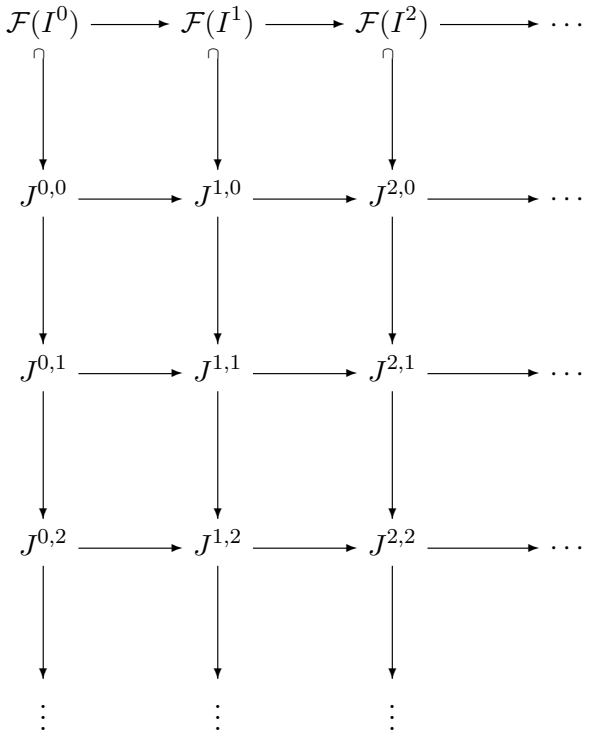
Definition 5.17. Let $\mathcal{G} : \mathcal{B} \to \mathcal{C}$ be a left exact functor between two abelian categories, where $\mathcal{B}$ has enough injectives. An object $B \in \mathrm{Ob}(\mathbf{B})$ is said to be $\mathcal{G}$*-acyclic* if $R^i\mathcal{G}(B) = 0$ for all $i > 0$.

Theorem 5.18 (Grothendieck)**.** *Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ be abelian categories, and suppose that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{C}$ be additive left exact functors, and suppose that $\mathcal{F}$ takes injective objects of $\mathcal{A}$ to $\mathcal{G}$-acyclic objects of $\mathcal{B}$. Then for each object $A \in \mathrm{Ob}(\mathcal{A})$ there exists a spectral sequence*

$$ E_2^{p,q} = (R^q\mathcal{G} \circ R^p\mathcal{F})(A) \Rightarrow R^{p+q}(\mathcal{G} \circ \mathcal{F})(A). $$

Proof. Let A be an object of $\mathcal{A}$, and let $A \to I^0 \to I^1 \to I^2 \to \cdots$ be a resolution of A by injective objects. We denote it by $\mathbf{I}$. Suppose we can construct a "resolution of the resolution $\mathcal{F}(\mathbf{I})$," namely a commutative diagram of objects of $\mathcal{B}$ as follows, in which each row is a cochain complex (i.e. the composition of two consecutive maps is zero) and for each $i \geq 0$,

the p-th column is an injective resolution of $\mathcal{F}(I^p)$:



A minor annoyance is that this diagram is not a double complex, since the squares are commutative rather than anti-commutative. We remedy this by changing the sign of the vertical arrows in every other column, i.e. by replacing $d' : J^{p,q} \to J^{p,q+1}$ with $(-1)^p d'$. Clearly the columns remain injective resolutions. Now apply $\mathcal{G}$ to this diagram, to produce a double complex of objects of $\mathcal{C}$. We write $K^{p,q}$ for $\mathcal{G}(J^{p,q})$. Since the $\mathcal{F}(I^q)$ are $\mathcal{G}$-acyclic by assumption, the columns of the double complex $\{K^{p,q}\}$ remain exact. In particular, considering the two spectral sequences associated to this double complex, we find that ${}'E_1^{p,q} = H^q(\mathbf{K}^{p,\bullet}) = 0$ whenever $q > 0$, whereas ${}'E_1^{p,0} = \mathcal{G}(\mathcal{F}(I^p))$. Thus the horizontal maps of the top row of our complex induce

$$ {}'E_2^{p,q} = \begin{cases} R^p(\mathcal{G} \circ \mathcal{F})(A) & : q = 0 \\ 0 & : q > 0. \end{cases} $$

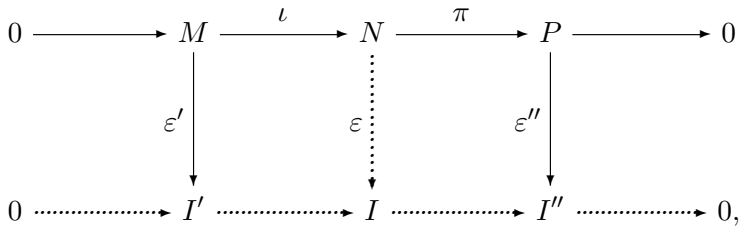
It is clear that the differentials of ${}'E_2$ are all zero maps, so ${}'E_\infty^{p,q} = {}'E_2^{p,q}$ for all p, q. Thus, for every i we see that $H^i(\mathbf{C})$ has only one non-zero graded piece, where $\mathbf{C}$ is the filtered complex defined as in (8) from the double complex $\{K^{p,q}\}$. We conclude for all p that

$$ H^p(\mathbf{C}) = R^p(\mathcal{G} \circ \mathcal{F})(A). \tag{9} $$

To compute the left-hand side of this equation in a different way, we will use the second spectral sequence associated to $\mathbf{K}$. It is determined by the cohomology of the rows of $\mathbf{K}$, over which we don't have much control in the generality in which we have worked so far in this proof. We will need to construct the diagram of resolutions $\{J^{p,q}\}$ in a rather specific way; thereby we will also prove that such diagrams exist.

Subclaim 5.19. *Suppose that $0 \to M \to N \to P \to 0$ is a short exact sequence of objects in some abelian category. Suppose that we are given embeddings $\varepsilon' : M \hookrightarrow I'$ and $\varepsilon'' : P \hookrightarrow I''$,*

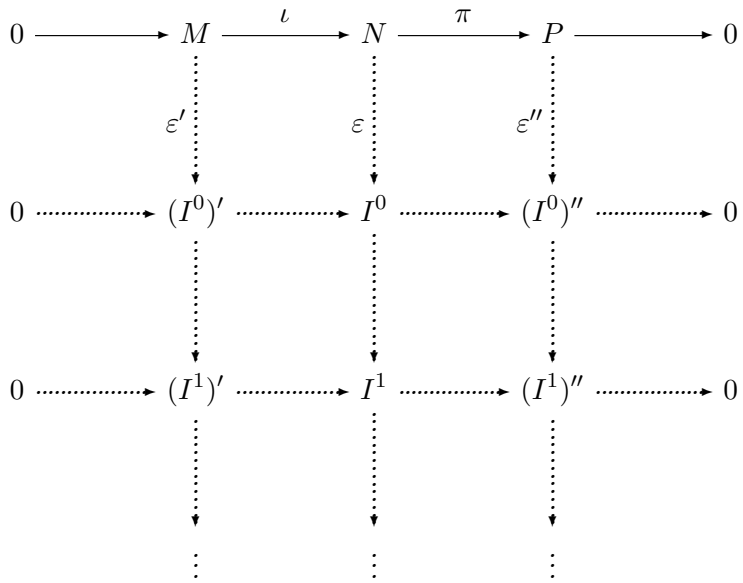
where I' and I'' are injective objects. Then we can complete this picture to a commutative diagram



where I is injective, $\varepsilon : N \to I$ is an embedding, and the bottom row is exact.

Proof. Let $I = I' \oplus I''$; clearly this is an injective object. The maps on the bottom row will be the standard embedding and projection. It remains to define ε. By injectivity of I' there exists a map $\varphi : N \to I'$ such that $\varphi \circ \iota = \varepsilon'$. The diagram above already includes a map from N to I'', and we take these to be the components of $\varepsilon = (\varphi, \varepsilon'' \circ \pi)$. This works. □

Subclaim 5.20. *Suppose that $0 \to M \to N \to P \to 0$ is a short exact sequence and $0 \to M \to \mathbf{I}'$ and $0 \to P \to \mathbf{I}''$ are injective resolutions. This data can be extended to a commutative diagram*



where the columns are injective resolutions and the rows are short exact sequences.

Proof. We are given embeddings $\varepsilon' : M \hookrightarrow (I^0)'$ and $\varepsilon'' : P \hookrightarrow (I^0)''$ of M and P into injective objects. By the previous subclaim we can fill in the second row of the diagram. Recalling that the map ε factors through the inclusion $(I^0)' \to I^0$, we check that the cokernels naturally lie in a short exact sequence

$$0 \to \operatorname{coker} \varepsilon' \to \operatorname{coker} \varepsilon \to \operatorname{coker} \varepsilon'' \to 0.$$

The injective resolutions we have been given provide for embeddings of $\operatorname{coker} \varepsilon'$ and $\operatorname{coker} \varepsilon''$ into injective objects $(I^1)'$ and $(I^1)''$, respectively. We apply the previous subclaim again and continue forever. □

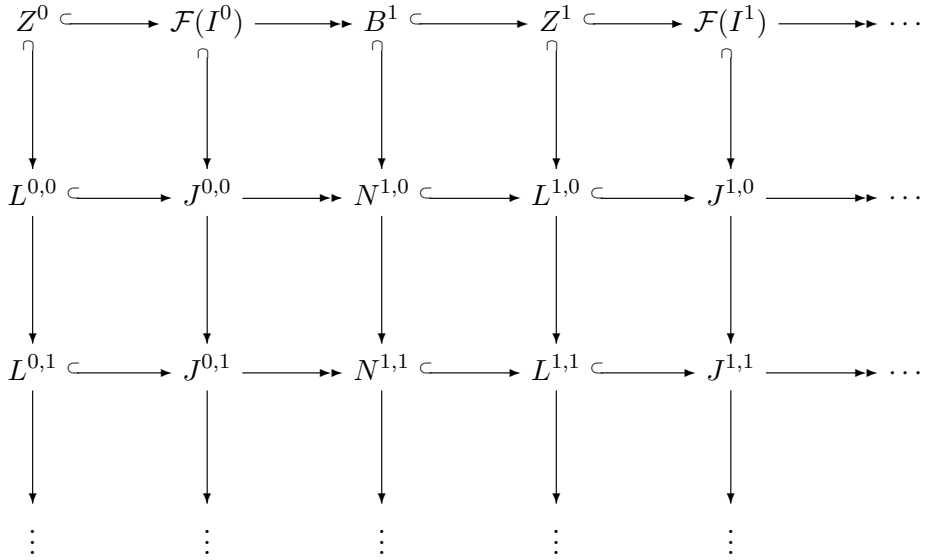
Now we return to the proof of Grothendieck's theorem. We have constructed an injective resolution $0 \to A \to \mathbf{I}$ of $A \in \mathrm{Ob}(\mathcal{A})$ and applied the functor $\mathcal{F}$ to it to obtain a complex

$$0 \to \mathcal{F}(A) \to \mathcal{F}(I^0) \xrightarrow{d^0} \mathcal{F}(I^1) \xrightarrow{d^1} \cdots$$

Defining $B^i = \operatorname{im} d^{i-1}$ and $Z^i = \ker d^i$ for each $i \geq 0$, we refine this sequence to a sequence

$$Z^0 \hookrightarrow \mathcal{F}(I^0) \twoheadrightarrow B^1 \hookrightarrow Z^1 \hookrightarrow \mathcal{F}(I^1) \twoheadrightarrow B^2 \hookrightarrow Z^2 \hookrightarrow \mathcal{F}(I^2) \cdots$$

which, by construction, is exact at each $\mathcal{F}(I^i)$. We wish to construct a diagram of the form



whose columns are injective resolutions and whose rows are exact at each $J^{p,q}$. Applying the previous subclaim to the short exact sequence $0 \to Z^0 \to \mathcal{F}(I^0) \to B^1 \to 0$, we get the first three columns. Choose an arbitrary injective resolution of Z^1/B^1 and apply the previous subclaim to the sequence $0 \to B^1 \to Z^1 \to Z^1/B^1 \to 0$ to get a resolution of Z^1. Choose an arbitrary resolution of B^2 and apply the previous lemma to $0 \to Z^1 \to \mathcal{F}(I^1) \to B^2 \to 0$ to finish the next three columns of our diagram, with exactness at each $J^{1,q}$. Continue forever.

Remark 5.21. An important consequence of our construction is that, for each $p > 0$,

$$Z^p/B^p \hookrightarrow L^{p,0}/N^{p,0} \to L^{p,1}/N^{p,1} \to L^{p,2}/N^{p,2} \to \cdots$$

is an injective resolution. Observe also that $Z^p/B^p = R^p\mathcal{F}(A)$.

Consider the diagram $\{J^{p,q}\}$ obtained from this construction. We apply $\mathcal{G}$ to it and consider the second spectral sequence associated to the double complex

$$\begin{array}{ccccccc}
\mathcal{G}(\mathcal{F}(I^0)) & \longrightarrow & \mathcal{G}(\mathcal{F}(I^1)) & \longrightarrow & \mathcal{G}(\mathcal{F}(I^2)) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
K^{0,0} & \longrightarrow & K^{1,0} & \longrightarrow & K^{2,0} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
K^{0,1} & \longrightarrow & K^{1,1} & \longrightarrow & K^{2,1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

where $K^{p,q} = \mathcal{G}(J^{p,q})$, and we have flipped the sign of the maps in every other column.

Since all the $N^{p,q}$ are injective, it is possible to complete each triangle

$$\begin{array}{ccc}
N^{p,q} & \hookrightarrow & L^{p,q} \\
\downarrow{\scriptstyle \mathrm{id}} & \swarrow & \\
N^{p,q} & &
\end{array}$$

It follows that the maps $\mathcal{G}(N^{p,q}) \to \mathcal{G}(L^{p,q})$ are all injections, and similarly for the maps $\mathcal{G}(L^{p,q}) \to \mathcal{G}(J^{p,q}) = K^{p,q}$. By construction, $J^{p,q} = L^{p,q} \oplus N^{p,q}$, so the maps $J^{p,q} \twoheadrightarrow N^{p+1,q}$ have sections, and thus the maps $K^{p,q} \to \mathcal{G}(N^{p+1,q})$ are surjective. Finally, we still have that $\ker(K^{p,q} \to \mathcal{G}(N^{p+1,q})) = \operatorname{im}(\mathcal{G}(L^{p,q}) \to K^{p,q})$.

Since the second spectral sequence of the double complex $\mathbf{K}$ computes the cohomology of the rows, the considerations above imply that

$$''E_1^{p,q} = \mathcal{G}(L^{p,q})/\mathcal{G}(N^{p,q}) = \mathcal{G}(L^{p,q}/N^{p,q}),$$

where the second equality follows from our construction and $N^{0,q} = 0$. By Remark 5.21 it follows that

$$''E_2^{p,q} = R^q\mathcal{G}(Z^p/B^p) = R^q\mathcal{G}(R^p\mathcal{F}(A)).$$

We know that this sequence abuts to $H(\mathbf{C}) = H(\operatorname{Tot}\mathbf{K})$, so by (9) we get $R^q\mathcal{G}(R^p\mathcal{F}(A)) \Rightarrow R^{p+q}(\mathcal{G}\circ\mathcal{F})(A)$, as claimed. □

5.5. **The Hochschild-Serre spectral sequence.** Having developed the tools we need from the theory of spectral sequences, we can finally specialize them to the study of G-modules.

Let G be a profinite group and $H \trianglelefteq G$ a normal subgroup. We apply Grothendieck's Theorem 5.18 to the following situation. Let $\mathcal{F} : \mathrm{Mod}_G \to \mathrm{Mod}_{G/H}$ be the functor sending a G-module M to the submodule M^H, with the obvious G/H-module structure. Let $\mathcal{G} : \mathrm{Mod}_{G/H} \to \mathrm{Ab}$ be the functor $M \mapsto M^{G/H}$. Both of these functors are left exact, and the

corresponding right derived functors are $R^p\mathcal{F} = H^p(H, \mathrm{Res}^G_H -)$ and $R^q\mathcal{G} = H^q(G/H, -)$. Both Mod_G and $\mathrm{Mod}_{G/H}$ have enough injectives.

Thus, to verify the hypotheses of Theorem 5.18 it remains to show that, for every injective G-module I, the G/H-module I^H is $\mathcal{G}$-acyclic. We claim that if I^H is, in fact, an injective G/H-module. Indeed, if we are given a diagram

$$\begin{array}{ccc} M & \hookrightarrow & N \\ \downarrow & & \\ I^H & & \end{array}$$

of G/H-modules, we can consider it as a diagram of G-modules via the natural projection $\pi : G \to G/H$:

$$\begin{array}{ccc} \pi^\times M & \hookrightarrow & \pi^\times N \\ \downarrow & \swarrow & \\ I^H \subset I & & \end{array}$$

By injectivity of I, we can fill in the dotted G-module map $N \to I$. Since H acts trivially on N, the image of this map is contained in I^H.

Definition 5.22. Let G be a profinite group and $H \trianglelefteq G$ a normal subgroup. In view of the preceding discussion, for every G-module M Theorem 5.18 provides us with an explicit spectral sequence

$$H^q(G/H, H^p(H, \mathrm{Res}^G_H M)) \Rightarrow H^{p+q}(G, M).$$

Thus is called the *(Lyndon)-Hochschild-Serre spectral sequence*.

Theorem 5.23. *Let G be a profinite group, let $H \trianglelefteq G$ be a normal subgroup, and let M be a G-module. The following sequence is exact:*

$$0 \to H^1(G/H, M^H) \overset{\mathrm{inf}}{\to} H^1(G, M) \overset{\mathrm{res}}{\to} H^1(H, \mathrm{Res}^G_H M)^{G/H} \to H^2(G/H, M^H) \overset{\mathrm{inf}}{\to} H^2(G, M).$$

Proof. We apply Corollary 5.13 to the Hochschild-Serre spectral sequence. It remains to check that the maps induced by the spectral sequence are indeed inflation and restriction. □

Corollary 5.24. *Let G be a profinite group, let $H \trianglelefteq G$ be a normal subgroup, and let M be a G-module. Suppose that $H^i(H, \mathrm{Res}^G_H M) = 0$ for all $i > 0$. Then $H^i(G/H, M^H) \simeq H^i(G, M)$ for all $i > 0$.*

Proof. If $i = 1$ this is immediate from the inflation-restriction sequence. Otherwise, our hypothesis implies that the Hochschild-Serre spectral sequence for M satisfies condition $(*)_n$ for all $n \geq 1$.

□

6. The Brauer group

If L/K is a finite Galois extension with Galois group $G = \mathrm{Gal}(L/K)$, then $H^0(G, L^\times) = K^\times$, whereas $H^1(G, L^\times) = 0$ by Hilbert 90. Our next aim is to understand $H^2(G, L^\times)$, which turns out to be most conveniently done in terms of the theory of central simple algebras.

6.1. **Non-abelian cohomology.** Let G be a profinite group, and let M be a group, not necessarily abelian. We write the operation of M multiplicatively. For the purposes of this section, M will be called a *(discrete) G-module* if it is equipped with a G-action such that $g(m_1m_2) = (gm_1)(gm_2)$ for all $g \in G$ and $m_1, m_2 \in M$, and if $\mathrm{stab}_G(m)$ is an open subgroup of G for every $m \in M$.

Example 6.1. Let L/K be a finite Galois extension. For every $n \geq 1$, the Galois group $G = \mathrm{Gal}(L/K)$ acts on $\mathrm{GL}_n(L)$ by acting on the matrix elements. Since matrix multiplication is given by polynomials in the matrix elements, it is respected by the action of G. Thus, the group $M = \mathrm{GL}_n(L)$, which is of course non-abelian if $n > 1$, is naturally a G-module.

We would like to have a cohomology theory in this situation. We still have a G-invariants functor $M \mapsto M^G$ from the category of G-modules to the category of groups. Unfortunately, the category of groups is not abelian, so most of the standard constructions of homological algebra don't work. However, for small i we can try to mimic the definition of $H^i(G, M)$ "by hand" and see what we get.

Definition 6.2. Let G be a profinite group and M a G-module.

(1) Set $H^0(G, M) = M^G$.
(2) Let $Z^1(G, M)$ be the set of functions $\psi : G \to M$ satisfying $\psi(g_1g_2) = \psi(g_1) \cdot (g_1(\psi(g_2)))$. Note that there is no natural group structure on this set, but it does have a distinguished element, namely the trivial function given by $\psi(g) = e_M$ for all $g \in G$.
(3) Given $\psi, \eta \in Z^1(G, M)$, say that $\psi \sim \eta$ if there exists an element $m \in M$ such that $\psi(g) = m^{-1}\eta(g) \cdot gm$ for all $g \in G$. This is clearly an equivalence relation, and we define $H^1(G, M)$ to be the set of equivalence classes $Z^1(G, M)/\sim$. Again, $H^1(G, M)$ is a pointed set, namely a set with a distinguished element.

Observe that if M is abelian, then these definitions coincide (as pointed sets) with the cohomology groups we have defined already. Moreover, if PtSet denotes the category of pointed sets, where the morphisms are set maps $A \to B$ mapping the distinguished element of A to that of B, then $H^1(G, -)$ is a naturally a functor from the category of G-modules to PtSet (exercise: prove this!).

If $f : A \to B$ is a morphism of pointed sets, then its image is a pointed subset of B. If we set the kernel of f to be the preimage of the distinguished element of B, then clearly $\ker f$ is a pointed subset of A. Thus, we have a notion of exact sequences of pointed sets.

Proposition 6.3. *Let G be a profinite group, and let $0 \to M \xrightarrow{\varepsilon} N \xrightarrow{\pi} P \to 0$ be an exact sequence of G-modules. Then there is an exact sequence of pointed sets*

$$0 \to H^0(G, M) \to H^0(G, N) \to H^0(G, P) \xrightarrow{\delta^0} H^1(G, M) \to H^1(G, N) \to H^1(G, P),$$

where, for $p \in P^G$, we set $\delta^0(p)$ to be the equivalence class of the 1-cocycle $m \mapsto \tilde{p}^{-1} \cdot g\tilde{p}$. Here $\tilde{p}$ is an arbitrary lift of p to N.

Furthermore, suppose that $\varepsilon(M)$ lies in the center of N. In this case M is abelian, so $H^2(G, M)$ is defined. Then the exact sequence above can be extended by the map $H^1(G, P) \xrightarrow{\delta^1} H^2(G, M)$, where, for $\psi \in Z^1(G, P)$,

$$\delta^1([\psi]) = [(g_1, g_2) \mapsto \widetilde{\psi(g_1)} \cdot g_1\widetilde{\psi(g_2)} \cdot \widetilde{\psi(g_1g_2)}^{-1}].$$

Proof. Explicit computation. □

Proposition 6.4. *Let L/K be a finite Galois extension with Galois group $G = \mathrm{Gal}(L/K)$, and let V be a finite-dimensional L-vector space endowed with a semi-linear G-action: for every $\sigma \in G$, $a \in L$, and $v \in V$ we have $\sigma(av) = \sigma(a) \cdot \sigma(v)$. Then $\dim_K V^G = \dim_L V$ and the natural map*

$$\begin{array}{rcl} V^G \otimes_K L & \stackrel{\gamma}{\to} & V \\ v \otimes a & \mapsto & av \end{array}$$

is an isomorphism of L-vector spaces.

Proof. First we will show that γ is surjective. Suppose not, and let $u \in V$ be an element that is not contained in $\mathrm{im}\,\gamma$. Then there exists a subspace $W \subset V$ such that $\mathrm{im}\,\gamma \subset W$ and $V \simeq W \oplus Lu$, and projection onto the second component provides a non-zero linear functional $\lambda : V \to L$ such that $\lambda(\mathrm{im}\,\gamma) = 0$. For any $a \in L$ and $v \in V$, the element $w = \sum_{\sigma \in G} \sigma(av)$ lies in V^G. Thus $w = \gamma(w \otimes 1)$, and hence

$$0 = \lambda(w) = \sum_{\sigma \in G} \lambda(\sigma(av)) = \sigma(a) \sum_{\sigma \in G} \lambda(\sigma(v)).$$

In particular, for all $a \in L$ we have $\sum_{\sigma \in G} \lambda(\sigma(v)) \cdot \sigma(a) = 0$. By Dedekind's Lemma this means that $\lambda(\sigma(v)) = 0$ for all $\sigma \in G$ and all $v \in V$, contradicting $\lambda \neq 0$.

Hence γ is surjective. In view of this, to establish the remaining parts of our claim it suffices to show that $\dim_K V^G \leq \dim_L V$. Let $\{e_1, \ldots, e_r\}$ be a K-basis of V^G. We claim that these vectors are linearly independent over L. Set

$$B = \{(b_1, \ldots, b_r) \in L^r : \sum_{i=1}^{r} b_i e_i = 0\}.$$

Clearly B is an L-linear subspace of L^r. We know that $B \cap K^r = \{0\}$, since the e_i are linearly independent over K. Also, since the e_i are G-invariants, we observe that if $b = (b_1, \ldots, b_r) \in B$ and $\sigma \in G$, then $(\sigma(b_1), \ldots, \sigma(b_r)) \in B$. Thus, for all $a \in L$ and all $b \in B$, we have $(\mathrm{Tr}_{L/K}(ab_1), \ldots, \mathrm{Tr}_{L/K}(ab_r)) \in B \cap K^r = \{0\}$.

Recall from field theory that a finite extension L/K is separable if and only if the trace map $\mathrm{Tr}_{L/K}$ is not identically zero. (If $\mathrm{char}\, K = 0$, then any finite extension is separable, and indeed $\mathrm{Tr}_{L/K}(1) = [L : K] \neq 0$. In positive characteristic, proving this claim takes a bit more work.) Since our extension L/K is Galois, it is separable, and hence we must have $b_i = 0$ for all $1 \leq i \leq r$. Thus $B = \{0\}$, and hence $\{e_1, \ldots, e_r\}$ is linearly independent over L. □

We now deduce a corollary, which may be viewed as a non-abelian version of Hilbert 90.

Corollary 6.5. *Let L/K be a finite Galois extension with group $G = \mathrm{Gal}(L/K)$. Then $H^1(G, \mathrm{GL}_n(L))$ has one element for all $n \geq 1$.*

Proof. Let $\psi \in Z^1(G, \mathrm{GL}_n(L))$. Observe that $V = L^n$ carries a semilinear G-action by $v_\sigma = (\sigma(a_1), \ldots, \sigma(a_n))$, where $\sigma \in G$ and $v = (a_1, \ldots, a_n) \in V$. We consider a modified G-action, setting $\sigma(v) = \psi(\sigma) \cdot v_\sigma$. This is clearly semilinear, and it is an action because

$$(\tau\sigma)(v) = \psi(\tau\sigma) \cdot v_{\tau\sigma} = \psi(\tau) \cdot \tau(\psi(\sigma)) \cdot (v_\sigma)_\tau = \psi(\tau) \cdot (\psi(\sigma) \cdot v_\sigma)_\tau = \tau(\sigma(v)),$$

where the second equality holds because ψ is a 1-cocycle.

By Proposition 6.4, V has an L-basis $\{e_1, \ldots, e_n\}$ consisting of G-invariants. Let $A \in \mathrm{GL}_n(L)$ be the matrix whose columns are $e_1, \ldots, e_n$. Since $\psi(\sigma)(e_i)_\sigma = e_i$ for all $\sigma \in G$ and $1 \leq i \leq n$, and since the columns of the matrix $\sigma(A)$ are the $(e_i)_\sigma$, we find that

$A^{-1}\psi(\sigma)\sigma(A) = A^{-1}A = I_n$ for all $\sigma \in G$, where I_n is the identity matrix. Thus ψ is equivalent to the constant function $\sigma \mapsto I_n$ and lies in the distinguished class of $H^1(G, \mathrm{GL}_n(L))$. □

For every $a \in L^\times$, let $\varepsilon(a) \in \mathrm{GL}_n(L)$ denote the scalar matrix aI_n. Then the image of $\varepsilon : L^\times \hookrightarrow \mathrm{GL}_n(L)$ is the center of $\mathrm{GL}_n(L)$, and the cokernel is, by definition, the projective linear group $\mathrm{PGL}_n(L)$.

Lemma 6.6. *Let L/K be a finite Galois extension with group G, let $n = [L : K]$, and consider the short exact sequence $0 \to L^\times \xrightarrow{\varepsilon} \mathrm{GL}_n(L) \to \mathrm{PGL}_n(L) \to 0$ of G-modules. The map*

$$\delta^1 : H^1(G, \mathrm{PGL}_n(L)) \to H^2(G, L^\times)$$

defined in Proposition 6.3 is a surjection of pointed sets.

Proof. Let $V = L[G]$ be the n-dimensional L-vector space spanned by the elements $\{e_\sigma : \sigma \in G\}$. If $A \in \mathrm{GL}_n(L)$, denote by $\overline{A}$ its image in $\mathrm{PGL}_n(L)$.

Let $\psi \in Z^2(G, L^\times)$. For each $\sigma \in G$, define $\varphi(\sigma) \in \mathrm{Aut}_L V \simeq \mathrm{GL}_n(L)$ by $\varphi(\sigma)e_\tau = \psi(\sigma, \tau)e_{\sigma\tau}$ for all $\tau \in G$. For every $\eta \in G$ we have by definition that $\varphi(\sigma\tau)e_\eta = \psi(\sigma\tau, \eta)e_{\sigma\tau\eta}$, whereas

$$(\varphi(\sigma) \cdot \sigma\varphi(\tau))e_\eta = \varphi(\sigma)(\sigma(\psi(\tau, \eta))e_{\tau\eta}) = \sigma(\psi(\tau, \eta))\psi(\sigma, \tau\eta)e_{\sigma\tau\eta} = \psi(\sigma, \tau)\psi(\sigma\tau, \eta)e_{\sigma\tau\eta},$$

where the last equality holds because it follows from (5) that

$$\sigma(\psi(\tau, \eta)) \cdot \psi(\sigma\tau, \eta)^{-1}\psi(\sigma, \tau\eta)\psi(\sigma, \tau)^{-1} = 0 \tag{10}$$

for all $\sigma, \tau, \eta \in G$. Hence, although the map $\varphi : G \to \mathrm{GL}_n(L)$ need not be a 1-cocycle, the map $\overline{\varphi} : G \to \mathrm{PGL}_n(L)$ given by $\overline{\varphi}(\sigma) = \overline{\varphi(\sigma)}$ is a 1-cocycle.

Finally, it follows from (10) that $\varphi(\sigma) \cdot \sigma\varphi(\tau) \cdot \varphi(\sigma\tau)^{-1}e_\eta = \psi(\sigma, \sigma^{-1}\eta) \cdot \sigma\psi(\tau, \tau^{-1}\sigma^{-1}\eta) \cdot \psi(\sigma\tau, \tau^{-1}\sigma^{-1}\eta)e_\eta = \psi(\sigma, \tau)e_\eta$. Hence

$$\delta^1([\overline{\varphi}]) = [(\sigma, \tau) \mapsto \varphi(\sigma) \cdot \sigma\varphi(\tau) \cdot \varphi(\sigma\tau)^{-1}] = [\psi]$$

and δ^1 is surjective. □

Remark 6.7. It follows from Corollary 6.5 that δ^1 has trivial kernel for all $n \geq 1$. However, since δ^1 is only a morphism of pointed sets and not of groups, this does not imply that it is injective. Nevertheless, we will manage below to get some mileage out of this observation.

We state the following result, which follows immediately from the Skolem-Noether Theorem. The precise statement and proof of Skolem-Noether (Theorem 6.21 below) will have to wait until we have developed the theory of central simple algebras.

Lemma 6.8 (Skolem-Noether)**.** *Let L be a field, and let $f, g : M_m(L) \to M_m(L)$ be two L-algebra homomorphisms. Then there exists an invertible matrix $b \in M_m(L)$ such that $g(a) = b \cdot f(a) \cdot b^{-1}$ for all $a \in A$.*

Lemma 6.9. *Let L/K be a finite Galois extension with Galois group $G = \mathrm{Gal}(L/K)$, and for every $m \geq 1$ let $A(L/K, m)$ denote the set of K-isomorphism classes of central simple K-algebras A such that $A \otimes_K L \simeq M_m(L)$. Then $A(L/K, m) \simeq H^1(G, \mathrm{PGL}_m(L))$ as pointed sets, where the distinguished element of $A(L/K, m)$ is (the isomorphism class of) $M_m(K)$.*

Proof. Let A be a K-algebra such that $A \otimes_K L \simeq M_m(L)$, and let $\sigma \in G$. On the one hand, σ acts on matrices $c \in M_m(L)$ by acting on each matrix element; we denote the image by $\sigma(c)$. On the other hand, σ acts on $A \otimes_K L$ by sending $c = \sum_{j=1}^r a_j \otimes b_j$ to $(1 \otimes \sigma)c = \sum_{j=1}^r a_j \otimes \sigma(b_j)$, where $a_j \in A$ and $b_j \in L$.

The maps $c \mapsto \sigma(c)$ and $c \mapsto (1 \otimes \sigma)c$ are semilinear, not linear, over L. However, the map $f : M_m(L) \to M_m(L)$ given by $f(c) = \sigma^{-1}((1 \otimes \sigma)(c))$ is L-linear. Since $M_m(L)$ is a simple L-algebra with center L, we can apply Lemma 6.8, taking g to be the identity map. We obtain an invertible matrix $b \in \mathrm{GL}_m(L)$ such that $c = b \cdot \sigma^{-1}(1 \otimes \sigma)c \cdot b^{-1}$ for all $c \in M_m(L)$. Hence $(1 \otimes \sigma)c = \Psi(\sigma)\sigma(c)\Psi(\sigma)^{-1}$, where $\Psi(\sigma) = \sigma(b^{-1})$. Note that $\Psi(\sigma)$ is not well-defined, but $\overline{\Psi(\sigma)} \in \mathrm{PGL}_m(L)$ is.

We claim that $(\sigma \mapsto \overline{\Psi(\sigma)}) \in Z^1(G, \mathrm{PGL}_m(L))$. Indeed,

$$(1 \otimes \sigma\tau)c = (1 \otimes \sigma)((1 \otimes \tau)c) = \Psi(\sigma)\sigma(\Psi(\tau)\tau(c)\Psi(\tau)^{-1})\Psi(\sigma)^{-1}.$$

Thus we get a map $A(L/K, m) \to H^1(G, \mathrm{PGL}_m(L))$.

Conversely, given a cocycle $(\sigma \mapsto \overline{\Psi(\sigma)})$, we can define an action of G on $M_m(L)$ by $(1 \otimes \sigma)c = \Psi(\sigma)\sigma(c)\Psi(\sigma)^{-1}$. The set of elements $c \in M_m(L)$ that are invariant under every $1 \otimes \sigma$ is a central simple K-algebra A, which satisfies $A \otimes_K L \simeq M_m(L)$ by Proposition 6.4. One checks that this gives a well-defined map $H^1(G, \mathrm{PGL}_m(L)) \to A(L/K, m)$: indeed, any 1-cocycle that is cohomologous to $(\sigma \mapsto \overline{\Psi(\sigma)})$ will give rise to a K-algebra of G-invariants that is conjugate to, and hence isomorphic to, A. It is obvious from the construction that this map is inverse to the map $A(L/K, m) \to H^1(G, \mathrm{PGL}_m(L))$ that was defined above.

Finally, if $A = M_m(K)$, then the two actions $c \mapsto \sigma(c)$ and $c \mapsto (1 \otimes \sigma)(c)$ coincide, and thus we may take $\Psi(\sigma)$ to be the identity matrix for every $\sigma \in G$. Thus our maps preserve distinguished elements. □

6.2. **Central simple algebras.** Before we can proceed, we'll need a crash course in the theory of central simple algebras. We'll now do a bit of beautiful pure ring theory, although for our purposes in this course it is a tool for computing cohomology. As Rowen writes about central simple algebras (*Ring Theory*, Volume II, p.187): "There is some question among experts as to whether this theory belongs more properly to ring theory, field theory, cohomology theory, or algebraic K-theory."

Let F be a field.

Definition 6.10. (1) An *F-algebra* is a non-commutative ring A, equipped with a ring homomorphism $F \to Z(A)$. Here $Z(A)$ denotes the center of A.
(2) An F-algebra A is called *simple* if has no non-zero proper two-sided ideals. It is called *central simple* if $F \simeq Z(A)$.
(3) An F-algebra A is called a *division algebra* if $A \setminus \{0\}$ is a group under multiplication.
(4) Given an F-algebra A, we let A^{op} be the algebra with the same underlying abelian group as A, but with a multiplication operation $*$ given by $a * b = ba$.

Lemma 6.11 (Schur)**.** *Let M and N be simple A-modules (no non-trivial submodules). If $f \in \mathrm{Hom}_A(M, N)$, then either $f = 0$ or f is an isomorphism. In particular, if M is a simple A-module, then $\mathrm{End}_A(M)$ is a division algebra.*

Proof. If $f \in \mathrm{Hom}_A(M, N)$, then $\ker f$ is a submodule of M and $\mathrm{im}\, f$ is a submodule of N. This implies the first claim. Hence any non-zero element of $\mathrm{End}_A(M)$ is an isomorphism, so it has a multiplicative inverse. □

In the sequel we shall only have cause to work with algebras that have finite dimension over F. In order to avoid writing this condition over and over, we shall make it a running hypothesis: *from now on, all F-algebras are assumed to be finite-dimensional.*

Theorem 6.12 (Jacobson Density Theorem)**.** *Let A be a finite-dimensional F-algebra, and let M be a simple A-module such that $\dim_F(M) < \infty$. Let $D = \mathrm{End}_A(M)$. Suppose that $m_1, \ldots, m_r \in M$ are linearly independent over D, and let $n_1, \ldots, n_r \in M$. Then there exists an element $a \in A$ such that $am_i = n_i$ for all $1 \leq i \leq r$.*

Proof. Many of the standard results in linear algebra work over division rings as well as fields, with the same proofs. For instance, any module over a division ring is free, and any linearly-independent set extends to a basis. Thus there exists a D-submodule $N \subset M$ such that $M = Dm_1 \oplus Dm_2 \oplus \cdots \oplus Dm_r \oplus N$. In particular, there exists $\varphi \in \mathrm{End}_D(M)$ such that $\varphi(m_i) = n_i$ for all $1 \leq i \leq r$.

Consider the element $m = (m_1, \ldots, m_r) \in M^r$. Since M^r is a semisimple A-module, being a direct sum of simple modules, any submodule is a direct summand. Thus there exists an A-submodule $P \subset M^r$ such that $M^r = Am \oplus P$. Let $\pi \in \mathrm{End}_A(M^r) = M_r(D)$ be projection onto the component Am.

The map $\varphi^r : M^r \to M^r$ given by $\varphi^r(x_1, \ldots, x_r) = (\varphi(x_1), \ldots, \varphi(x_r))$ is a map of $M_r(D)$-modules, and thus

$$(n_1, \ldots, n_r) = \varphi^r(m_1, \ldots, m_r) = \varphi^r\pi(m_1, \ldots, m_r) = \pi\varphi^r(m_1, \ldots, m_r) = \pi(n_1, \ldots, n_r).$$

Hence $(n_1, \ldots, n_r) \in Am$, which is exactly our claim. □

Corollary 6.13. *Suppose that A is a central simple F-algebra. Then $A \otimes_F A^{\mathrm{op}} \simeq M_n(F)$, where $n = \dim_F(A)$.*

Proof. We wish to prove that $A \otimes_F A^{\mathrm{op}} \simeq \mathrm{End}_F(A)$. Let $\{a_1, \ldots, a_n\}$ be an F-basis of A. Since $A \otimes_F A^{\mathrm{op}}$ acts on A by $(x \otimes y)a = xay$, for $a, x \in A$ and $y \in A^{\mathrm{op}}$, we get an F-linear map $f : A \otimes_F A^{\mathrm{op}} \to \mathrm{End}_F(A)$. Since the two algebras are both of dimension n^2 over F, it suffices to show that f is surjective.

Note that A is a simple $A \otimes_F A^{\mathrm{op}}$-module, since any submodule would be a two-sided ideal. Observe that $\mathrm{End}_A(A) \simeq A^{\mathrm{op}}$ via the map $\theta \mapsto \theta(1)$. Thus any element of $\mathrm{End}_{A \otimes_F A^{\mathrm{op}}}(A)$ is also of the form $a \mapsto ab$ for some $b \in A$. In order for this to be compatible with the A^{op}-action as well, we must have $b \in Z(A) \simeq F$. Thus $D = \mathrm{End}_{A \otimes_F A^{\mathrm{op}}}(A) \simeq F$, so that $a_1, \ldots, a_n$ are linearly independent over D.

Let $\varphi \in \mathrm{End}_F(A)$. By the Jacobson density theorem, there exists an element $c \in A \otimes_F A^{\mathrm{op}}$ such that $ca_i = \varphi(a_i)$ for all $1 \leq i \leq n$. Then $f(c) = \varphi$, so f is indeed surjective. □

Corollary 6.14. *Let A be a central simple F-algebra and B any simple F-algebra. Then $A \otimes_F B$ is a simple F-algebra.*

Proof. As in the proof of Corollary 6.13, let $\{a_1, \ldots, a_n\}$ be an F-basis of A. By that corollary, for each $1 \leq i \leq n$ there exists an element $c_i \in A \otimes_F A^{\mathrm{op}}$ such that

$$c_i(a_j) = \delta_{ij} = \begin{cases} 1 & : i = j \\ 0 & : i \neq j. \end{cases}$$

Suppose that $I \subset A \otimes_F B$ is a two-sided ideal. Let $\sum_{j=1}^n a_j \otimes b_j \in I$. Then $\sum_{j=1}^n c_i(a_j) \otimes b_j = 1 \otimes b_i \in I$, for all $1 \leq i \leq n$, since I is a two-sided ideal. However, $1 \otimes b_i \in F \otimes_F B$.

Since $J = I \cap (F \otimes_F B)$ is a two-sided ideal of $F \otimes_F B \simeq B$, we have either $J = 0$ or $J = B$. In the first case, $b_i = 0$ for all i, and hence $I = 0$. In the second case, $1 \otimes b \in I$ for every $b \in B$, so clearly $I = A \otimes_F B$. □

Corollary 6.15. *If A and B are central simple F-algebras, then so is $A \otimes_F B$.*

Proof. By the previous corollary, it suffices to verify that $Z(A \otimes_F B) \simeq F$. As before, let $\{a_1, \ldots, a_n\}$ be an F-basis of A, and suppose that $\sum_{j=1}^n a_j \otimes b_j \in Z(A \otimes_F B)$. Then $1 \otimes b$ commutes with this element for all $b \in B$, so that $\sum_{j=1}^n a_j \otimes (b_j b - b b_j) = 0$ for every $b \in B$. Since the a_j are F-linearly independent, it follows that $b_j b - b b_j = 0$ for all $b \in B$, hence that $b_j \in Z(B) = F$. Hence $c = \alpha \otimes 1$ for some $\alpha \in A$. Since c commutes with $a \otimes 1$ for every $a \in A$, we find that $\alpha \in Z(A) = F$, hence $c \in F$. □

Recall, by Wedderburn's theorem, that every simple F algebra is isomorphic to $M_r(D)$ for some division algebra D and some integer r. Moreover, every simple $M_r(D)$-module is isomorphic to D^r, with the natural action of $M_r(D)$.

Definition 6.16. Let A and B be two central simple F-algebras. We say that they are *equivalent* if there exists a division algebra D such that $A \simeq M_r(D)$ and $B \simeq M_s(D)$. Let $\mathrm{Br}(F)$ be the set of equivalence classes of central simple F-algebras.

Lemma 6.17. *The set* $\mathrm{Br}(F)$ *is an abelian group under the operation* $[A][B] = [A \otimes_F B]$.

Proof. The only bit of the proof that is non-trivial at this point is to verify that the operation is well-defined. This follows from the fact that $M_r(D) \otimes_F M_s(D') \simeq M_{rs}(D \otimes D')$. If we view F as an algebra over itself, then the class $[F] \in \mathrm{Br}(F)$ is an identity element. Since $[A][A^{\mathrm{op}}] = [A \otimes_F A^{\mathrm{op}}] = [M_{\dim_F(A)}(F)] = [F]$, every element of $\mathrm{Br}(F)$ has an inverse. □

Definition 6.18. Let L/K be an extension of fields. There is a natural homomorphism of groups

$$\begin{array}{rcl} \mathrm{Br}(K) & \to & \mathrm{Br}(L) \\ [A] & \mapsto & [A \otimes_K L]. \end{array}$$

We write $\mathrm{Br}(L/K)$ for the kernel.

Lemma 6.19. *Let D be a division algebra, and let $r \geq 1$. Consider D^r as a $M_r(D)$-module in the natural way. Then* $\mathrm{End}_{M_r(D)}(D^r) \simeq D^{\mathrm{op}}$.

Proof. We write elements of D^r as columns. Let $f \in \mathrm{End}_{M_r(D)}(D^r)$, and let $\varepsilon = (1, 0, \ldots, 0)^T \in D^r$. Since the $M_r(D)$-orbit of ε is all of D^r, we see that f is determined by $f(\varepsilon)$. If $C \in M_r(D)$ is any matrix whose first column consists entirely of zeroes, then $C\varepsilon = (0, \ldots, 0)^T$ and hence $Cf(\varepsilon) = (0, \ldots, 0)^T$. It follows that $f(\varepsilon) = (d, 0, \ldots, 0)^T$ for some $d \in D$. Thus, for any $d_1, \ldots, d_r \in D$, we must have

$$f\begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_r \end{pmatrix} = f\left(\begin{pmatrix} d_1 & 0 & \cdots & 0 \\ d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ d_r & 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right) = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ d_r & 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} d \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} d_1 d \\ d_2 d \\ \vdots \\ d_r d \end{pmatrix}.$$

On the other hand, any f as above is indeed a $M_r(D)$-endomorphism of D^r, since $M_r(D)$ acts by left multiplication on the components of elements of D^r, and this commutes with right multiplication by d. Identifying the element f as above with $d \in D$, we clearly obtain an isomorphism $\mathrm{End}_{M_r(D)}(D^r) \simeq D^{\mathrm{op}}$. □

Proposition 6.20 (Double Centralizer Theorem)**.** *Suppose that A is a central simple F-algebra, and let $B \subset A$ be a simple subalgebra. Write $C_A(B)$ for the centralizer of B, namely the set of all $a \in A$ such that $ab = ba$ for every $b \in B$. Then $C_A(B)$ is a simple F-algebra. Moreover,* $\dim_F C_A(B) = \frac{\dim_F(A)}{\dim_F(B)}$ *and* $C_A(C_A(B)) = B$.

Proof. Since $B \otimes_F A^{\mathrm{op}}$ is simple by Corollary 6.14, we have $B \otimes_F A^{\mathrm{op}} \simeq M_r(D)$ for some division algebra D and some $r \geq 1$ by Wedderburn's theorem. The algebra A is a $B \otimes_F A^{\mathrm{op}}$-module in the obvious way; thus it is semisimple. Since any simple $M_r(D)$-module is isomorphic to D^r by Wedderburn's theorem, we have $A \simeq (D^r)^s$ as a $M_r(D)$-module, for some integer s.

Now observe that any $f \in \mathrm{End}_{B\otimes_F A^{\mathrm{op}}}(A)$ is determined by $f(1)$. Moreover, if $f(1) = c$, then for any $a \in A$ and $b \in B$ we have the equality $bca = (b \otimes a)f(1) = f((b \otimes a)1) = f(ba) = f((1 \otimes ba)1) = (1 \otimes ba)f(1) = cba$, which forces $c \in C_B(A)$. This gives an isomorphism $C_A(B) \simeq \mathrm{End}_{B\otimes_F A^{\mathrm{op}}}(A)$. But $\mathrm{End}_{B\otimes_F A^{\mathrm{op}}}(A) \simeq \mathrm{End}_{M_r(D)}((D^r)^s) \simeq M_s(\mathrm{End}_{M_r(D)}(D^r)) \simeq M_s(D^{\mathrm{op}})$, where the last isomorphism comes from Lemma 6.19. Hence $C_A(B)$ is simple.

It follows from the previous paragraph that

$$r^2 \dim_F D = \dim_F M_r(D) = \dim_F(B \otimes_F A^{\mathrm{op}}) = (\dim_F A)(\dim_F B).$$

On the other hand, $A \simeq (D^r)^s$, so $\dim_F A = rs \dim_F D$. This in turn implies that $\dim_F B = \frac{r}{s}$. We have also shown that $C_A(B) \simeq M_s(D^{\mathrm{op}})$, hence $\dim_F C_A(B) = s^2 \dim_F D$, and the claimed relation of dimensions follows immediately.

Finally, since $C_A(B)$ is simple, the second part of the claim implies that $\dim_F C_A(C_A(B)) = \frac{\dim_F A}{\dim_F C_A(B)} = \dim_F(B)$. Clearly $B \subseteq C_A(C_A(B))$. Since these two F-algebras have the same dimension, they are equal. □

Theorem 6.21 (Skolem-Noether)**.** *Let F be a field, and let A and B be two simple F-algebras. Suppose that $Z(B) = F$ and that B has finite dimension over F. Let $f, g : A \to B$ be two F-algebra homomorphisms. Then there exists a unit $b \in B$ such that $g(a) = b \cdot f(a) \cdot b^{-1}$ for all $a \in A$.*

Proof. First we treat the special case $B = M_n(F) = \mathrm{End}_F(F^n)$. In this case, we can view the maps f and g as specifying two A-actions on the vector space F^n. Let V_f and V_g be the corresponding A-modules. Since there is only one simple A-module up to isomorphism, every finite-dimensional A-module is a direct sum of simple A-modules, and $\dim_F V_f = \dim_F V_g = n$, we see that V_f and V_g must be isomorphic as A-modules. Let $b : V_f \to V_g$ be an A-module isomorphism. Forgetting the A-module structure, we have that $b : F^n \to F^n$ is a linear transformation and hence $b \in M_n(F) = B$. Since B respects the A-module structure, for all $a \in A$ we have $bf(a) = g(a)b$ as claimed.

Now consider the general case. Since B is central simple, we know that $B \otimes_F B^{\mathrm{op}} \simeq M_n(F)$ by Corollary 6.13, where $n = \dim_F(B)$. Moreover, $A \otimes_F B^{\mathrm{op}}$ is a simple F algebra by Corollary 6.14. We obtain a map

$$\begin{aligned} f \otimes 1 : A \otimes_F B^{\mathrm{op}} &\to B \otimes_F B^{\mathrm{op}} \simeq M_n(F) \\ a \otimes c &\mapsto f(a) \otimes c, \end{aligned}$$

for all $a \in A$ and $c \in B^{\mathrm{op}}$. We define $g \otimes 1$ similarly. By the case that we have proved already, there exists an element $\beta \in B \otimes_F B^{\mathrm{op}}$ such that

$$(g \otimes 1)(a \otimes c) = \beta \cdot (f \otimes 1)(a \otimes c) \cdot \beta^{-1} \tag{11}$$

for all $a \in A$ and $c \in B^{\mathrm{op}}$. Taking $a = 1$, we find that $\beta(1 \otimes c)\beta^{-1} = 1 \otimes c$, namely that $b \in C_{B\otimes_F B^{\mathrm{op}}}(F \otimes_F B^{\mathrm{op}}) = B \otimes_F F$. To establish the last equality, note that $B \otimes_F F$ is obviously contained in the centralizer, and it has the same dimension as the centralizer by the Double Centralizer Theorem. Hence $\beta = b \otimes 1$ for some $b \in B$, and taking $c = 1$ in (11), we see that b has the property we want. □

Note that the matrix algebras $M_m(F)$ are central simple F-algebras, and Lemma 6.8, which was already used above, is just the previous theorem in the case $A = B = M_m(F)$. Having repaid our Skolem-Noether debt, we deduce some further corollaries of the Double Centralizer Theorem.

Corollary 6.22. *Let D be a central division F-algebra. Then $\dim_F D$ is a square, and any maximal subfield of D has degree $\sqrt{\dim_F D}$ over F.*

Proof. Observe that D does have subfields, since F is one. Let L be a maximal subfield of D. Since L is commutative, we have $L \subseteq C_D(L)$. This inclusion is in fact an equality; otherwise, we could take $x \in C_D(L) \setminus L$, and the subalgebra $L(x)$ would be a commutative division algebra, hence a field, contradicting the maximality of L. Hence $L = C_D(L)$. Then the Double Centralizer Theorem tells us that $\dim_F L = \frac{\dim_F D}{\dim_F L}$, which completes the proof. □

Corollary 6.23. *Let A be a central simple F-algebra, and let L be a maximal subfield of A. Then $A \otimes_F L \simeq M_n(L)$. In other words, $[A] \in \mathrm{Br}(L/F)$.*

Proof. Note that maximal subfields of A are the same as maximal subfields of A^{op}. Let L be a maximal subfield. By the Double Centralizer Theorem, the centralizer $C_{A^{\mathrm{op}}}(L)$ is simple, and hence Wedderburn implies that $C_{A^{\mathrm{op}}}(L) \simeq M_r(D)$ for some division algebra D and some $r \geq 1$. Since L commutes with every element of $C_{A^{\mathrm{op}}}(L)$, we have $L \subseteq Z(M_r(D)) = D$. Now maximality of L implies that $L = D$: if there were an element $x \in D \setminus L$, then $L(x)$ would be a larger subfield of A^{op}. Hence $C_{A^{\mathrm{op}}}(L) \simeq M_r(L)$. However, the proof of the Double Centralizer Theorem shows that if $C_{A^{\mathrm{op}}}(L)$ is a matrix algebra over L, then $L \otimes_F A$ is a matrix algebra over $L^{\mathrm{op}} = L$, which is what we want. □

Using the tools we have developed, we can finally understand the structure of the cohomology group $H^2(\mathrm{Gal}(L/K), L^\times)$. Observe that if $m|q$ are two natural numbers, then the diagonal embedding $\Delta : \mathrm{GL}_m(L) \hookrightarrow \mathrm{GL}_q(L)$ sending $A \in \mathrm{GL}_m(L)$ to the block-diagonal matrix $\mathrm{diag}(A, A, \ldots, A)$ maps scalar matrices, and only scalar matrices, to scalar matrices in $\mathrm{GL}_q(L)$. Thus it induces an embedding $\mathrm{PGL}_m(L) \hookrightarrow \mathrm{PGL}_q(L)$ and hence a map $\Delta : H^1(G, \mathrm{PGL}_m(L)) \to H^1(G, \mathrm{PGL}_q(L))$. Recall that $A(L/K, m)$ denotes the set of K-isomorphism classes of central simple K-algebras A such that $A \otimes_K L \simeq M_m(L)$. We want to understand the map $A(L/K, m) \to A(L/K, q)$ that corresponds to Δ under the bijection of Lemma 6.9.

Let $(\sigma \mapsto \overline{\Psi(\sigma)}) \in Z^1(G, \mathrm{PGL}_m(L))$ be a 1-cocycle. Recall that in Lemma 6.9 we defined an action of G on $M_m(L)$ as follows: an element $\sigma \in G$ sends $c \in M_m(L)$ to the matrix $(1 \otimes \sigma)(c) = \Psi(\sigma)\sigma(c)\Psi(\sigma)^{-1}$, where $\sigma(c)$ denotes the action of G on matrix elements as in Example 6.1. Then the element of $A(L/K, m)$ associated to our cocycle is (the isomorphism class of) the algebra A of G-invariants under this new action.

It is easy to check that the algebra associated to the cocyle $(\sigma \mapsto \overline{\mathrm{diag}(\Psi(\sigma), \ldots, \Psi(\sigma))}) \in Z^1(G, \mathrm{PGL}_q(L))$ consists of all matrices of the form

$$c = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1d} \\ c_{21} & c_{22} & \cdots & c_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ c_{d1} & c_{d2} & \cdots & c_{dd} \end{pmatrix},$$

where $d = \frac{q}{m}$ and each $c_{ij} \in M_m(L)$ is contained in A. Thus, if a class $\psi \in H^1(G, \mathrm{PGL}_m(L))$ corresponds to an algebra isomorphic to $M_r(D)$ for some division algebra D and some $r \geq 1$, then $\Delta(\psi) \in H^1(G, \mathrm{PGL}_q(L))$ corresponds to $M_{rd}(D)$. We have thus proved the following.

Lemma 6.24. *The bijections of Lemma 6.9 induce a bijection of pointed sets*

$$\varinjlim H^1(G, \mathrm{PGL}_m(L)) \to \mathrm{Br}(L/K). \tag{12}$$

Proof. The content of the claim is that the direct limit $\varinjlim A(L/K, m)$ induced by the maps of Lemma 6.9 is exactly $\mathrm{Br}(L/K)$, as a set. This is immediate from the calculation we just did. □

Since $\mathrm{Br}(L/K)$ is a group under the operation $[A][B] = [A \otimes_K L]$, it follows that the injective limit on the left-hand side of (12) has a natural group structure even though the sets $H^1(G, \mathrm{PGL}_m(L))$ do not. Moreover, recalling the maps $\delta^1 : H^1(G, \mathrm{PGL}_m(L)) \to H^2(G, L^\times)$ from Lemma 6.6, the universal property of the injective limit and the previous lemma produce a map $\mathrm{Br}(L/K) \to H^2(G, L^\times)$, which we continue to label δ^1.

Theorem 6.25. *The map $\delta^1 : \mathrm{Br}(L/K) \to H^2(G, L^\times)$ is an isomorphism of abelian groups.*

Proof. It is enough to show that δ^1 is a group homomorphism. Indeed, δ^1 is surjective by Lemma 6.6, and by Corollary 6.5 it has trivial kernel, which for group homomorphisms implies injectivity.

So let A and B be two central simple K-algebras such that $[A], [B] \in \mathrm{Br}(L/K)$. Let $A \in A(L/K, m)$ and $B \in A(L/K, m')$. If A and B correspond to the cosets $(\sigma \mapsto \overline{\Psi(\sigma)})$ and $(\sigma \mapsto \overline{\Phi(\sigma)})$ in $H^1(G, \mathrm{PGL}_m(L))$ and $H^1(G, \mathrm{PGL}_{m'}(L))$, respectively, then it is easy to see that $A \otimes_K B$ corresponds to $(\sigma \mapsto \overline{\Xi(\sigma)})$, where $\Xi(\sigma) \in M_{mm'}(L)$ corresponds to the endomorphism of $L^{mm'} = L^m \otimes_L L^{m'}$ satisfying

$$(\Xi(\sigma))(v \otimes v') = (\Psi(\sigma))v \otimes \Phi(\sigma)v'$$

for all $v \in L^m$ and $v' \in L'$. By the definition of δ^1, we have

$$\Xi(g_1) \circ g_1 \Xi(g_2) \circ \Xi(g_1 g_2)^{-1} = \delta^1([A \otimes_K B]) I_{mm'}$$

as automorphisms of $L^m \otimes_L L^{m'}$, for any $g_1, g_2 \in G$. Applying both sides to a vector of the form $v \otimes v'$, we find that $\delta^1([A \otimes_K L]) = \delta^1([A])\delta^1([B])$, as claimed. □

Corollary 6.26. *Let K be a field, and let G_K be its absolute Galois group. Then $\mathrm{Br}(K) \simeq H^2(G_K, \overline{K}^\times)$.*

Proof. On one hand, $H^2(G_K, \overline{K}^\times) = \varinjlim H^2(\mathrm{Gal}(L/K), L^\times)$ by Proposition 4.10, where L/K runs over finite Galois extensions. On the other hand, by Corollary 6.23, $\mathrm{Br}(K)$ is the union of the $\mathrm{Br}(L/K)$, and so $\mathrm{Br}(K) = \varinjlim \mathrm{Br}(L/K)$, where the connecting maps are inclusions. Finally, one checks that the maps $\delta^1 : \mathrm{Br}(L/K) \xrightarrow{\sim} H^2(\mathrm{Gal}(L/K), L^\times)$ are compatible with the injective systems. □

Corollary 6.27. *Let L/K be a finite Galois extension of degree $n = [L : K]$. Then $\mathrm{Br}(L/K)$ is a torsion group with exponent dividing n.*

Proof. Let $G = \mathrm{Gal}(L/K)$. Then $\{e\}$ is an open subgroup of G, and as in the proof of Corollary 4.13, we observe that the composition

$$H^2(G, L^\times) \xrightarrow{\mathrm{res}} H^2(\{e\}, \mathrm{Res}^G_{\{e\}} L^\times) \xrightarrow{\mathrm{cor}} H^2(G, L^\times)$$

is zero since the middle group is trivial. On the other hand, this composition is multiplication by $[G : \{e\}] = n$ by Lemma 4.4. Thus all elements of $H^2(G, L^\times) \simeq \mathrm{Br}(L/K)$ are killed by multiplication by n. □

7. Examples of Brauer groups and the invariant map

7.1. **Trivial Brauer groups.** So far we have not explicitly computed a single Brauer group. We will now remedy this situation, developing useful techniques along the way. Some of the arguments we give are much more complicated than is necessary to compute the relevant Brauer groups, but the idea is to introduce tools that will serve us well later in the course, and in later life.

Proposition 7.1. *If K is an algebraically closed field, then $\mathrm{Br}(K)$ is trivial.*

Proof. Let A be a central simple K-algebra. By Corollary 6.23 we see that $[A] \in \mathrm{Br}(L/K)$ for a finite extension L/K. Since K has no nontrivial finite extensions, it follows that $[A] \in \mathrm{Br}(K/K) = 0$. □

Lemma 7.2. *Let D be a finite division ring. Then D is a field.*

Proof. Let $K = Z(D)$ be the center of D. Clearly this is a field, and D is a central division K-algebra. Suppose, by way of contradiction, that D is not a field, so that $n = \sqrt{\dim_K D} > 1$. Any element $x \in D$ is contained in the subfield $K(x) \subset D$ and hence in some maximal subfield of D. By Corollary 6.22, all maximal subfields of D have degree n over K and thus have cardinality $|K|^n$.

Let L and L' be two maximal subfields of D. Since they have the same finite cardinality, there exists an isomorphism $\iota : L \to L'$, which may be taken to be K-linear. Now we apply Skolem-Noether to the two maps $f, g : L \to D$, where f is the natural inclusion of L in D, and g is the composition of ι with the natural inclusion of L' in D. We find that L and L' are conjugate in D, and hence that the group $D^\times$ is the union of the conjugates of the subgroup $L^\times$. Moreover, since D is not a field and hence $D \neq L$, we have that $L^\times$ is a proper subgroup of $D^\times$. We have arrived at a contradiction, since a finite group cannot be a union of conjugates of a proper subgroup.

Indeed, if G is a finite group and H is a subgroup, then the number of conjugates of H is $[G : N_G(H)]$. Since all the conjugates contain the identity element of G, we find that the cardinality of the union of the subgroups conjugate to H is at most $[G : N_G(H)](|H|-1)+1 \leq [G : H](|H| - 1) + 1$, and this is strictly smaller than $|G|$ if H is proper. □

Proposition 7.3. *If K is a finite field, then $\mathrm{Br}(K)$ is trivial.*

Proof. Let A be a central simple K-algebra. Then, by Wedderburn's Theorem, we have $A \simeq M_r(D)$, where D is a division ring that is finite-dimensional over K, hence finite. Thus D is a field by the previous lemma, so that $D = Z(A) = K$. □

7.2. **Brauer groups of local fields.** We now begin to consider one of the most interesting cases, namely that of local fields. The study of their Brauer groups will lead to a deeper understanding of their Galois cohomology. From this point onwards in the course, we will rely on standard theorems from algebraic number theory.

Definition 7.4. Let L/K be an extension of fields. We say that a simple K-algebra splits over L if $A \otimes_K L \simeq M_r(L)$ for some $r \geq 1$.

Proposition 7.5. *Let $K/\mathbb{Q}_p$ be a finite extension and let D be a division K-algebra. There exists an* unramified *finite extension L/K such that D splits over L.*

Proof. Let k be the residue field of K, and let $n = \sqrt{\dim_K(D)}$. Let $|\cdot|_K$ denote the (multiplicative) valuation of K. We claim that one can construct a non-Archimedean valuation $|\cdot|_D : D \to \mathbb{R}_{\geq 0}$ such that

- For any $x \in D$, we have $|x|_D = 0$ if and only if $x = 0$.
- For any $x, y \in D$, we have $|xy|_D = |x|_D|y|_D$.
- For any $x, y \in D$, we have $|x + y|_D \leq \max\{|x|_D, |y|_D\}$.

Indeed, for every $x \in D$, we define $|x|_D = |\det A_x|_K$, where A_x is a matrix, with entries in K, representing the K-linear map

$$\begin{array}{ccc} D & \to & D \\ y & \mapsto & yx \end{array}$$

with respect to some K-basis of D. It is clear that $|\cdot|_D$, thus defined, satisfies the first two of the three desired properties, and that $|x|_D = |x|_K^{\dim_K(D)}$ whenever $x \in K$. Note that the third property is equivalent to the claim that $|1 + z|_D \leq 1$ whenever $z \in D$ is such that $|z|_D \leq 1$. As in the previous proof, $K(z)$ is a subfield of D. Judiciously choosing a K-basis of D of the form $\alpha_i\beta_j$, where $\{\alpha_i\}$ is a $K(z)$-basis of D and $\{\beta_j\}$ is a K-basis of $K(z)$, we see that, if $t \in K(z)$, then A_z is a block-diagonal matrix and that

$$|t|_D = |\det A_t|_K = |\mathrm{N}_{K(z)/K}(t)|_K^{\dim_{K(z)}(D)}. \tag{13}$$

We know from algebraic number theory that the right-hand side of (13) defines a multiplicative valuation of $K(z)$. Considering $t = z$ and $t = 1 + z$, we obtain the third property as well.

As in the commutative case, we define $\mathcal{O}_D = \{x \in D : |x|_D \leq 1\}$. By our three properties, this is a ring, and $\mathfrak{m}_D = \{x \in D : |x|_D < 1\}$ is a two-sided ideal. It is a maximal ideal since all elements of $\mathcal{O}_D \setminus \mathfrak{m}_D$ have inverses in $\mathcal{O}_D$, and hence the quotient $\Delta = \mathcal{O}_D/\mathfrak{m}_D$ is a division ring. For every subfield $K \subset L \subset D$ we have $\mathcal{O}_D \cap L = \mathcal{O}_L$ by (13), and thus $\mathcal{O}_D$ consists precisely of the elements of D that are integral over K. The same argument as in the commutative case shows that (the underlying abelian group of) $\mathcal{O}_D$ is a finitely generated $\mathcal{O}_K$-module. Since $\mathcal{O}_D$ is torsion-free and $\mathcal{O}_K$ is a principal ideal domain, $\mathcal{O}_D$ is a free $\mathcal{O}_K$-module, and its rank must be $\dim_K D = n^2$, since $\mathcal{O}_D \otimes_{\mathcal{O}_K} K = D$. Since $\mathfrak{m}_K \subset \mathfrak{m}_D$, we find that Δ is naturally a finite-dimensional vector space over k. Hence Δ is finite, and thus it is a field by Lemma 7.2. Set $f = \dim_k \Delta$.

Since all extensions of finite fields are finite and separable and hence simple, there exists an element $\overline{\delta} \in \Delta$ such that $\Delta = k(\overline{\delta})$. Let $\delta \in \mathcal{O}_D$ be a lift of $\overline{\delta}$. Then $K(\delta)$ is a subfield of D with residue field Δ. Thus $f \leq [K(\delta) : K] \leq n$, where the second inequality is Corollary 6.22.

Since $\mathcal{O}_D$ is discretely valued, we can prove exactly as for commutative discretely valued rings that any two-sided ideal of $\mathcal{O}_D$ is principal, generated by an element of maximal valuation, and hence is either zero or of the form $\mathfrak{m}_D^r$ for some $r \geq 0$. Thus we can define the ramification index of D/k, as in the commutative case, to be the natural number e satisfying $\mathfrak{m}_K\mathcal{O}_D = \mathfrak{m}_D^e$. Finally, if π_D and π_K are uniformizers (i.e. elements of maximal valuation) in $\mathcal{O}_D$ and $\mathcal{O}_K$, respectively, then (13) implies that

$$|\pi_D|_D = |\mathrm{N}_{K(\pi_D)/K}(\pi_D)|_K^{\dim_{K(\pi_D)}(D)} = |\pi_K^{\frac{[K(\pi_D):K]}{e(K(\pi_D)/K)}}|_K^{\dim_{K(\pi_D)}(D)} = |\pi_K|_D^{\frac{1}{e(K(\pi_D)/K)}},$$

where $e(K(\pi_D)/K)$ is the ramification index of the field extension $K(\pi_D)/K$. Hence $e = e(K(\pi_D)/K) \leq [K(\pi_D) : K] \leq n$. Now $ef = \mathrm{rank}_{\mathcal{O}_K}\mathcal{O}_D = n^2$; one can check that the proof for the commutative case given in, say, Proposition II.6.8 of Neukirch's *Algebraic Number Theory* transfers verbatim to our case. However, we have already shown that $f \leq n$ and $e \leq n$. Hence $e = f = n$. In particular, we must have $[K(\delta) : K] = f = [\Delta : k]$, so that $K(\delta)/K$ is an unramified extension. However, $[K(\delta) : K] = n$, so $D \otimes_K K(\delta)$ splits over $K(\delta)$ by Corollaries 6.22 and 6.23. □

For the rest of this section, K will always denote a local field. Let k denote the residue field $\mathcal{O}_K/\mathfrak{m}_K$, and let K^{nr} denoted the maximal unramified extension of K; this is the compositum of all the unramified extensions of K inside some fixed algebraic closure. Recall that K^{nr}/K is an infinite Galois extension, and that $\mathrm{Gal}(K^{\mathrm{nr}}/K)$ is isomorphic in a standard way to the absolute Galois group $G_k = \mathrm{Gal}(\overline{k}/k)$ of the residue field k. The kernel of the projection $G_K \twoheadrightarrow G_k$ is $I_K = \mathrm{Gal}(\overline{K}/K^{\mathrm{nr}})$, which is called the *inertia subgroup* of G_K. Furthermore, G_k has a dense cyclic subgroup generated by the *arithmetic Frobenius* element $\mathrm{Frob}_K : \overline{k} \to \overline{k}$, where $\mathrm{Frob}_K(x) = x^q$ for all $x \in \overline{k}$ and q is the cardinality of k. Be aware that some books use the notation Frob_K for the geometric Frobenius, which is the inverse of the arithmetic Frobenius.

Corollary 7.6. *Let $K/\mathbb{Q}_p$ be a finite extension. Then $\mathrm{Br}(K) = \mathrm{Br}(K^{\mathrm{nr}}/K) \simeq H^2(G_k, (K^{\mathrm{nr}})^\times)$.*

Proof. The first equality is the content of Proposition 7.5. The isomorphism of $\mathrm{Br}(K^{\mathrm{nr}}/K)$ with $H^2(G_k, (K^{\mathrm{nr}})^\times)$ is proved in exactly the same way as Corollary 6.26. □

The next result often provides a handy way to verify that cohomology groups of a G-module M vanish, if we can find a filtration of M with tractable graded pieces. In principle we could have proved it much earlier, but we had no need for it until now.

Lemma 7.7. *Let G be a finite group and let M be a G-module. Suppose that we have a descending filtration of M by submodules:*

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$$

such that $\bigcap_{j=0}^{\infty} M_j = 0$. Suppose that M is complete for the topology defined by the filtration $\{M_j\}$ (cosets of the submodules M_j form a base of open sets). Let $i \geq 0$ and suppose that $H^i(G, M_j/M_{j+1}) = 0$ for all $j \geq 0$. Then $H^i(G, M) = 0$.

Proof. Let $\psi_0 : G^i \to M$ be an i-cocycle; our aim is to show that it is also a coboundary. Composing ψ_0 with the natural projection $M = M_0 \twoheadrightarrow M_0/M_1$, we get a cocycle $\overline{\psi_0} \in Z^i(G, M_0/M_1)$. Since $H^i(G, M_0/M_1) = 0$ by assumption, there exists a cochain $\overline{\varphi_1} \in C^{i-1}(G, M_0/M_1)$ such that $d_{i-1}(\overline{\varphi_1}) = \overline{\psi_0}$. Let $\varphi_1 : G^{i-1} \to M_0$ be any lift of $\overline{\varphi_1}$; since G is finite, φ_1 is automatically continuous and thus a cochain. Now define $\psi_1 = \psi_0 - d_{i-1}(\varphi_1)$. Then $d_i\psi_1 = 0$ and the image of ψ_1 lies in M_1, so $\psi_1 \in Z^i(G, M_1)$.

We can now play the same game with ψ_1. Continuing this process, we obtain a sequence of cocycles $\psi_j \in Z^i(G, M_j)$ and of cochains $\varphi_j \in Z^i(G, M_{j-1})$ such that for each $j \geq 0$ we have $\psi_j = d_{i-1}(\varphi_{j+1}) + \psi_{j+1}$. Now define $\varphi = \sum_{j=1}^{\infty} \varphi_j$. By our assumptions on M the series converges. The difference $\psi_0 - d_{i-1}\varphi$ is congruent to zero modulo every M_j, hence it is zero. We have proved that ψ_0 is a coboundary, as claimed. □

Our next goal is to obtain an explicit description of the Brauer group $\mathrm{Br}(K)$ of a finite extension $K/\mathbb{Q}_p$. To this end, we prepare some final tools. First of all, let $v_K : K^\times \twoheadrightarrow \mathbb{Z}$ be

the normalized additive valuation of K: if $x \in K^\times$ and $m \in \mathbb{Z}$, then $v_K(x) = m$ if and only if $x = u\pi_K^m$, where $u \in \mathcal{O}_K^\times$. We continue to denote the unique extension of v_K to $(K^{\mathrm{nr}})^\times$ by v_K. Since π_K is still a uniformizer in K^{nr}, we have that $v_K((K^{\mathrm{nr}})^\times) = \mathbb{Z}$. Since the action of G_k on (K^{nr}) preserves the ideals of $\mathcal{O}_{K^{\mathrm{nr}}}$ and hence preserves valuation, we see that $v_K : (K^{\mathrm{nr}})^\times \to \mathbb{Z}$ is a G_k-module map, where G_k acts on $\mathbb{Z}$ trivially. This induces a natural map on cohomology:

$$v_K : \mathrm{Br}(K) \simeq H^2(G_k, (K^{\mathrm{nr}})^\times) \to H^2(G_k, \mathbb{Z}).$$

Next consider the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ of abelian groups. We view all three groups as G_k-modules, with a trivial action of G_k. Looking at the following bit of the long exact cohomology sequence:

$$\cdots \to H^1(G_k, \mathbb{Q}) \to H^1(G_k, \mathbb{Q}/\mathbb{Z}) \to H^2(G_k, \mathbb{Z}) \to H^2(G_k, \mathbb{Q}) \to \cdots$$

and observing that the leftmost and rightmost groups vanish by Corollary 4.13, we conclude that $H^2(G_k, \mathbb{Z}) \simeq H^1(G_k, \mathbb{Q}/\mathbb{Z})$.

Finally we have a map $\gamma : H^1(G_k, \mathbb{Q}/\mathbb{Z})$ given by $\gamma([\psi]) = \psi(\mathrm{Frob}_K)$ for every $\psi \in Z^1(G_k, \mathbb{Q}/\mathbb{Z})$. This is well-defined: since G_k acts trivially on $\mathbb{Q}/\mathbb{Z}$, there are no non-zero 1-coboundaries, and so every cohomology class in $H^1(G_k, \mathbb{Q}/\mathbb{Z})$ contains only one 1-cocycle. Furthermore, again because G_k acts trivially on $\mathbb{Q}/\mathbb{Z}$, the 1-cocycles are nothing more than continuous group homomorphisms $G_k \to \mathbb{Q}/\mathbb{Z}$. This implies that γ is an isomorphism. Indeed, γ is injective because a *continuous* homomorphism $G_k \to \mathbb{Q}/\mathbb{Z}$ is determined by its restriction to the dense cyclic subgroup $\langle \mathrm{Frob}_K \rangle$. It is surjective because any homomorphism $\langle \mathrm{Frob}_K \rangle \to \mathbb{Q}/\mathbb{Z}$ can be extended to G_k by the injectivity of $\mathbb{Q}/\mathbb{Z}$. (Why can we always extend to a *continuous* homomorphism?)

Putting all this together, we define the *Hasse invariant* $\mathrm{Inv}_K : \mathrm{Br}(K) \to \mathbb{Q}/\mathbb{Z}$ as the composition

$$\mathrm{Br}(K) \simeq H^2(G_k, (K^{\mathrm{nr}})^\times) \xrightarrow{v_K} H^2(G_k, \mathbb{Z}) \simeq H^1(G_k, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\gamma} \mathbb{Q}/\mathbb{Z}.$$

Remark 7.8. To visualize the Hasse invariant map, observe that any non-trivial element of $\mathbb{Q}/\mathbb{Z}$ contains a representative $\frac{a}{b} \in \mathbb{Q}$ such that $0 < a < b$ and $(a, b) = 1$. Let K_b/K be the unique unramified extension of K of degree b.

Proposition 7.9. *Let $K/\mathbb{Q}_p$ be a finite extension. The map $\mathrm{Inv}_K : \mathrm{Br}(K) \to \mathbb{Q}/\mathbb{Z}$ is an isomorphism of abelian groups.*

Proof. We defined Inv_K as a composition of four maps, and we already know that three of them are isomorphisms. So it suffices to show that $v_K : H^2(G_k, (K^{\mathrm{nr}})^\times) \to H^2(G_k, \mathbb{Z})$ is an isomorphism. The short exact sequence $0 \to \mathcal{O}_{K^{\mathrm{nr}}}^\times \to (K^{\mathrm{nr}})^\times \xrightarrow{v_K} \mathbb{Z} \to 0$ of G_k-modules produces the fragment

$$\cdots H^2(G_k, \mathcal{O}_{K^{\mathrm{nr}}}^\times) \to H^2(G_k, (K^{\mathrm{nr}})^\times) \xrightarrow{v_K} H^2(G_k, \mathbb{Z}) \to H^3(G_k, \mathcal{O}_{K^{\mathrm{nr}}}^\times) \to \cdots$$

of the long exact cohomology sequence, and thus it suffices to prove that $H^i(G_k, \mathcal{O}_{K^{\mathrm{nr}}}^\times) = 0$ for $i \in \{2, 3\}$. Proposition 4.10 tells us that

$$H^i(G_k, \mathcal{O}_{K^{\mathrm{nr}}}^\times) = \varinjlim H^i(\mathrm{Gal}(\ell/k), \mathcal{O}_L^\times),$$

where L/K runs over all finite unramified Galois extensions and ℓ is the residue field of L; since L/K is unramified, note that $\mathrm{Gal}(\ell/k) \simeq \mathrm{Gal}(L/K)$. Therefore it suffices to show that $H^i(\mathrm{Gal}(\ell/k), \mathcal{O}_L^\times) = 0$ for any finite unramified Galois L/K and $i \in \{2, 3\}$.

So let L/K be a finite unramified extension and consider the following filtration of $\mathcal{O}_L^\times$ by $\mathrm{Gal}(\ell/k)$-submodules:

$$\mathcal{O}_L^\times \supset 1+\mathfrak{m}_L \supset 1+\mathfrak{m}_L^2 \supset 1+\mathfrak{m}_L^3 \supset \cdots$$

where $\mathfrak{m}_L$ is the maximal ideal of $\mathcal{O}_L$. Since $\mathcal{O}_L^\times/(1+\mathfrak{m}_L) \simeq \ell^\times$ and $(1+\mathfrak{m}_L^j)/(1+\mathfrak{m}_L^{j+1}) \simeq \ell$ for all $j \geq 1$, by Lemma 7.7 it suffices to show that $H^i(\mathrm{Gal}(\ell/k), \ell) = 0$ and $H^i(\mathrm{Gal}(\ell/k), \ell^\times) = 0$ for $i \in \{2, 3\}$.

We know that $H^i(\mathrm{Gal}(\ell/k), \ell) = 0$ for all $i \geq 1$ by Corollary 3.13, and furthermore we know that $H^2(\mathrm{Gal}(\ell/k), \ell^\times) = \mathrm{Br}(\ell/k) = 0$, where the first equality is Theorem 6.25 and the second is immediate from Proposition 7.3. It remains only to prove that $H^3(\mathrm{Gal}(\ell/k), \ell^\times) = 0$. This will be established, in two different ways, in the following sections. □

Remark 7.10. It is possible to give far shorter and more direct proofs of the isomorphism $\mathrm{Br}(K) \simeq \mathbb{Q}/\mathbb{Z}$, but we have chosen one that allows us to practice cohomological techniques that are useful in a wide variety of situations.

7.3. **Tate cohomology.** Let G be a *finite* group and let M be a G-module. In this case, M is equipped with a norm map $\mathrm{N} : M \to M$ given by

$$\mathrm{N}(m) = \sum_{g \in G} gm \tag{14}$$

for all $m \in M$. We now define a slightly modified version of cohomology as follows:

Definition 7.11. Let G be a finite group and M a G-module. Then we set

$$\hat{H}^i(G, M) = \begin{cases} M^G/N(M) & : i = 0 \\ H^i(G, M) & : i > 0. \end{cases}$$

Proposition 7.12. *Let G be a finite group, and let $0 \to A \to B \to C \to 0$ be a short exact sequence of G modules. There is a long exact sequence*

$$\hat{H}^0(G, A) \to \hat{H}^0(G, B) \to \hat{H}^0(G, C) \xrightarrow{\delta^0} \hat{H}^1(G, A) \to \hat{H}^1(G, B) \to \hat{H}^1(G, C) \to \hat{H}^2(G, A) \to \cdots$$

where, if $c \in C^G$ belongs to the class $[c] \in \hat{H}^0(G, C)$, then $\delta^0([c]) = [g \mapsto g\tilde{c} - \tilde{c}]$, where $\tilde{c}$ is an arbitrary lift of c to B.

Proof. We only have to check exactness at the first four groups, since from then onwards this is the usual long exact cohomology sequence. The checking is straightforward. □

Remark 7.13. Tate also defined $\hat{H}^i(G, M)$ for $i < 0$. These groups see the homology of the G-module M and allow the exact sequence of the previous proposition to be continued infinitely to the left, accounting for the failure of the map $\hat{H}^0(G, A) \to \hat{H}^0(G, B)$ to be injective. Since we have not introduced homology in this course, we do not consider Tate cohomology in negative degree – but see the exercises!

Now let G be a finite cyclic group of order n, and let $\sigma \in G$ be a generator. Let M be a G-module and define a map $D : M \to M$ by $\mathrm{D}(m) = \sigma(m) - m$ for all $m \in M$. We define a complex $\mathcal{K}(M)$ as follows:

$$\cdots \xrightarrow{\partial^{-2}} M^{-1} \xrightarrow{\partial^{-1}} M^0 \xrightarrow{\partial^0} M^1 \xrightarrow{\partial^1} M^2 \xrightarrow{\partial^2} \cdots$$

where M^i is a copy of M for each $i \in \mathbb{Z}$, and the maps $\partial^i : M^i \to M^{i+1}$ are given by

$$\partial^i(m) = \begin{cases} \mathrm{D}(m) & : m \text{ even} \\ \mathrm{N}(m) & : m \text{ odd.} \end{cases}$$

for all $m \in M$. Note that this is indeed a complex, since for any $m \in M$ we have

$$\mathrm{N}(\mathrm{D}(m)) = \mathrm{D}(\mathrm{N}(m)) = (\sigma - 1)(1 + \sigma + \sigma^2 + \cdots + \sigma^{n-1})m = (\sigma^n - 1)m = 0.$$

Here we implicitly treat M as a module over the group ring $\mathbb{Z}[G]$. Let $H^i(\mathcal{K}(M)) = (\ker \partial^i)/(\operatorname{im} \partial^{i-1})$ denote the cohomology of this complex. It is manifestly clear that the abelian groups $H^i(\mathcal{K}(M))$ depend only on the parity of i.

Proposition 7.14. *Let G be a finite cyclic group and M a G-module. Given a generator $\sigma \in G$, let $\mathcal{K}(M)$ be the complex defined as above. Then $H^i(\mathcal{K}(M)) \simeq \hat{H}^i(G, M)$ for all $i \geq 0$. In particular, $H^i(G, M) \simeq H^{i+2}(G, M)$ for all $i \geq 1$.*

Proof. Since G is cyclic, we have $M^G = \{m \in M : \sigma(m) = m\}$. It follows immediately that $H^0(\mathcal{K}(M)) = M^G/N(M) = \hat{H}^0(G, M)$. Similarly, we have already observed in Example 3.17 that there is an isomorphism

$$\begin{array}{rcl} Z^1(G, M) & \xrightarrow{\sim} & \ker \mathrm{N} \\ \psi & \mapsto & \psi(\sigma). \end{array}$$

Under this isomorphism, $B^1(G, M)$ corresponds to the set of elements of M of the form $\sigma(m) - m$, namely to $\operatorname{im} \mathrm{D}$. Thus $\hat{H}^1(G, M) = H^1(G, M) \simeq H^1(\mathcal{K}(M))$.

It is easy to see that $H^i(\mathcal{K}(-)) : \mathrm{Mod}_G \to \mathrm{Ab}$ is a functor for every $i \geq 0$. Moreover, if $0 \to M \to N \to P \to 0$ is a short exact sequence of G-modules, then we clearly get a short exact sequence of complexes $0 \to \mathcal{K}(M) \to \mathcal{K}(N) \to \mathcal{K}(P)$, and by the Zigzag Lemma this produces a long exact cohomology sequence. Thus $\{H^i(\mathcal{K}(-))\}$ is a δ-functor. We claim that it is universal; this implies our proposition.

As usual, we prove universality by means of Lemma 3.9, so we need to show that the functors $H^i(\mathcal{K}(-))$ are effaceable for all $i \geq 1$. In the course of proving Corollary 2.12, we showed that any G-module M embeds in a module of the form $\mathrm{Ind}_{\{e\}}^G I$, where I is an injective abelian group. Let $f \in (\mathrm{Ind}_{\{e\}}^G I)^G$. This means that f is a constant function sending every $g \in G$ to some fixed element $\iota \in I$. Define $h_\iota \in \mathrm{Ind}_{\{e\}}^G I$ by

$$h_\iota(\sigma) = \begin{cases} \iota & : \sigma = e \\ 0 & : \sigma \neq e. \end{cases}$$

Then it is clear that $\mathrm{N}(f_\iota) = f$. Hence $\hat{H}^0(G, \mathrm{Ind}_{\{e\}}^G I) = H^0(\mathcal{K}(\mathrm{Ind}_{\{e\}}^G I)) = 0$, which in turn implies that $H^i(\mathcal{K}(\mathrm{Ind}_{\{e\}}^G I)) = 0$ for any even i.

Similarly, suppose that $f \in \mathrm{Ind}_{\{e\}}^G I$ lies in the kernel of N. This means that $\sum_{\tau \in G} f(\tau) = 0$. Thus the element $h \in \mathrm{Ind}_{\{e\}}^G I$ given by $h(\sigma^j) = \sigma_{k=0}^{j-1} f(\sigma^k)$ for all $j \geq 0$ is well-defined. One checks that $f(\tau) = h(\tau\sigma) - h(\tau)$ for all $\tau \in G$ and hence that $f \in \operatorname{im} \mathrm{D}$. It follows that $H^i(\mathcal{K}(\mathrm{Ind}_{\{e\}}^G I)) = 0$ for all odd i, and we have obtained the required effaceability. □

Corollary 7.15. *Let ℓ/k be an extension of finite fields. Then $H^i(\mathrm{Gal}(\ell/k), \ell^\times) = 0$ for any odd $i \geq 1$.*

Proof. If $i \geq 1$ is odd, then $H^i(\mathrm{Gal}(\ell/k), \ell^\times) \simeq H^1(\mathrm{Gal}(\ell/k), \ell^\times)$ by Proposition 7.14. Now $H^1(\mathrm{Gal}(\ell/k), \ell^\times) = 0$ by Hilbert 90. □

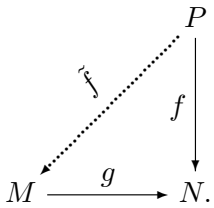
In particular, we have finally established that $H^3(\mathrm{Gal}(\ell/k), \ell^\times) = 0$, which was the last remaining missing piece in our proof that $\mathrm{Inv}_K : \mathrm{Br}(K) \to \mathbb{Q}/\mathbb{Z}$ is an isomorphism for any finite extension $K/\mathbb{Q}_p$.

Exercises

(1) Let ℓ/k be an extension of finite fields. Show that the norm map $\mathrm{N}_{\ell/k} : \ell \to k$ is surjective and use this to obtain another proof of Proposition 7.3.

(2) Let G be a profinite group, and let M be a G-module. Let $M' \subseteq M$ be the submodule generated by all elements of the form $gm - m$, for $g \in G$ and $m \in M$. Define the *G-coinvariants* of M to be $M_G = M/M'$. Prove that the functor $M \mapsto M_G$ is a right exact functor $\mathrm{Mod}_G \to \mathrm{Ab}$.

(3) A G-module P is called *projective* if, given a surjection $g : M \twoheadrightarrow N$ of G-modules and a map $f : P \to N$, there exists $\tilde{f} : P \to M$ completing the triangle:



This is dual to the notion of an injective G-module from Definition 2.9. Prove that every G-module M has a projective resolution, namely an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

(4) Let M be a G-module. Choose a projective resolution as above and apply the G-coinvariants functor to get a complex

$$\cdots \to (P_2)_G \to (P_1)_G \to (P_0)_G.$$

The homology of this complex is called the homology of M and denoted $H_i(G, M)$. Prove that it is independent of the choice of projective resolution.

(5) Let G be a finite group and let M be a G-module. Observe that the map $N : M \to M$ of (14) above vanishes on M' and thus defines a map $N : M_G \to M^G$. Define Tate cohomology in negative degrees by

$$\hat{H}^i(G, M) = \begin{cases} \ker N \subseteq M_G & : i = -1 \\ H_{-(i+1)}(G, M) & : i \leq -2. \end{cases}$$

Given an exact sequence $0 \to M \to N \to C$ of G-modules, show that there is long exact sequence that extends infinitely in both directions:

$$\cdots \to \hat{H}^{-2}(G, C) \to \hat{H}^{-1}(G, M) \to \hat{H}^{-1}(G, N) \to \hat{H}^{-1}(G, C) \to \hat{H}^0(G, M) \to$$
$$\hat{H}^0(G, N) \to \hat{H}^0(G, C) \to \hat{H}^1(G, M) \to \cdots$$

(6) If G is finite cyclic and M is a G-module, prove that $\hat{H}^i(G, M) \simeq \hat{H}^{i+2}(G, M)$ for any $i \in \mathbb{Z}$.

8. Cohomological dimension

Our next big aim in this course is to understand the cohomology of local fields, namely to obtain an explicit description of $H^i(G_K, M)$, where $K/\mathbb{Q}_p$ is a finite extension and G_K, as usual, is the absolute Galois group of K. It turns out that essentially nothing of interest happens when $i > 2$. In this section we will prove this result and fit it into the more general framework of the theory of cohomological dimension.

8.1. **Basic properties.** If G is any abelian group and $n \geq 1$, then we denote $G[n] = \{g \in G : ng = 0\}$. Of course, $G[n]$ is a subgroup of G. The following fact is basic but essential.

Lemma 8.1. *Let G be a profinite group and M a torsion G-module. Then $H^i(G, M)$ is a torsion group for any $i \geq 0$.*

Proof. This is obvious for $H^0(G, M) = M^G$, so we assume $i \geq 1$. Clearly it suffices to show that $Z^i(G, M)$ is a torsion group. Since G^i is compact, any continuous map $\varphi : G^i \to M$ has finite image. Let $\{m_1, \ldots, m_r\}$ be the image of φ. Since M is torsion, there exist integers $n_1, \ldots, n_r$ such that $n_i m_i = 0$ for all $1 \leq i \leq r$. Let $n = \operatorname{lcm}(n_1, \ldots, n_r)$. Then $n\varphi = 0$. □

Definition 8.2. Let G be a profinite group and p a prime. The *cohomological dimension of G at p*, denoted $\operatorname{cd}_p(G)$, is defined to be the largest i such that there exists a torsion G-module M satisfying $H^i(G, M)[p] \neq 0$. We say that $\operatorname{cd}_p(G) = \infty$ if the set of such i is unbounded.

The *cohomological dimension of G* is then defined to be

$$\operatorname{cd}(G) = \sup_p \operatorname{cd}_p(G).$$

Since $H^i(G, M)$ is a torsion group whenever M is torsion, by Lemma 8.1, we observe that $\operatorname{cd}(G)$ is, equivalently, the largest degree i for which there exists a torsion G-module M satisfying $H^i(G, M) \neq 0$.

Example 8.3. It is immediate from Lemma 3.12 that $\operatorname{cd}(\{e\}) = 0$.

Definition 8.4. Let $K/\mathbb{Q}_p$ be a finite extension. For any $n \geq 1$, we set $\mu_n \subset \overline{K}^\times$ to be the subgroup of n-th roots of unity. This is naturally a discrete G-module, where G is any closed subgroup of G_K.

Example 8.5. Let $K/\mathbb{Q}_p$ be a finite extension. Then $\operatorname{cd}_\ell(G_K) \geq 2$ for all primes ℓ. Indeed, for any $n \geq 1$ we may consider the short exact sequence

$$0 \to \mu_n \to \overline{K}^\times \overset{x \mapsto x^n}{\to} \overline{K}^\times \to 0 \tag{15}$$

of G_K-modules. This gives rise to the following short exact sequence of cohomology groups:

$$H^1(G_K, \overline{K}^\times) \to H^2(G_K, \mu_n) \to H^2(G_K, \overline{K}^\times) \to H^2(G_K, \overline{K}^\times).$$

The leftmost group is trivial by Hilbert 90. Thus $H^2(G_K, \mu_n)$ is the kernel of $H^2(G_K, \overline{K}^\times) \to H^2(G_K, \overline{K}^\times)$. This map is just multiplication by n. Hence it follows from Corollary 6.26 and Proposition 7.9 that $H^2(G_K, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$. Clearly μ_n is a torsion module, and if $\ell | n$ then $H^2(G_K, \mu_n)[\ell] \neq 0$.

The reader may well wonder whether the cohomological dimension of a profinite group G tells us anything at all about the cohomology of non-torsion modules. It turns out that it tells us quite a lot.

Definition 8.6. Let G be a profinite group and p a prime. The *strict cohomological dimension of G at p*, denoted $\mathrm{scd}_p(G)$, is defined to be the largest i such that there exists a G-module M satisfying $H^i(G,M)[p] \neq 0$. As usual, we say that $\mathrm{cd}_p(G) = \infty$ if the set of such i is unbounded. Similarly, we define $\mathrm{scd}(G) = \sup_p \mathrm{scd}_p(G)$.

Proposition 8.7. *Let G be a profinite group. Then $\mathrm{cd}_p(G) \leq \mathrm{scd}_p(G) \leq \mathrm{cd}_p(G)+1$ for any prime p.*

Proof. The first inequality is obvious, so we only prove the second. Moreover, there is nothing to prove if $\mathrm{cd}_p(G) = \infty$, so we assume that $n = \mathrm{cd}_p(G)$ is finite. We need to show that $H^q(G,M)[p] = 0$ for all G-modules M whenever $q > n+1$. Consider the short exact sequence $0 \to M[p] \to M \xrightarrow{\pi} pM \to 0$ arising from multiplication by p. Since $M[p]$ is a torsion module, we have $H^q(G,M[p]) = H^q(G,M[p])[p] = 0$, and thus $H^q(G,M) \xrightarrow{\pi} H^q(G,pM)$ is injective.

On the other hand, consider the short exact sequence $0 \to pM \xrightarrow{\varepsilon} M \to M/pM \to 0$. Again M/pM is a p-torsion module, hence $H^{q-1}(G,M/pM) = H^{q-1}(G,M/pM)[p] = 0$, so that $H^q(G,pM) \xrightarrow{\varepsilon} H^q(G,M)$ is injective. Clearly the composition $\varepsilon \circ \pi : M \to M$ is just multiplication by p, and the same is true of the induced maps on cohomology. But if multiplication by p is injective on $H^q(G,M)$, then $H^q(G,M)[p] = 0$. □

Proposition 8.8. *Let G be a profinite group, and let $H \subseteq G$ be a closed normal subgroup. Then $\mathrm{cd}(G) \leq \mathrm{cd}(H) + \mathrm{cd}(G/H)$.*

Proof. Let M be a torsion G-module and let $i > \mathrm{cd}(H) + \mathrm{cd}(G/H)$. If $p, q \in \mathbb{N}$ are integers such that $p+q = i$, then either $p > \mathrm{cd}(H)$, in which case $H^p(H, \mathrm{Res}^G_H M) = 0$ and hence

$$H^q(G/H, H^p(H, \mathrm{Res}^G_H M)) = 0, \tag{16}$$

or else $q > \mathrm{cd}(G/H)$, in which case (16) still holds because $H^p(H, \mathrm{Res}^G_H M)$ is a torsion module by Lemma 8.1. Thus we have shown that the Hochschild-Serre spectral sequence of Definition 5.22 associated to the triple (G,H,M) satisfies $E_2^{p,q} = 0$ for all p,q such that $p+q = i$. Since $E_{r+1}^{p,q}$ is a subquotient of $E_r^{p,q}$ for any $r \geq 2$, it follows that $E_\infty^{p,q} = 0$ for all p,q such that $p,q = i$. By the abutment of the Hochschild-Serre spectral sequence, we have $F^p H^i(G,M) = 0$ for all $0 \leq p \leq i$, and hence $H^i(G,M) = 0$. □

8.2. **The cohomological dimension of inertia.** Let $K/\mathbb{Q}_p$ be a finite extension. We can now state precisely the claim to which we alluded at the beginning of this chapter. We would like to prove that $\mathrm{cd}(G_K) = 2$. We already know, by Example 8.5, that $\mathrm{cd}(G_K) \geq 2$. By Proposition 8.8 it suffices to show that $\mathrm{cd}(I_K) = \mathrm{cd}(G_k) = 1$. We prove these equalities separately. The beginning of this section introduces notions and lemmas that could have been proved much earlier in the course, but we did not then have need for them.

Definition 8.9. Let p be a prime. A *pro-p group* is a profinite group whose finite quotients are all p-groups.

It follows from the proof of Theorem 1.7 that a profinite group G is pro-p if and only if $G \simeq \varprojlim_I G_i$, where the G_i are all finite p-groups. The following fun fact is foundational in mod p representation theory and accounts for much of its difference from representation theory over fields of characteristic zero.

Lemma 8.10. *Let G be a pro-p group and let M be a G-module. Suppose that M has the structure of an $\mathbb{F}_p$-vector space, or that M is finite and that its cardinality is a power of p. Then $M^G \neq \{0\}$.*

Proof. If M is an $\mathbb{F}_p$-vector space and $m \in M$ is a non-zero element, then, since $\mathrm{stab}_G(m)$ has finite index in G, the G-orbit of m spans a finite-dimensional subspace. Thus we may assume without loss of generality that M is finite of p-power cardinality. In this case, $U = \bigcap_{m\in M} \mathrm{stab}_G(m)$ is an open subgroup of G, hence of finite index. Thus the action of G on M factors through the quotient G/H, where $H \subseteq U$ is an open normal subgroup, and we may assume without loss of generality that G is a finite p-group.

The finite group M decomposes into a disjoint union of G-orbits, whose cardinalities all divide $|G|$ and so are powers of p. Since $|M|$ is divisible by p, it follows that $|M^G|$, which is just the number of G-orbits of cardinality 1, must be divisible by p. In particular, $|M^G| > 1$. □

Corollary 8.11. *Let G be a profinite group, let $H \trianglelefteq G$ be a pro-p normal subgroup, and let M be a G-module which has either an $\mathbb{F}_p$-vector space structure or finite p-power cardinality. Suppose that M is a simple G-module, in the sense that it has no proper non-trivial G-submodules. Then H acts trivially on M.*

Proof. Observe that M^H is a G-submodule of M. Indeed, for any $g \in G$, $h \in H$, and $m \in M^H$ we have $hgm = g(g^{-1}hg)m = gm$ and hence $gm \in M^H$. By the previous lemma, $M^H \neq \{0\}$. Therefore $M^H = M$. □

Definition 8.12. Let G be a profinite group and let p be prime. A *pro-p-Sylow subgroup* of G is a maximal closed pro-p subgroup.

Remark 8.13. Suppose that $G \simeq \varprojlim_I G_i$, where $\{G_i\}_{i\in I}$ is a projective system of finite groups. Let $H \subseteq G$ be a pro-p-Sylow subgroup. It is a pleasant exercise to show that there exist p-Sylow subgroups $H_i \subseteq G_i$ for every $i \in I$ that are compatible, in the sense that $\varphi_{ij}(H_i) \subseteq H_j$ whenever $i, j \in I$ satisfy $i \geq j$, and that $H \simeq \varprojlim_I H_i$. It then follows from the usual Sylow theorems for finite groups that any two pro-p-Sylow subgroups of G are conjugate.

Lemma 8.14. *Let G be a profinite group, let p be prime, and let M be a G-module of finite p-power cardinality. Let $H \subseteq G$ be a pro-p-Sylow subgroup. Then for any $i \geq 0$ the restriction map* $\mathrm{res} : H^i(G, M) \to H^i(H, M)$ *is injective.*

Proof. As in the proof of Corollary 4.13 we can reduce to the case where G is finite. Indeed, we saw in that proof that $H^i(G, M) \simeq \varinjlim H^i(G/U_j, M^{U_j})$, where $\{U_j\}$ is the family of open normal subgroups of G. If H_j is the image of H under the natural projection $G \twoheadrightarrow G/U_j$, then $H_j \subseteq G/U_j$ is a p-Sylow subgroup. Moreover, $H \simeq \varprojlim H_j$, and the restriction map $\mathrm{res} : H^i(G, M) \to H^i(H, M)$ arises from the maps $H^i(G/U_j, M^{U_j}) \overset{\mathrm{res}}{\to} H^i(H_j, M^{U_j}) \to H^i(H, M)$.

So let G be a finite group. Then any p-Sylow subgroup $H \subseteq G$ is open. Since $Z^i(G, M)$ is a finite group of exponent $|M|$, it has p-power order, and hence so does $H^i(G, M)$. By Lemma 4.4, the composition $H^i(G, M) \overset{\mathrm{res}}{\to} H^i(H, M) \overset{\mathrm{cor}}{\to} H^i(G, M)$ is multiplication by the index $[G : H]$. This is an isomorphism of $H^i(G, M)$, since $[G : H]$ is prime to p and hence to the order of $H^i(G, M)$. Thus the restriction map is injective, and the corestriction is surjective. □

Lemma 8.15. *Let K be a finite extension of $\mathbb{Q}_p$ and $n \geq 1$. Then $H^2(I_K, \mu_n) = 0$.*

Proof. An immediate consequence of Corollary 7.6 is that $H^2(I_K, \overline{K}^\times) \simeq \mathrm{Br}(K^{\mathrm{nr}}) = 0$. Moreover, $H^1(I_K, \overline{K}^\times) = 0$ by Hilbert 90 and Proposition 4.10. Considering the long exact cohomology sequence arising from the short exact sequence (15), whose terms are now viewed as I_K-modules, we obtain the desired result. □

Corollary 8.16. *Let $K/\mathbb{Q}_p$ be a finite extension, and let M be a torsion I_K-module. Then $H^i(I_K, M) = 0$ for any $i \geq 2$.*

Proof. By Lemma 4.15 it suffices to show that $H^2(I_K, M)$ for any I_K-module M of finite cardinality. Since the abelian group M decomposes into a direct sum of groups of prime power order, since each of these direct summands is clearly preserved by the action of I_K, and since cohomology commutes with direct sums, we may assume without loss of generality that M has prime power order. So let $|M| = \ell^r$, where ℓ is a prime number.

If $H \subset I_K$ is a pro-ℓ-Sylow subgroup, then by Lemma 8.14 it suffices to prove that $H^2(H, M) = 0$. We show this by induction on r. If $r = 1$, then M has prime order and is thus a simple H-module. It follows from Corollary 8.11 that there is only one possible H-action on M, namely the trivial one. Thus $H^2(H, M) = H^2(H, \mu_\ell)$.

Let $F \subset \overline{K}$ be the fixed field of H. If L/K is a finite extension, then $[I_K : I_L] = [\mathrm{Gal}(\overline{K}/K^{\mathrm{nr}}) : \mathrm{Gal}(\overline{K} : L^{\mathrm{nr}})] = [L^{\mathrm{nr}} : K^{\mathrm{nr}}] = e_{L/K}$, where $e_{L/K}$ is the ramification index of L/K. In particular, if $\ell \nmid e_{L/K}$ then we must have $H \subseteq I_L$, since otherwise the image of H under the natural projection $I_K \twoheadrightarrow I_K/I_L$ would be a non-trivial subgroup of ℓ-power order. Thus

$$H \subseteq \bigcap_{L/K \text{ finite}, \ell \nmid e_{L/K}} I_L. \tag{17}$$

On the other hand, suppose that $x \in F$. Then $\ell \nmid e_{K(x)/K}$, hence F is contained in the compositum of the fields L^{nr}, where L runs over all finite extensions of K satisfying $\ell \nmid e_{L/K}$. This implies the inverse inclusion to (17). If we order these fields by inclusion, then the groups I_L naturally from a projective system whose connecting homomorphisms are inclusions. Then $H = \varprojlim I_L$, so that $H^2(H, \mu_\ell) = \varinjlim H^2(I_L, \mu_\ell)$. Hence $H^2(H, \mu_\ell) = 0$ by Lemma 8.15.

Now suppose the claim is known for $|M| \leq \ell^{r-1}$. Let $|M| = \ell^r$. Let $N \subseteq M$ be a simple H-submodule; this exists by the finiteness of M. By Lemma 8.10, the action of H on N is trivial, so that any subgroup of N is an H-submodule. Since any finite ℓ-group has a subgroup of order ℓ, it follows that $|N| = \ell$. Now the short exact sequence $0 \to N \to M \to M/N \to 0$ of H-modules gives rise to the exact sequence

$$H^2(H, N) \to H^2(H, M) \to H^2(H, M/N),$$

where the two outer groups are trivial by induction. Therefore $H^2(H, M) = 0$ and we are done.

Note that this induction would fail if we tried to work with I_K directly rather than with a pro-ℓ-Sylow subgroup, since then there could be simple G-modules of order ℓ^r for $r > 1$. □

8.3. **Cohomological dimension of G_k.**

Definition 8.17. Consider the projective system of groups $G_n = \mathbb{Z}/n\mathbb{Z}$, where $n \geq 1$. We say that $n \geq m$ if $m|n$. In this case, let $\varphi_{nm} : G_n \to G_m$ be the natural projection $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$. The projective limit of this system is denoted $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$.

Proposition 8.18. *If k is any finite field, then* $\mathrm{Gal}(\overline{k}/k) \simeq \hat{\mathbb{Z}}$.

Proof. This follows from the fact that k has a unique extension of any degree $n \geq 1$ inside a fixed algebraic closure $\overline{k}$. Moreover, this extension is Galois with a cyclic Galois group. □

Proposition 8.19. *Let M be a torsion $\hat{\mathbb{Z}}$-module. Then $H^i(\hat{\mathbb{Z}}, M) = 0$ for any $i \geq 2$.*

Proof. By Lemma 4.15 it suffices to show that $H^2(\hat{\mathbb{Z}}, M) = 0$ for any $\hat{\mathbb{Z}}$-module M of finite cardinality. So suppose that M has finite cardinality. For every $m \geq 1$, let $G_m \subset \hat{\mathbb{Z}}$ denote the kernel of the natural surjection $\hat{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}/m\mathbb{Z}$. It follows from the definition of the profinite topology that the family $\{G_m\}$ is a base of open neighborhoods of the identity of $\hat{\mathbb{Z}}$. Hence $M = \bigcup_{m=1}^{\infty} M^{G_m}$ and thus $H^2(\hat{\mathbb{Z}}, M) = \varinjlim H^2(\mathbb{Z}/m\mathbb{Z}, M^{G_m})$ by Proposition 4.10.

The groups $H^2(\mathbb{Z}/m\mathbb{Z}, M^{G_m})$ need not be trivial, but we will show that their injective limit is trivial.

FINISH THE PROOF! □

8.4. **Finite G_K-modules.** We are now in a position to make an observation about the cohomology of G_K-modules that will lay the groundwork for local duality, which will be developed in the next section.

Proposition 8.20. *Let $K/\mathbb{Q}_p$ be a finite extension. Then* $\mathrm{cd}(G_K) = 2$.

Proof. As noted above, Example 8.5 shows that $\mathrm{cd}(G_K) \geq 2$, so it suffices to prove that $\mathrm{cd}(G_K) \leq 2$. Consider the closed normal subgroup $H = I_K \subset G_K$. Since $G/H \simeq G_k$, it suffices by Proposition 8.8 to show that $\mathrm{cd}(I_K) \leq 1$ and $\mathrm{cd}(G_k) \leq 1$. The first of these statements is Corollary 8.16, and the second is Proposition 8.19. □

Corollary 8.21. *Let $K/\mathbb{Q}_p$ be a finite extension, and let M be a G_K-module of finite cardinality. Then $H^i(G_K, M)$ is a finite group for all $i \geq 0$.*

Proof. Observe that $\bigcap_{m \in M} \mathrm{stab}_G(m)$ is a finite intersection of open subgroups of G_K, hence open. Similarly, there exists an open subgroup of G_K that acts trivially on $\mu_{|M|}$. Recall that $\{G_L\}$, where L/K runs over all finite Galois extensions, is a base of open neighborhoods of the identity of G_K. Therefore we can find a finite Galois extension L/K such that G_L is contained in the intersection of these two open subgroups and so acts trivially both on M and on $\mu_{|M|}$, and thus also on μ_n for any $n||M|$. By the structure theorem of abelian groups, we have $M \simeq \mu_{n_1} \oplus \cdots \oplus \mu_{n_r}$ as abelian groups for some integers n_j such that $n_1 n_2 \cdots n_r = |M|$. This is also an isomorphism of G_L-modules, since G_L acts trivially on both sides.

We claim that $H^i(G_L, M) \simeq \oplus_{j=1}^{r} H^i(G_L, \mu_{n_j})$ is finite for all $i \geq 0$. Of course it is enough to showing that $H^i(G_L, \mu_n)$ is finite for all $i \geq 0$ and all $n \geq 1$. If $i \geq 3$, then $H^i(G_L, \mu_n) = 0$ by Proposition 8.20. Note also that $H^0(G_L, \mu_n) = \mu_n^{G_L} = \mu_n$ is finite, and that $H^2(G_L, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$ by Example 8.5 (recall that this used all the machinery of the Brauer group), so in particular $H^2(G_L, \mu_n)$ is finite. Finally, viewing the short exact sequence (15) as a sequence of G_L-modules, we obtain the following bit of the long exact cohomology sequence:

$$H^0(G_L, \overline{K}^\times) \to H^0(G_L, \overline{K}^\times) \to H^1(G_L, \mu_n) \to H^1(G_L, \overline{K}^\times).$$

The rightmost group vanishes by Hilbert 90, and the leftmost map is raising to the power n. Hence $H^1(G_L, \mu_n) \simeq L^\times/(L^\times)^n$. This is finite because the surjection $L^\times \to (L^\times)^n$ given by $x \mapsto x^n$ has the finite kernel $\mu_n \cap L$.

Thus $H^i(G_L, M)$ is indeed finite for all $i \geq 0$. Consider the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^q(\mathrm{Gal}(L/K), H^p(G_L, \mathrm{Res}_{G_L}^{G_K} M)) \Rightarrow H^{p+q}(G_K, M).$$

Observe that each $E_2^{p,q}$ is finite: since $\mathrm{Gal}(L/K)$ and $H^p(G_L, \mathrm{Res}_{G_L}^{G_K} M)$ are both finite, there are only finite many possible q-cocycles. Thus each $E_\infty^{p,q}$ is also finite. Since the filtration of

each $H^i(G_K, M)$ produces only finitely many graded pieces, each of which is isomorphic to $E_\infty^{i-q,q}$ for some $0 \leq q \leq i$, we see that $H^i(G_K, M)$ is finite for any $i \geq 0$, as claimed. □

One of the statements reached in the course of the proof of Corollary 8.21 is important in its own right; it encapsulates the results of Kummer theory. We state it as a proposition of its own.

Proposition 8.22. *Let $n \geq 1$, and let K be a field whose characteristic is prime to n. Then $H^1(G_K, \mu_n) \simeq K^\times/(K^\times)^n$.*

Proof. This was established in the proof of Corollary 8.21, where K was called L. While the setup there assumes that K is a p-adic field and that G_K acts trivially on μ_n, i.e. that $\mu_n \subset K^\times$, none of this figures in the proof of our statement. We only require that $K(\mu_n)$ be separable over K, in order to obtain a G_K-action on μ_n, and this is ensured by our condition on the characteristic of K. The diligent reader will notice that this statement appeared as an exercise a few chapters ago. □

Remark 8.23. While Proposition 8.22 is a very general statement, the proof of Corollary 8.21 made use of all of our work on the Brauer group of a finite extension $K/\mathbb{Q}_p$ and is thus specific to the p-adic setting. Indeed, Corollary 8.21 may fail for other fields. For instance, let $K = \mathbb{Q}$, and let $M = \mathbb{Z}/2\mathbb{Z}$, where the $G_\mathbb{Q}$-action on M is trivial. Then the non-trivial elements of $H^1(G_K, M) = \mathrm{Hom}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}/2\mathbb{Z})$ correspond to quadratic extensions of $\mathbb{Q}$, and there are infinitely many of these.

Exercises

(1) Let G be a profinite group. Prove that $\mathrm{cd}(G) = 0$ if and only if $G = \{e\}$.

9. The cup product and local duality

One can define a cup product in the context of group cohomology, analogous to that appearing in algebraic topology. This will turn out to be a very useful tool.

9.1. **The cup product.** Let G be a profinite group, and let M and N be two G-modules. Viewing their underlying abelian groups as $\mathbb{Z}$-modules, we can form the tensor product $M \otimes_\mathbb{Z} N$; hereafter we will omit $\mathbb{Z}$ from the notation. Then $M \otimes N$ carries a G-action determined by $g(m \otimes n) = gm \otimes gn$ for any $g \in G$, $m \in M$, and $n \in N$. It is easy to see that the action is discrete, so that $M \otimes N$ is a G-module. There is a natural map $M^G \times N^G \to (M \otimes N)^G$ given by $(m, n) \mapsto m \otimes n$. The *cup product* will generalize this to a map

$$\begin{aligned} H^i(G, M) \times H^j(G, N) &\to H^{i+j}(G, M \otimes N) \\ (\varphi, \psi) &\mapsto \varphi \cup \psi. \end{aligned}$$

Recall the spaces $\mathcal{C}^i(G, M)$ of Definition 3.6. These are spaces of continuous functions $G^{i+1} \to M$, with a G-action as defined there. Consider the map

$$\mathcal{C}^i(G, M) \times \mathcal{C}^j(G, N) \xrightarrow{\cup} \mathcal{C}^{i+j}(G, M \otimes N) \tag{18}$$

sending the pair (φ, ψ) to the function $\varphi \cup \psi \in \mathcal{C}^{i+j}(G, M \otimes N)$ given by

$$(\varphi \cup \psi)(g_0, \ldots, g_{i+j}) = \varphi(g_0, \ldots, g_i) \otimes \psi(g_i, \ldots, g_{i+j}).$$

Our first task is to check that the map of (18) does induce induce a map on cohomology. One verifies immediately that this map is G-equivariant, where G acts diagonally on the

left-hand side. Thus G-invariants are preserved, and we get a map $C^i(G,M)\times C^j(G,N)\stackrel{\cup}{\to} C^{i+j}(G,M\otimes N)$ of inhomogeneous cochains by the correspondence (4).

Lemma 9.1. *Let M and N be G-modules, let $i,j\geq 0$, and let $\varphi\in C^i(G,M)$ and $\psi\in C^j(G,N)$ be cochains. Then*

$$d_{i+j}(\varphi\cup\psi)=d_i\varphi\cup\psi+(-1)^i\varphi\cup d_j\psi.$$

Proof. Recall the maps $f_i:\mathcal{C}^i(G,M)\to\mathcal{C}^{i+1}(G,M)$ defined by (2); these correspond to the boundary maps d_i. It is straightforward to check that, for any $\varphi\in\mathcal{C}^i(G,M)$ and $\psi\in\mathcal{C}^j(G,N)$ and any $g_0,\dots,g_{i+j+1}\in G$, we have

$$\begin{aligned}(f_{i+j}(\varphi\cup\psi))(g_0,\dots,g_{i+j+1}) &= (f_i\varphi)(g_0,\dots,g_{i+1})\otimes\psi(g_{i+1},\dots,g_{i+j+1})+\\ &\quad (-1)^i\varphi(g_0,\dots,g_i)\otimes(f_j\psi)(g_i,\dots,g_{i+j+1}).\end{aligned}$$

In other words, $f_{i+j}(\varphi\cup\psi)=f_i\varphi\cup\psi+(-1)^i\varphi\cup f_j\psi$. Since the correspondence (4) is linear, the claim follows immediately. □

Corollary 9.2. *Let M and N be G-modules. For any $i,j\geq 0$, the map of* (18) *induces a cup product map on cohomology:*

$$H^i(G,M)\times H^j(G,N)\stackrel{\cup}{\to}H^{i+j}(G,M\otimes N).$$

Proof. It is immediate from Lemma 9.1 that if $\varphi\in Z^i(G,M)$ and $\psi\in Z^j(G,N)$, then $d_{i+j}(\varphi\cup\psi)=0$, so that $\varphi\cup\psi\in Z^{i+j}(G,M\otimes N)$. Moreover, if $\varphi\in B^i(G,M)$, then there exists $\eta\in C^{i-1}(G,M)$ such that $\varphi=d_{i-1}\eta$. The lemma then implies that

$$\varphi\cup\psi=d_{i-1}\eta\cup\psi=d_{i-1}\eta\cup\psi+(-1)^{i-1}\eta\cup d_j\psi=d_{i+j-1}(\eta\cup\psi)\in B^{i+j}(G,M\otimes N).$$

An analogous observation holds if ψ is a j-coboundary and φ is any i-cocyle. Hence the cup product induces a map on cohomology. □

We now observe several basic properties of the cup product. Most of them can be established by simple manipulation of cocycles.

Proposition 9.3. *Let M and N be G-modules.*

1. *The cup product $H^0(G,M)\times H^0(G,N)\stackrel{\cup}{\to}H^0(G,M\otimes N)$ is just the natural map $M^G\times N^G\to(M\otimes N)^G$ given by $(m,n)\mapsto m\otimes n$.*
2. *If $\varphi\in C^i(G,M)$ and $\psi\in C^j(G,N)$, then the inhomogeneous cochain $\varphi\cup\psi\in C^{i+j}(G,M\otimes N)$ is given explicitly as*
$$(\varphi\cup\psi)(g_1,\dots,g_{i+j})=\varphi(g_1,\dots,g_i)\otimes g_1g_2\cdots g_i\psi(g_{i+1},\dots,g_{i+j}).$$
3. *If $m\in M^G=H^0(G,M)$ and $\psi\in Z^j(G,N)$, then $m\cup[\psi]=[m\otimes\psi]\in H^j(G,M\otimes N)$, where $m\otimes\psi\in Z^j(G,M\otimes N)$ is the cocycle $(g_1,\dots,g_j)\mapsto m\otimes\psi(g_1,\dots,g_j)$.*
4. *If $\varphi\in Z^1(G,M)$ and $\psi\in Z^1(G,N)$, then $[\varphi]\cup[\psi]=[\eta]$, where $\eta\in Z^2(G,M\otimes N)$ is the cocycle $\eta(g_1,g_2)=\varphi(g_1)\otimes g_1\psi(g_2)$.*
5. *The cup product is associative: if P is a third G-module, then for any $\varphi\in H^i(G,M)$, $\psi\in H^j(G,N)$, and $\eta\in H^k(G,P)$ we have $(\varphi\cup\psi)\cup\eta=\varphi\cup(\psi\cup\eta)$ as elements of $H^{i+j+k}(G,M\otimes N\otimes P)$.*
6. *Let $H\subset G$ be a closed subgroup. If $\varphi\in H^i(G,M)$ and $\psi\in H^j(G,N)$, then*
$$\operatorname{res}^G_H\varphi\cup\operatorname{res}^G_H\psi=\operatorname{res}^G_H(\varphi\cup\psi).$$

If, in addition, H is normal and $\varphi \in H^i(G/H, M^H)$ and $\psi \in H^j(G/H, N^H)$, then

$$\inf \varphi \cup \inf \psi = \inf (\varphi \cup \psi).$$

Proof. The first claim is obvious from the definition of the cup product. The second follows from an easy computation using (4). The third and fourth claims are special cases of the second. The fifth and sixth claims follow from the second and the explicit descriptions of the restriction and inflation maps from Lemma 4.8. □

Lemma 9.4. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of G-modules. Observe that the corresponding long exact sequence of cohomology groups produces a map $\delta^i : H^i(G, M'') \to H^{i+1}(G, M')$ for each $i \geq 0$. Let N be a G-module such that the sequence $0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ is also short exact. As above, we obtain maps $\delta^i : H^i(G, M'' \otimes N) \to H^{i+1}(G, M')$. Let $\varphi'' \in H^i(G, M'')$ and $\psi \in H^j(G, N)$ for some $i, j \geq 0$. Then*

$$(\delta^i \varphi'') \cup \psi = \delta^{i+j}(\varphi'' \cup \psi).$$

Proof. This is obtained from a computation on cocycles using Proposition 9.3(2) and the explicit description of the connecting maps in Lemma 4.7. □

Analogously we obtain the following statement.

Lemma 9.5. *Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of G-modules. Let M be a G-module, and suppose that the sequence $0 \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$ is also short exact. Let $\varphi \in H^i(G, M)$ and $\psi'' \in H^j(G, N'')$ for some $i, j \geq 0$. Then*

$$\varphi \cup (\delta^j \psi'') = (-1)^i \delta^{i+j}(\varphi \cup \psi''),$$

where the δ^i are connecting maps arising from the appropriate long exact sequences.

Using the preceding lemmas, we can establish that the cup product is commutative up to sign. This property is called *anti-commutativity*.

Proposition 9.6. *Let M and N be G-modules. If $\varphi \in H^i(G, M)$ and $\psi \in H^j(G, N)$, then $\varphi \cup \psi = (-1)^{ij}(\psi \cup \varphi)$ as elements of $H^{i+j}(G, M \otimes N)$, where we identify $M \otimes N$ with $N \otimes M$ by means of the isomorphism*

$$\begin{array}{rcl} M \otimes N & \xrightarrow{\sim} & N \otimes M \\ m \otimes n & \mapsto & n \otimes m. \end{array}$$

Proof. We will use a dimension-shifting argument to prove the proposition by induction on the pair (i, j). If $(i, j) = (0, 0)$, the claim is obvious. Now suppose it known for the pair $(i - 1, j)$ and arbitrary G-modules. Consider the short exact sequence

$$0 \to M \xrightarrow{\iota} \mathrm{Ind}_{\{e\}}^G \mathrm{Res}_{\{e\}}^G M \to C \to 0, \tag{19}$$

where $\iota(m)$ is the constant function $G \to M$ with image $\{m\}$, for every $m \in M$, and C is the cokernel of ι. We study the associated long exact sequence; since $\mathrm{Ind}_{\{e\}}^G \mathrm{Res}_{\{e\}}^G M$ is acyclic by Lemma 3.12, the connecting maps $\delta^{i-1} : H^{i-1}(G, C) \to H^i(G, M)$ are surjective for all $i \geq 1$. Let φ and ψ be as in the statement of this proposition; then there exists $\gamma \in H^{i-1}(G, C)$ such that $\varphi = \delta^{i-1}(\gamma)$.

Suppose we knew that tensoring with N preserved the exactness of the above sequence. Then we could conclude

$$\begin{aligned}\varphi \cup \psi = \delta^{i-1}(\gamma) \cup \psi = \delta^{i+j-1}(\gamma \cup \psi) &= (-1)^{(i-1)j}\delta^{i+j-1}(\psi \cup \gamma) = \\ &(-1)^{(i-1)j}(-1)^{j}(\psi \cup \delta^{i-1}(\gamma)) = (-1)^{ij}(\psi \cup \varphi).\end{aligned}$$

Here the second equality is Lemma 9.4, the third comes from the inductive hypothesis, and the fourth is Lemma 9.5. Hence we would obtain the claim for (i, j).

Similarly, if the claim were known for $(i, j-1)$, it could be proved for (i, j) by an analogous argument that exchanges the roles of M and N. Since we already know the claim for $(0, 0)$, these two results imply it for arbitrary (i, j). Thus, it remains only to prove that the short exact sequence (19) remains exact after tensoring with an arbitrary G-module N. Since the functor $- \otimes N$ is always right exact (see, for instance, Proposition XVI.2.6 in Lang's *Algebra*, 3rd ed.), it suffices to show that the map $\iota \otimes 1 : M \otimes N \to (\mathrm{Ind}_{\{e\}}^{G}\mathrm{Res}_{\{e\}}^{G} M) \otimes N$ is injective. It is an exercise to check that the following map is an isomorphism:

$$\begin{aligned}(\mathrm{Ind}_{\{e\}}^{G}\mathrm{Res}_{\{e\}}^{G} M) \otimes N &\xrightarrow{\sim} \mathrm{Ind}_{\{e\}}^{G}\mathrm{Res}_{\{e\}}^{G}(M \otimes N) \\ f \otimes n &\mapsto (g \mapsto f(g) \otimes n).\end{aligned}$$

The injectivity of $\iota \otimes 1$ follows straightforwardly. □

Definition 9.7. Let $K/\mathbb{Q}_p$ be a finite extension.

(1) Define $\mu_\infty = \bigcup_{n \geq 1} \mu_n \subset \overline{K}^{\times}$ to be the G_K-submodule consisting of all roots of unity. Note that $\mu_\infty \simeq \mathbb{Q}/\mathbb{Z}$ as a group.
(2) Let M be a G_K-module. Its *dual module* is $M^* = \mathrm{Hom}(M, \mu_\infty)$ with the G_K-action defined by $(\sigma f)(m) = \sigma(f(\sigma^{-1}m))$ for all $\sigma \in G_K$, $f \in M^*$, and $m \in M$.
(3) Let A be a finite abelian group. Its *Pontryagin dual* is $A^\vee = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$.

For any G_K-module M, the evaluation map $M \times M^* \to \mu_\infty$ given by $(m, f) \mapsto f(m)$ is clearly bilinear over $\mathbb{Z}$, and thus it factors through a map $\alpha : M \otimes M^* \to \mu_\infty$. Moreover, α is G_K-equivariant. Indeed, for all $\sigma \in G_K$ we have

$$\alpha(\sigma(m \otimes f)) = \alpha(\sigma m \otimes \sigma f) = (\sigma f)(\sigma m) = \sigma(f(\sigma^{-1}\sigma m)) = \sigma(f(m)) = \sigma(\alpha(m \otimes f)).$$

Thus α induces a map $\alpha^* : H^i(G_K, M \otimes M^*) \to H^i(G_K, \mu_\infty)$ on cohomology for all $i \geq 0$.

Definition 9.8. Let $K/\mathbb{Q}_p$ be a finite extension and let M be a G_K-module. For each $i \in \{0, 1, 2\}$ we define the pairing $\langle , \rangle_K : H^i(G_K, M) \times H^{2-i}(G_K, M^*) \to \mathbb{Q}/\mathbb{Z}$ as the composition

$$H^i(G_K, M) \times H^{2-i}(G_K, M^*) \xrightarrow{\cup} H^2(G_K, M \otimes M^*) \xrightarrow{\alpha^*} H^2(G_K, \mu_\infty) \xrightarrow{\mathrm{Inv}_K} \mathbb{Q}/\mathbb{Z}.$$

The image under this pairing of a pair $(\varphi, \psi) \in H^i(G_K, M) \times H^{2-i}(G_K, M^*)$ is denoted $\langle \varphi, \psi \rangle_K$.

Now suppose that M is a finite G_K-module. Clearly M^* is also finite, and the groups $H^i(G_K, M)$ and $H^{2-i}(G_K, M^*)$ are finite by Corollary 8.21. The pairing defined above gives rise to maps

$$\begin{aligned}A : H^{2-i}(G_K, M^*) &\to H^i(G_K, M)^\vee & B : H^i(G_K, M) &\to (H^{2-i}(G_K, M^*)^\vee \\ \psi &\mapsto (\varphi \mapsto \langle \varphi, \psi \rangle_K) & \varphi &\mapsto (\psi \mapsto \langle \varphi, \psi \rangle_K).\end{aligned}$$

The pairing $\langle , \rangle_K$ is called a *perfect pairing* if the maps A and B are isomorphisms for each $i \in \{0, 1, 2\}$.