PRO-ISOMORPHIC ZETA FUNCTIONS OF NILPOTENT GROUPS AND LIE RINGS UNDER BASE EXTENSION

MARK N. BERMAN AND MICHAEL M. SCHEIN

ABSTRACT. We consider pro-isomorphic zeta functions of the groups $\Gamma(\mathcal{O}_K)$, where Γ is a nilpotent group scheme defined over $\mathbb{Z}$ and K varies over all number fields. Under certain conditions, we show that these functions have a fine Euler decomposition with factors indexed by primes of K and depending only on the structure of Γ and the corresponding localization of K. Explicit computations are given for a number of families of nilpotent groups.

1. INTRODUCTION

The field of subgroup growth studies connections between the structural features of a finitely generated group and its lattice of subgroups of finite index. For instance, a celebrated theorem [22, 23] characterizes, in terms of their structural properties, the finitely generated groups having 'polynomial subgroup growth,' namely those for which the number of subgroups of index n grows polynomially in n.

We are interested in the confluence of two independent branches of this broad and rich topic. On the one hand, we study pro-isomorphic zeta functions of groups - these are analytic functions which keep track of finite index subgroups of a given group in a certain specialized setting. On the other hand, we consider base extension - a process whereby we enlarge the group by means of a field extension in a controlled way - and then ask what effect this has on the subgroup growth. Each of these aspects of subgroup growth is of independent interest, and their intersection is largely uncharted territory.

Let G be a finitely generated group, and let $\mathcal{S}$ be a collection of subgroups of G. For each positive integer n we define

$$a_n^{\mathcal{S}} = \#\{H \in \mathcal{S} \mid [G:H] = n\},$$

which is finite since G is finitely generated. Define the $\mathcal{S}$-zeta function of G as

$$\zeta_G^{\mathcal{S}}(s) = \sum_{n=1}^{\infty} a_n^{\mathcal{S}} n^{-s} = \sum_{\substack{H \in \mathcal{S} \\ [G:H] < \infty}} [G:H]^{-s}, \tag{1}$$

where s is a complex variable. Similarly, for each prime p define the local $\mathcal{S}$-zeta function of G at p to be the analogous sum running over subgroups of p-power index:

$$\zeta_{G,p}^{\mathcal{S}}(s) = \sum_{k=0}^{\infty} a_{p^k}^{\mathcal{S}} p^{-ks}. \tag{2}$$

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Natural examples occur when $\mathcal{S}$ is the set of all subgroups or of all normal subgroups of G. This paper studies another interesting case, where $\mathcal{S}$ is the collection of all *pro-isomorphic* subgroups of G (a subgroup $H \leq G$ is called pro-isomorphic if its profinite completion is isomorphic to that of G). One writes $\zeta^{\leq}_G(s)$, $\zeta^{\triangleleft}_G(s)$, and $\zeta^{\wedge}_G(s)$ for the zeta function (1), where $\mathcal{S}$ is the set of all subgroups, normal subgroups, and pro-isomorphic subgroups, respectively. This notation carries over to the local zeta functions (2). A rich theory of zeta functions of groups has been developed in the special case where G is finitely generated, torsion-free, and nilpotent – hereafter, a $\mathcal{T}$-group.

A *Lie ring* over a ring R is a free R-module of finite rank endowed with an R-bilinear anti-commutative multiplication satisfying the Jacobi identity. We refer to Lie rings over $\mathbb{Z}$ simply as Lie rings. It is convenient to "linearize" by translating counting problems for subgroups of a $\mathcal{T}$-group to counting problems for subrings of a Lie ring; techniques of linear algebra may then be applied. Given a Lie ring $\mathcal{L}$, let $b_n(\mathcal{L})$ be the number of Lie subrings $\mathcal{M} \leq \mathcal{L}$ of index n such that, for any prime p, there is an isomorphism $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ of Lie rings over $\mathbb{Z}_p$. Define the global and local pro-isomorphic zeta functions

$$\zeta^{\wedge}_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} b_n(\mathcal{L}) n^{-s}, \qquad \zeta^{\wedge}_{\mathcal{L},p}(s) = \sum_{k=0}^{\infty} b_{p^k}(\mathcal{L}) p^{-ks}.$$

There is an Euler decomposition $\zeta^{\wedge}_{\mathcal{L}}(s) = \prod_p \zeta^{\wedge}_{\mathcal{L},p}(s)$, which is essentially a consequence of the Chinese remainder theorem. For every $\mathcal{T}$-group G there exists a Lie ring $\mathcal{L}(G)$ such that for almost all primes (and for all primes if G is of class two) the equality $\zeta^{\wedge}_{G,p}(s) = \zeta^{\wedge}_{\mathcal{L}(G),p}(s)$ holds [16, §4]; see Section 2.1 below for more details. In this paper we will work directly with zeta functions of Lie rings rather than those of groups.

If K is a number field of degree $d = [K : \mathbb{Q}]$ with ring of integers $\mathcal{O}_K$, and $\mathcal{L}$ is a Lie ring of rank n, then we may view $\mathcal{L} \otimes_{\mathbb{Z}} \mathcal{O}_K$ as a Lie ring over $\mathbb{Z}$ of rank dn. Our central aim is to investigate how the local pro-isomorphic zeta functions $\zeta^{\wedge}_{\mathcal{L}\otimes_{\mathbb{Z}}\mathcal{O}_K,p}(s)$ vary with K and p. Prior to our work, complete answers to this question were known only for free nilpotent Lie rings [16]. The idea of considering base extensions is implicit in work of du Sautoy and Lubotzky [13]; see Remark 3.16 below for a discussion of the relation between the our results and those of [13].

We will now illustrate our main results. Since the precise definition of the class of Lie rings to which our method applies is quite technical, we content ourselves for now with a list of examples.

Theorem 1.1. *Let $\mathcal{L}$ be a Lie ring of one of the following types:*

- *Free nilpotent Lie rings; see Section 4.1.*
- *Higher Heisenberg Lie rings; see Section 4.2.*
- *Nilpotent Lie rings $\mathcal{L}_{m,n}$ introduced in* [5] *generalizing the Grenham Lie rings; see Section 4.3.*
- *Maximal class Lie rings of rank $c+1$ and nilpotency class c; see Section 4.4.*
- *A filiform Lie ring of nilpotency class 4; see Section 4.5.*
- *A Lie ring of nilpotency class 4 constructed in* [3] *whose local pro-isomorphic zeta functions do not satisfy functional equations; see Section 4.6.*

Let $d \in \mathbb{N}$. Then there exists an explicit rational function $W_{\mathcal{L},d}(X,Y) \in \mathbb{Q}(X,Y)$, depending only on $\mathcal{L}$ and d, such that for any number field K of degree d and any *rational prime p, the following holds:*

$$\zeta^{\wedge}_{(\mathcal{L}\otimes_{\mathbb{Z}}\mathcal{O}_K),p}(s) = \prod_{\mathfrak{p}|p} W_{\mathcal{L},d}(q_{\mathfrak{p}}, q_{\mathfrak{p}}^{-s}),$$

where $q_{\mathfrak{p}}$ *is the cardinality of the residue field* $\mathcal{O}_K/\mathfrak{p}$.

For each Lie ring in the list, we compute the explicit functions $W_{\mathcal{L},d}$ in Section 4. In the first case mentioned, that of free nilpotent Lie rings, this result was obtained more than thirty years ago by Grunewald, Segal, and Smith [16, Theorem 7.1]. An important property of the pro-isomorphic zeta functions $\zeta^{\wedge}_{\mathcal{L}\otimes_{\mathbb{Z}}\mathcal{O}_K}(s)$ that is evident from Theorem 1.1 is the existence of a fine Euler decomposition, namely a decomposition into a product whose factors run over primes of K rather than rational primes. Another property is finite uniformity. More precisely, fix a number field K of degree d, and let $\mathbf{e} = (e_1, \ldots, e_r)$ and $\mathbf{f} = (f_1, \ldots, f_r)$ be r-tuples satisfying $\sum_{i=1}^r e_i f_i = d$. We say that a prime p has ramification type $(\mathbf{e}, \mathbf{f})$ in K if $p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$, where $[\mathcal{O}_K/\mathfrak{p}_i : \mathbb{F}_p] = f_i$ for all $1 \leq i \leq r$. Noting that $q_{\mathfrak{p}} = p^{[\mathcal{O}_K/\mathfrak{p}:\mathbb{F}_p]}$ for any prime $\mathfrak{p}$ of K dividing p, we observe the following:

Corollary 1.2. *Let* $\mathcal{L}$ *be a Lie ring to which Theorem 1.1 applies, and let* $\mathbf{e}, \mathbf{f} \in \mathbb{N}^r$. *There exists a rational function* $W_{\mathcal{L},\mathbf{e},\mathbf{f}}(X,Y) \in \mathbb{Q}(X,Y)$ *such that for any number field of degree* $d = \sum_{i=1}^r e_i f_i$ *and for any prime* p *of ramification type* $(\mathbf{e}, \mathbf{f})$ *in* K, *the following holds:*

$$\zeta^{\wedge}_{(\mathcal{L}\otimes_{\mathbb{Z}}\mathcal{O}_K),p}(s) = W_{\mathcal{L},\mathbf{e},\mathbf{f}}(p, p^{-s}).$$

Proof. Set $W_{\mathcal{L},\mathbf{e},\mathbf{f}}(X,Y) = \prod_{i=1}^r W_{\mathcal{L},d}(X^{f_i}, Y^{f_i})$ and apply Theorem 1.1. □

A (global) zeta function whose local factors are described by finitely many rational functions is called finitely uniform. Finitely many ramification types of primes appear in any given number field K, so Corollary 1.2 implies the finite uniformity of $\zeta^{\wedge}_{\mathcal{L}\otimes_{\mathbb{Z}}\mathcal{O}_K}(s)$. There exist Lie rings whose zeta functions counting subrings and ideals are not finitely uniform [11]; it is not known whether such examples occur in the pro-isomorphic setting.

Let $\mathcal{L}$ be a Lie ring, and let $W(X,Y) \in \mathbb{Q}(X,Y)$ be a rational function describing the local pro-isomorphic zeta function of $\mathcal{L}$ at p, in the sense that $\zeta^{\wedge}_{\mathcal{L},p}(s) = W(p, p^{-s})$; by [16, Theorem 1] such a rational function always exists. If a relation of the form $W(X^{-1}, Y^{-1}) = (-1)^c X^a Y^b W(X,Y)$ holds for some $a, b, c \in \mathbb{Z}$, then we say that $\zeta^{\wedge}_{\mathcal{L},p}(s)$ satisfies a functional equation with symmetry factor $(-1)^c p^{a-bs}$, written

$$\zeta^{\wedge}_{\mathcal{L},p}(s)|_{p\mapsto p^{-1}} = (-1)^c p^{a-bs} \zeta^{\wedge}_{\mathcal{L},p}(s).$$

See [37] for a more general notion of functional equations. The existence of functional equations apparently holds some important, although mysterious, clue about the structure of a Lie ring, or of an associated finitely generated group. Indeed, for other types of zeta functions some very general results were obtained by Voll [37, 38], in which local functional equations with specified symmetry factors were established for large classes of Lie rings; see also [21]. Nevertheless, there are known examples of Lie rings whose local ideal zeta functions do not satisfy functional equations; the first were found by Woodward [14, Theorems 2.32 and 2.74]. A Lie ring whose local pro-isomorphic zeta functions have no functional equation was constructed by Klopsch and the first author [3].

The first author, Klopsch and Onn have recently postulated the following conjecture about pro-isomorphic zeta functions as a first step to an understanding of functional equations for Lie rings. A graded Lie ring is one equipped with a decomposition

$$\mathcal{L} = \bigoplus_{i \geq 1} \mathcal{L}_i$$

of the underlying free $\mathbb{Z}$-module that is compatible with the Lie bracket, so that $[\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_{i+j}$ for all $i, j \in \mathbb{N}$.

Conjecture 1.3 ([4, Conjecture 1.3])**.** *Let $\mathcal{L}$ be a graded Lie ring. The local pro-isomorphic zeta functions $\zeta^{\wedge}_{\mathcal{L},p}(s)$ satisfy functional equations of the form*

$$\zeta^{\wedge}_{\mathcal{L},p}(s)|_{p \to p^{-1}} = (-1)^c p^{a-bs} \zeta^{\wedge}_{\mathcal{L},p}(s)$$

for almost all primes p. Furthermore, b is equal to $\sum_{i=1}^{\infty} \mathrm{rk}_{\mathbb{Z}}\, \gamma_i \mathcal{L}$, the sum of the ranks of the members of the lower central series of $\mathcal{L}$.

If $\mathcal{L}$ is a graded Lie ring, then $\mathcal{L} \otimes_{\mathbb{Z}} \mathcal{O}_K$ remains graded, as a Lie ring over $\mathbb{Z}$, for any number field K. The first four Lie rings $\mathcal{L}$ from the list of Theorem 1.1 are graded, and our computations show that all the Lie rings $\mathcal{L} \otimes_{\mathbb{Z}} \mathcal{O}_K$ satisfy Conjecture 1.3. In each case, there is a local functional equation for all primes p, but at the primes that ramify in K, of which there are finitely many, the constant b in the symmetry factor is different from the one specified in the conjecture; see Remark 4.2. The remaining two examples in Theorem 1.1 satisfy neither the hypotheses nor the conclusion of Conjecture 1.3.

It is interesting to compare Theorem 1.1 with analogous results for subring and ideal zeta functions under base extension. The most general statement currently available is [9, Theorem 4.21], which explicitly describes the local ideal zeta functions $\zeta^{\triangleleft}_{(\mathcal{L} \otimes_{\mathbb{Z}} \mathcal{O}_K),p}(s)$, when p is unramified in K, for a collection of nilpotent Lie rings $\mathcal{L}$ of class two that includes the free nilpotent Lie rings of class two and the higher Heisenberg Lie rings. It also includes the Grenham Lie rings, but not other members of the family studied in Section 4.3, and includes Lie rings not covered by Theorem 1.1. All these local ideal zeta functions satisfy functional equations with the symmetry factor specified by [37, Theorem C]. It is conjectured that *all* local ideal zeta functions of the Lie rings considered in [9] satisfy functional equations, and that the symmetry factors at ramified primes are different from the generic one. As mentioned above, Theorem 1.1 implies the analogue of this conjecture for pro-isomorphic zeta functions of some Lie rings. Finally, Theorem 1.1 covers two families of Lie rings of unbounded nilpotency class: the free nilpotent and the maximal class Lie rings. By contrast, no subring or ideal zeta functions are known for any nilpotent Lie ring of class five or greater.

A further important analytic property of the pro-isomorphic zeta function $\zeta^{\wedge}_{\mathcal{L}}(s)$ is its abscissa of convegence $\alpha^{\wedge}_{\mathcal{L}} = \inf S$, where S is the set of real numbers β such that $\zeta^{\wedge}_{\mathcal{L}}(s)$ converges on the right half-plane $\mathrm{Re}\, s > \beta$. Note that $\alpha^{\wedge}_{\mathcal{L}}$ is a property of the global zeta function, and not of its local components. The number $\alpha^{\wedge}_{\mathcal{L}}$ reflects the algebraic structure of $\mathcal{L}$ as the polynomial rate of growth of the sequence $s_n(\mathcal{L}) = b_1(\mathcal{L}) + \cdots + b_n(\mathcal{L})$. For many of the Lie rings $\mathcal{L}$ of Theorem 1.1, the abscissa of convergence $\alpha^{\wedge}_{\mathcal{L} \otimes_{\mathbb{Z}} \mathcal{O}_K}$ is given by a linear function in the degree $d = [K : \mathbb{Q}]$. While the abscissae of convergence of subring and ideal zeta functions are known always to be rational [12], this question remains open for the pro-isomorphic zeta functions.

1.1. **Overview.** We will now give a brief overview of the methods of this paper. A basic property of local pro-isomorphic zeta functions of Lie rings is that they are readily interpreted as certain explicit p-adic integrals. Indeed, let $\mathcal{L}$ be a Lie ring and p a prime. We write $\mathcal{L}_p = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ as before, and L_p for the $\mathbb{Q}_p$-Lie algebra $\mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Q}_p$. Let $\mathbf{G} = \mathfrak{Aut}\, L_p$ be the algebraic automorphism group of L_p, so that $\mathbf{G}(F) = \mathrm{Aut}_F(L_p \otimes_{\mathbb{Q}_p} F)$ for any field extension $F/\mathbb{Q}_p$. Let $\mathbf{G}(\mathbb{Z}_p)$ be the subgroup of $\mathbf{G}(\mathbb{Q}_p) = \mathrm{Aut}_{\mathbb{Q}_p} L_p$ consisting of elements that restrict to automorphisms of the $\mathbb{Z}_p$-lattice $\mathcal{L}_p$, and let $\mathbf{G}^+ \subset \mathbf{G}(\mathbb{Q}_p)$ be the submonoid of elements preserving $\mathcal{L}_p$. Throughout this paper, automorphisms act from the right. The map $\varphi \mapsto (\mathcal{L}_p)\varphi$ induces a bijection between the set $\mathbf{G}(\mathbb{Z}_p)\backslash\mathbf{G}^+$ of right cosets and the set of subrings of $\mathcal{L}_p$ that are isomorphic to $\mathcal{L}_p$ as Lie rings over $\mathbb{Z}_p$. Let μ_p be the right Haar measure on $\mathbf{G}(\mathbb{Q}_p)$, normalized so that $\mu_p(\mathbf{G}(\mathbb{Z}_p)) = 1$. Observe that $[\mathcal{L}_p : (\mathcal{L}_p)\varphi] = |\det \varphi|_{\mathbb{Q}_p}^{-1}$, where the multiplicative valuation on $\mathbb{Q}_p$ is normalized so that $|p|_{\mathbb{Q}_p} = p^{-1}$. We obtain the following observation, which is [16, Proposition 3.4].

Proposition 1.4 (Grunewald-Segal-Smith)**.** *For each prime p, we have the equality*

$$\zeta^{\wedge}_{\mathcal{L}_p}(s) = \int_{\mathbf{G}_p^+} |\det(g)|^s_{\mathbb{Q}_p} d\mu_p(g). \tag{3}$$

Integrals of the type appearing on the right-hand side of (3) have been studied by many authors, including Hey [18], Satake [28], Tamagawa [34], and Macdonald [25].

In order to study the local pro-isomorphic zeta functions of $\mathcal{L} \otimes_{\mathbb{Z}} \mathcal{O}_K$, it is thus essential to understand the automorphism groups of the $\mathbb{Q}_p$-Lie algebras $(\mathcal{L} \otimes_{\mathbb{Z}} \mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Q}_p = L_p \otimes_{\mathbb{Q}_p} (K \otimes_{\mathbb{Q}} \mathbb{Q}_p)$. The crucial tool of the present paper is a certain rigidity property that was already identified by Segal [32] and used in [16] for the study of the pro-isomorphic zeta functions of base extensions of the free nilpotent Lie rings. To illustrate this property, write R for the $\mathbb{Q}_p$-algebra $K \otimes_{\mathbb{Q}} \mathbb{Q}_p$, and observe that $\mathrm{Aut}_{\mathbb{Q}_p}(L_p \otimes_{\mathbb{Q}_p} R)$ certainly contains automorphisms of the following three types:

- R-linear automorphisms; these are understood if we know the algebraic automorphism group $\mathfrak{Aut}\, L_p$.
- Automorphisms that are trivial modulo the center of $L_p \otimes_{\mathbb{Q}_p} R$; these are easy to describe given the structure of $\mathcal{L}$.
- Automorphisms induced by $\mathbb{Q}_p$-automorphisms of R; these form a finite group.

If, for any finite-dimensional semisimple $\mathbb{Q}_p$-algebra R, the group $\mathrm{Aut}_{\mathbb{Q}_p}(L_p \otimes_{\mathbb{Q}_p} R)$ is generated by automorphisms of the above three types, i.e. is the smallest possible, then we say that L_p is rigid over its center; see Definition 3.7 below. In Theorem 3.8 below we generalize the rigidity criterion proved by Segal in [32] and thereby establish rigidity, in particular, for the Lie algebras arising from the Lie rings in Theorem 1.1.

Fix a decomposition $\mathbf{G} = \mathbf{H} \ltimes \mathbf{N}$, where $\mathbf{N}$ is the unipotent radical of $\mathbf{G}$ and $\mathbf{H} \subseteq \mathbf{G}$ is a reductive subgroup. The integral of Proposition 1.4 may be reformulated as an integral over a suitable submonoid of $\mathbf{H}(\mathbb{Q}_p)$, at the cost of replacing the integrand by a more complicated function. This simplified domain of integration allows the integral to be computed by means of a p-adic Bruhat decomposition. Moreover, du Sautoy and

Lubotzky [13] have shown that, under a series of simplifying hypotheses, the integrand may be expressed as a product of functions, each of which can often be computed explicitly. The main technical result of our paper is Corollary 3.15, which says, roughly speaking, that if L_p is rigid over its center and the hypotheses of [13] are satisfied, then the local pro-isomorphic zeta function $\zeta^{\wedge}_{\mathcal{L}\otimes_{\mathbb{Z}}\mathcal{O}_K,p}(s)$ splits into a product indexed by the primes of K dividing p, and that the computation of each factor is essentially the same as that of $\zeta^{\wedge}_{\mathcal{L},p}(s)$, with minor alterations depending on the degree $d = [K : \mathbb{Q}]$.

In the case of the higher Heisenberg Lie rings, the explicit formulation given in Theorem 4.10 below is new even in the case of $K = \mathbb{Q}$, except for several small cases. Along the way we prove a combinatorial identity involving the hyperoctahedral group, Lemma 4.7, that may have independent interest. For the remaining examples mentioned in Theorem 1.1, the pro-isomorphic local zeta functions in the case $K = \mathbb{Q}$ were known previously; their computation is straightforward in some cases and quite intricate in others. We apply Corollary 3.15 to show that the same calculations, with slight modifications, treat arbitrary number fields K.

1.2. **Organization of the paper.** In Section 2 below, we review the framework that will be used to analyze the integrals of Proposition 1.4; most of this material is due to du Sautoy and Lubotzky [13]. In Section 3, we define the rigidity property mentioned above and prove a sufficient criterion for it to hold; this criterion generalizes one of Segal [32], and its proof is a modification of Segal's argument. Rigidity is then combined with the setup of Section 2 to prove Corollary 3.15, the technical result discussed above. An example of a family of Lie rings that do not satisfy the rigidity property is given. Finally, in Section 4 we study the Lie rings listed in Theorem 1.1 and explicitly compute their pro-isomorphic zeta functions.

1.3. **Notation.** For any finite extension $F/\mathbb{Q}_p$, the valuation $|\cdot|_F$ is normalized so that the valuation of a uniformizer is $1/q$, where q is the cardinality of the residue field of F.

For any $n \in \mathbb{N}$, we set M_n to be the multiplicative algebraic monoid, over $\mathbb{Z}$, of $n \times n$ matrices. If $m, n \in \mathbb{N}$, then $M_{m,n}$ denotes the additive algebraic group, over $\mathbb{Z}$, of $m \times n$ matrices. We denote the set $\{1, 2, \ldots, n\}$ by $[n]$ and the set $\{0, 1, \ldots, n\}$ by $[n]_0$. If m and n are integers, we write $[m, n]$ for the set $\{k \in \mathbb{Z} : m \leq k \leq n\}$.

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2. Preliminaries

In this section we recall some basic facts about pro-isomorphic zeta functions and p-adic integrals that will be used throughout the paper.

2.1. **Linearization.** We briefly discuss the correspondence between $\mathcal{T}$-groups and Lie rings mentioned in the introduction. If G is a $\mathcal{T}$-group of nilpotency class two, so that $[G, G] \leq Z(G)$, then define the Lie ring

$$\mathcal{L}(G) = G/Z(G) \times Z(G), \tag{4}$$

with the natural multiplication $[(g_1Z(G), z_1), (g_2Z(G), z_2)] = (Z(G), [g_1, g_2])$. Then associating a finite-index subgroup $H \leq G$ with the subring $HZ(G)/Z(G) \times (H \cap Z(G))$ gives an inclusion- and index-preserving bijection between finite-index subgroups of G and finite-index subrings of $\mathcal{L}(G)$. Normal subgroups correspond to ideals of $\mathcal{L}(G)$ under this bijection, whereas pro-isomorphic subgroups are associated with subrings $\mathcal{M} \leq \mathcal{L}(G)$ such that $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \mathcal{L}(G) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for all primes p. For $\mathcal{T}$-groups G of arbitrary nilpotency class, the Malcev correspondence gives a Lie ring $\mathcal{L}(G)$ affording a similar bijection between finite-index subgroups of G and finite-index subrings of $\mathcal{L}(G)$, provided that the index is coprime to an effectively computable integer depending only on the Hirsch length of G; see [16, §4] for details.

Given a nilpotent Lie ring $\mathcal{L}$, define the subring and ideal zeta functions $\zeta_{\mathcal{L}}^{\leq}(s) = \sum_{n=1}^{\infty} b_n^{\leq}(\mathcal{L}) n^{-s}$ and $\zeta_{\mathcal{L}}^{\triangleleft}(s) = \sum_{n=1}^{\infty} b_n^{\triangleleft}(\mathcal{L}) n^{-s}$, where $b_n^{\leq}(\mathcal{L})$ and $b_n^{\triangleleft}(\mathcal{L})$ are the numbers of subrings and ideals of index n, respectively. These zeta functions have Euler decompositions, and for any $\mathcal{T}$-group G the equalities $\zeta_{G,p}^{\leq}(s) = \zeta_{\mathcal{L}(G),p}^{\leq}(s)$ and $\zeta_{G,p}^{\triangleleft}(s) = \zeta_{\mathcal{L}(G),p}^{\triangleleft}(s)$ are satisfied for all but finitely many primes p, or for all p if G is of class two.

2.2. **Simplification of the p-adic integral.** A framework for treating p-adic integrals as in (3), under favorable conditions, was described in [13, §2]. Since we will make use of these methods repeatedly, we recall the main ideas here.

Let E be a number field with ring of integers $\mathcal{O}_E$, and let $\mathfrak{p}$ be a finite place of E. Let $F/E_{\mathfrak{p}}$ be a finite extension of the localization $E_{\mathfrak{p}}$. We denote by $\mathcal{O}$ the valuation ring of F. Fix a uniformizer $\pi \in \mathcal{O}$, and let q be the cardinality of the residue field $\mathcal{O}/\pi\mathcal{O}$. Let $\mathbf{G} \subseteq \mathbf{GL}_n$ be an affine group scheme over E. Set $\mathbf{G}^\circ$ to be the connected component of the identity, and fix a decomposition $\mathbf{G}^\circ = \mathbf{N} \rtimes \mathbf{H}$, where $\mathbf{N}$ is the unipotent radical of $\mathbf{G}^\circ$ and $\mathbf{H}$ is reductive. Define the group $G = \mathbf{G}(F)$, the subgroup $\mathbf{G}(\mathcal{O}) = G \cap \mathrm{GL}_n(\mathcal{O})$, and the submonoid $G^+ = G \cap M_n(\mathcal{O})$. In addition, set $N = \mathbf{N}(F)$ and $H = \mathbf{H}(F)$. For any algebraic subgroup $\mathbf{S}$ of $\mathbf{G}$, set $\mathbf{S}(\mathcal{O}) = \mathbf{S}(F) \cap \mathbf{G}(\mathcal{O})$ and let $\mu_{\mathbf{S}(F)}$ denote the right Haar measure on $\mathbf{S}(F)$, normalized so that $\mu_{\mathbf{S}(F)}(\mathbf{S}(\mathcal{O})) = 1$. We are interested in computing the integral

$$\mathcal{Z}_{\mathbf{G},F}(s) = \int_{G^+} |\det g|_F^s \, d\mu_G(g). \tag{5}$$

We will now introduce three conditions under which such integrals are treated in [13, §2]. In view of Proposition 1.4, for the purposes of this article we are most interested in the case where $\mathcal{L}$ is a Lie ring over $\mathbb{Z}$ and $\mathbf{G} = \mathfrak{Aut}(\mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Q})$. The first two conditions hold for $\mathbf{G}(\mathbb{Q}_p)$ for almost all primes p by general considerations. A rigidity condition on $\mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Q}_p$ ensures that they will hold for $\mathbf{G}(\mathbb{Q}_p)$, where $F/\mathbb{Q}_p$ is any finite extension. The third condition, however, is much more restrictive. See Remark 3.13 below.

Assumption 2.1. We assume that $G = \mathbf{G}(\mathcal{O})\mathbf{G}^\circ(F)$.

It easily follows from Assumption 2.1 that $\mathcal{Z}_{\mathbf{G},F}(s) = \mathcal{Z}_{\mathbf{G}^\circ,F}(s)$ [13, Proposition 2.1]. Replace $\mathbf{G}$ with $\mathbf{G}^\circ$, so that $\mathbf{G}$ is a connected algebraic group.

Furthermore, we assume that the embedding $\mathbf{G} \subseteq \mathbf{GL}_n$ has a particularly convenient form. The group $\mathrm{GL}_n(F)$ naturally acts from the right on F^n; let $(e_1, \ldots, e_n)$ be the standard basis. Given a sequence $0 = d_0 < d_1 < \cdots d_t = n$, define U_i to be the F-linear span of $e_{d_{i-1}+1}, \dots, e_{d_i}$. Setting $V_i = U_i \oplus \cdots \oplus U_t$ for every $i \in [t]$, note that $V_1 = F^n$ and $V_{t+1} = (0)$.

Assumption 2.2. We assume that, for a suitable sequence as above, the following conditions are satisfied:

- The subspace U_i is H-stable for every $i \in [t]$.
- The subspace V_i is N-stable for every $i \in [t]$; moreover, N acts trivially on the quotient V_i/V_{i+1}.

Set $V = V_1$. Under our assumptions, the group $G = N \rtimes H$ acts on the quotient V/V_{i+1} for every $i \in [t]$; let $\psi_i' : G \to \mathrm{Aut}(V/V_{i+1})$ be the corresponding homomorphism. Since V/V_{i+1} is spanned by the images of $e_1, \dots, e_{d_i}$, there is a natural identification of $\mathrm{Aut}(V/V_{i+1})$ with $\mathrm{GL}_{d_i}(F)$. Putting $N_i = N \cap \ker \psi_i'$, we note that ψ_i' factors through a map $\psi_i : G/N_i \to \mathrm{Aut}(V/V_{i+1})$.

Let $\overline{n} \in N_i/N_{i+1}$ and $j \in [d_i]$. Then $e_j\overline{n} - e_j \in V_{i+1}$ is well-defined modulo V_{i+2}; recall that G acts on V from the right. Moreover, H acts on V_{i+1}/V_{i+2}. Hence for every $h \in H$ there is a map

$$\begin{aligned} \tau(h) : N_i/N_{i+1} &\hookrightarrow (V_{i+1}/V_{i+2})^{d_i} \\ \overline{n} &\mapsto ((e_1\overline{n} - e_1)h, \dots, (e_{d_i}\overline{n} - e_{d_i})h). \end{aligned}$$

We identify $(V_{i+1}/V_{i+2})^{d_i}$ with $M_{d_i, \dim U_{i+1}}(F)$ by viewing a d_i-tuple $(v_1, \dots, v_{d_i}) \in (V_{i+1}/V_{i+2})^{d_i}$ as the matrix whose j-th row is v_j, expressed in terms of the basis of V_{i+1}/V_{i+2} given by the images of $e_{d_i+1}, \dots, e_{d_{i+1}}$. Define the function $\theta_i^F : H \to \mathbb{R}$ by

$$\theta_i^F(h) = \mu_{N_i/N_{i+1}}(\{\overline{n} \in N_i/N_{i+1} : \tau(h)(\overline{n}) \in M_{d_i, d_{i+1}-d_i}(\mathcal{O})\}), \tag{6}$$

where $\mu_{N_i/N_{i+1}}$ is the right Haar measure on N_i/N_{i+1}, normalized so that the set $\psi_{i+1}^{-1}(\psi_{i+1}(N_i/N_{i+1}) \cap M_{d_i}(\mathcal{O}))$ has measure 1; recall that $\mathrm{Aut}(V/V_{i+1})$ has been identified with $\mathrm{GL}_{d_i}(F)$. Thus $\mu_{N_i/N_{i+1}}$ is identified with the additive Haar measure on $F^{d_i(d_{i+1}-d_i)}$, normalized on $\mathcal{O}^{d_i(d_{i+1}-d_i)}$. Note also that $\mu_G = \left(\prod_{i=1}^{c-1} \mu_{N_i/N_{i+1}}\right) \mu_H$. Define $H^+ = H \cap M_n(\mathcal{O})$ and $(G/N_i)^+ = \psi_i^{-1}(\psi_i(G/N_i) \cap M_{d_i}(\mathcal{O}))$.

Assumption 2.3. We say that *the lifting condition holds* if for every $i \in [2, t-1]$ and every $\overline{g} \in (NH^+/N_i) \cap (G/N_i)^+$ there exists $\gamma \in G^+$ such that $\overline{g} = \gamma N_i$; observe that the above always holds for $i = 1$, and hence the condition stated here is equivalent to [13, Assumption 2.3].

Proposition 2.4. [13, Theorem 2.2] *For $h \in H$ define*

$$\theta^F(h) = \mu_N(\{n \in N : nh \in G^+\}).$$

If Assumptions 2.1, 2.2, and 2.3 hold, then

$$\theta^F = \prod_{i=1}^{t-1} \theta_i^F.$$

Furthermore,

$$\mathcal{Z}_{\mathbf{G},F}(s) = \int_{H^+} |\det h|_F^s \left(\prod_{i=1}^{t-1} \theta_i^F(h)\right) d\mu_H.$$

The second claim of Proposition 2.4 follows from the first by the observation that every $g \in G^+$ decomposes uniquely as a product $g = nh$ with $h \in H^+$ and $n \in N$. Since $\det g = \det h$, it follows that

$$\mathscr{Z}_{\mathbf{G},F}(s) = \int_{H^+} |\det h|_F^s \theta^F(h) d\mu_H. \tag{7}$$

2.3. **Consequences of the p-adic Bruhat decomposition.** An important benefit of rewriting the p-adic integral of (3) in the form (7) is that the domain of integration of the latter is partitioned conveniently by the p-adic Bruhat decomposition. Under certain conditions, this may be used to evaluate the integral. We state here the results that will be needed later in the paper. The idea is essentially due to Igusa [19] and was developed by du Sautoy and Lubotzky [13, §5] and further by the first author [2, §4]; the reader is invited the consult these references for details.

To streamline the notation, we will replace $\mathbf{H} \times_E F$ with $\mathbf{H}$; thus we treat $\mathbf{H}$ as a reductive group over F. Recall that $\mathbf{H}(\mathcal{O}) = \mathbf{H}(F) \cap \mathbf{GL}_n(\mathcal{O})$, and similarly for algebraic subgroups of $\mathbf{H}$. Let $\mathbf{T} \subset \mathbf{H}$ be a maximal torus, and suppose that $\mathbf{T}$ splits over F; this allows us to fix an isomorphism $\kappa : \mathbf{T} \to \mathbb{G}_a^{\mathrm{rk}\,\mathbf{H}}$ defined over F. Let Φ be the root system of $\mathbf{H}$, and let $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ be a set of simple roots. Denote by Φ^+ the consequent set of positive roots. Let $\Xi = \mathrm{Hom}(\mathbb{G}_m, \mathbf{T})$ be the set of cocharacters of $\mathbf{T}$. Recall the natural pairing between characters and cocharacters: if $\beta \in \mathrm{Hom}(\mathbf{T}, \mathbb{G}_m)$ and $\xi \in \Xi$, then $\langle \beta, \xi \rangle$ is the integer satisfying $\beta(\xi(t)) = t^{\langle \beta, \xi \rangle}$ for all $t \in \mathbf{G}_m(F)$. For every root $\alpha \in \Phi$, let $\mathbf{U}_\alpha \subset \mathbf{H}$ be the corresponding root subgroup, and let $\psi_\alpha : \mathbb{G}_a \to \mathbf{U}_\alpha$ be an isomorphism; note that the isomorphisms κ and ψ_α may all be chosen to be defined over $\mathcal{O}$. It is crucial to assume that these data have "very good reduction" in the sense of [19, II.2], namely that $\mathbf{G}$, $\mathbf{T}$, and all the isomorphisms κ and ψ_α have good reduction modulo π; the latter condition means that reduction modulo π induces isomorphisms $\overline{\kappa} : \mathbf{T}(k) \to k^{\mathrm{rk}\,\mathbf{H}}$ and $\overline{\psi_\alpha} : k \to \mathbf{U}_\alpha(k)$, where $k = \mathcal{O}/(\pi)$ is the residue field.

Let $W = N_{\mathbf{H}}(\mathbf{T})/\mathbf{T}$ be the Weyl group; it acts by conjugation on the collection of root subgroups and hence on Φ. Define $\Xi^+ = \{\xi \in \Xi : \xi(\pi) \in \mathbf{H}(\mathcal{O})\}$, and for every $w \in W$ and $\alpha \in \Delta$ set

$$\delta_{w,\alpha} = \begin{cases} 1 & : \alpha \in \Delta \cap w(\Phi^-) \\ 0 & : \alpha \in \Delta \setminus w(\Phi^-). \end{cases}$$

Here $\Phi^- = \Phi \setminus \Phi^+$ is the set of negative roots. Then put

$$w\Xi_w^+ = \left\{ \xi \in \Xi^+ : \alpha(\xi(\pi)) \in \pi^{\delta_{w,\alpha}} \mathcal{O} \text{ for all } \alpha \in \Delta \right\}. \tag{8}$$

Partitioning the domain of integration according to the p-adic Bruhat decomposition and analyzing the behavior of the integrand on each piece, the first author established the following result, which is immediate from the proof of [2, Proposition 4.2] and generalizes [13, (5.4)].

Proposition 2.5. *Suppose that the maximal torus $\mathbf{T} \subset \mathbf{H}$ is F-split and that very good reduction holds. Then*

$$\int_{\mathbf{H}^+(F)} |\det h|_F^s \theta^F(h) d\mu_H(h) = \sum_{w \in W} q^{-\ell(w)} \sum_{\xi \in w\Xi_w^+} q^{\langle \prod_{\beta \in \Phi^+} \beta, \xi \rangle} |\det \xi(\pi)|_F^s \theta^F(\xi(\pi)),$$

where ℓ is the length function on W with respect to the Coxeter generating set corresponding to Δ.

Example 2.6. Let $\mathcal{L}$ be the abelian Lie ring $\mathbb{Z}^n$ of rank n. It is clear that all finite-index sublattices of $\mathcal{L}$ are ideals isomorphic to $\mathcal{L}$, so the pro-isomorphic zeta function of $\mathcal{L}$ coincides with the subring and ideal zeta functions. Moreover, the algebraic automorphism group of $L = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}^n$ is GL_n. Thus

$$\zeta_{\mathbb{Z}_p^n}(s) = \int_{\mathrm{GL}_n^+(\mathbb{Q}_p)} |\det A|_{\mathbb{Q}_p}^s \, d\mu_{\mathrm{GL}_n(\mathbb{Q}_p)}(A) = \prod_{i=0}^{n-1} \frac{1}{1-p^{i-s}}.$$

It is a simple exercise to derive the second equality from Proposition 2.5. See also [16, Proposition 1.1] and [24, Theorem 15.1] for an assortment of proofs not relying on Proposition 2.5.

We will consider two applications of Proposition 2.5 below, in Sections 4.2 and 4.3. In both cases the hypothesis of very good reduction is readily verified, and we will not address it explicitly.

3. Rigidity

3.1. **A rigidity criterion.** Let L be a finite-dimensional non-abelian Lie algebra over a field k, and let R be a finite-dimensional semisimple commutative k-algebra. As in [32], for any subset $X \subseteq L$ and ideal $I \leq L$, we set

$$C_{L/I}(X) = \{x \in L : [x, X] \subseteq I\},$$

where $[x, X] = \{[x, y] : y \in X\}$. Note that $I + kx \subseteq C_{L/[I,L]}(C_{L/[I,L]}(x))$ for any $x \in L$ and any ideal $I \leq L$. If $Z \leq \gamma_2 L$ is a verbal ideal, write $Z_1 = [Z, L]$ and define

$$\mathcal{Y}(Z) = \{x \in L \setminus Z : C_{L/Z_1}(C_{L/Z_1}(x)) = Z + kx\}. \tag{9}$$

For instance, if $Z = Z(L)$ is the center of L, then $Z_1 = 0$ and $x \in \mathcal{Y}(Z)$ if and only if x is not central and the set of elements of L commuting with everything with which x commutes is as small as possible, namely $Z + kx$. All tensor products in this section are over k. The following easy statement is noted after [32, Lemma 1].

Remark 3.1. Let E/k be a field extension of arbitrary dimension. Then, for any ideal $I \leq L$ and any $X \subseteq L$, we have $C_{L \otimes E/I \otimes E}(X \otimes 1) = C_{L/I}(X) \otimes E$ since centralizers are defined by linear conditions over E. Hence the same conclusion holds for semisimple k-algebras R as above.

The following lemma somewhat weakens the hypotheses of [32, Lemma 1].

Lemma 3.2. *Suppose that L/Z_1 is indecomposable, that $\dim_k L/Z_1 > 1$, and that $\mathcal{Y}(Z)$ generates L as a Lie algebra. Then there exists a unique epimorphism*

$$\widehat{\ }: \mathrm{Aut}_k(L \otimes R) \to \mathrm{Aut}_k R$$

such that

$$\varphi(\alpha w) \equiv \widehat{\varphi}(\alpha) \cdot \varphi(w) \bmod Z \otimes R$$

for any $\varphi \in \mathrm{Aut}_k(L \otimes R)$ and any elements $\alpha \in R$ and $w \in L \otimes R$. Moreover, the natural injection $\mathrm{Aut}_k R \to \mathrm{Aut}_k(L \otimes R)$ splits $\widehat{\ }$.

Proof. Fix an automorphism $\varphi \in \mathrm{Aut}_k(L \otimes R)$, and consider an element $x \in \mathcal{Y}(Z)$. For any subset $X \subseteq L$, write $X \otimes 1 = \{x \otimes 1 : x \in X\} \subseteq L \otimes R$. Then $C_{(L\otimes R)/(I\otimes R)}(X \otimes R) = C_{(L\otimes R)/(I\otimes R)}(X \otimes 1) = C_{L/I}(X) \otimes R$ for any ideal $I \leq L$ by Remark 3.1. In particular,

$$C_{(L\otimes R)/(Z_1\otimes R)}(C_{(L\otimes R)/(Z_1\otimes R)}(x \otimes 1)) = C_{(L\otimes R)/(Z_1\otimes R)}(C_{L/Z_1}(x) \otimes R) = C_{L/Z_1}(C_{L/Z_1}(x)) \otimes R = (kx + Z) \otimes R = (x \otimes 1)R + Z \otimes R,$$

since we assumed $x \in \mathcal{Y}(Z)$. Hence, $x \otimes 1 \in \mathcal{Y}(Z \otimes R)$. By assumption Z is verbal. Thus $Z \otimes R$ is stable under φ, and so is $\mathcal{Y}(Z \otimes R)$. Therefore,

$$C_{(L\otimes R)/(Z_1\otimes R)}(C_{(L\otimes R)/(Z_1\otimes R)}(\varphi(x \otimes 1))) = (\varphi(x \otimes 1))R + Z \otimes R.$$

Let $\alpha \in R$. Since $\varphi(x\otimes\alpha) \in C_{(L\otimes R)/(Z_1\otimes R)}(C_{(L\otimes R)/(Z_1\otimes R)}(\varphi(x\otimes 1)))$, there is a uniquely defined element $\beta_x(\alpha) \in R$ such that $\varphi(x \otimes \alpha) \equiv \beta_x(\alpha)\varphi(x \otimes 1) \bmod Z \otimes R$. The map $\beta_x : R \to R$ is clearly k-linear and bijective.

Since $\mathcal{Y}(Z)$ generates L, we may fix a subset $\mathcal{S} \subset \mathcal{Y}(Z) \cap (L \setminus \gamma_2 L)$ such that $\mathcal{S}$ generates L. If $x \in \mathcal{S}$, then there exists $y \in \mathcal{S}$ such that $[x, y] \notin Z_1$; otherwise, $kx + Z_1$ would be a proper direct summand of L/Z_1, contradicting our hypotheses. For any $\alpha \in R$ we have

$$\beta_x(\alpha)\varphi([x, y] \otimes 1) = [\beta_x(\alpha)\varphi(x \otimes 1), \varphi(y \otimes 1)] \equiv \varphi([x \otimes \alpha, y \otimes 1]) = \varphi([x \otimes 1, y \otimes \alpha]) \equiv [\varphi(x \otimes 1), \beta_y(\alpha)\varphi(y \otimes 1)] = \beta_y(\alpha)\varphi([x, y] \otimes 1) \bmod Z_1 \otimes R. \tag{10}$$

Observe that the image of $[x, y] \otimes R$ in $(L \otimes R)/(Z_1 \otimes R)$ is a $(\dim_k R)$-dimensional k-subspace. The same is true of the image of $\varphi([x, y] \otimes R)$. Since this image lies inside the image of $(\varphi([x, y] \otimes 1))R$, which is at most $(\dim_k R)$-dimensional, it follows that $\mathrm{Ann}_R(\varphi([x, y] \otimes 1)) = 0$. Hence $\beta_x(\alpha) = \beta_y(\alpha)$ for all $\alpha \in R$. Further, we claim that $\beta_x \in \mathrm{Aut}_k R$. To establish this, it remains only to show that β_x is multiplicative. Indeed, for y as above and $\alpha_1, \alpha_2 \in R$, we see that

$$\beta_x(\alpha_1\alpha_2)\varphi([x, y] \otimes 1) \equiv \varphi([x, y] \otimes \alpha_1\alpha_2) = \varphi([x \otimes \alpha_1, y \otimes \alpha_2]) \equiv [\beta_x(\alpha_1)\varphi(x \otimes 1), \beta_x(\alpha_2)\varphi(y \otimes 1)] \equiv \beta_x(\alpha_1)\beta_x(\alpha_2)\varphi([x, y] \otimes 1) \bmod Z_1 \otimes R,$$

and hence $\beta_x(\alpha_1\alpha_2) = \beta_x(\alpha_1)\beta_x(\alpha_2)$ as above.

For any $\beta \in \mathrm{Aut}_k R$, set

$$\mathcal{S}_\beta = \{x \in \mathcal{S} : \varphi(x \otimes \alpha) \equiv \beta(\alpha)\varphi(x \otimes 1) \bmod Z \otimes R \,\text{for all}\, \alpha \in R\},$$

and define L_β to be the subalgebra of L generated by $\mathcal{S}_\beta$. It is evident from the above that any $x \in L_\beta$ satisfies $\varphi(x \otimes \alpha) \equiv \beta(\alpha)\varphi(x \otimes 1) \bmod Z \otimes R$ for all $\alpha \in R$, and that the same congruence holds modulo $Z_1 \otimes R$ if $x \in \gamma_2 L_\beta$. Moreover, if $\beta \neq \beta'$, then $[L_\beta, L_{\beta'}] \subseteq Z_1$ by (10). Writing $\overline{L}_\beta$ for the image of L_β in L/Z_1, we find that

$$L/Z_1 = \bigoplus_{\beta \in \mathrm{Aut}_k R} \overline{L}_\beta$$

as k-Lie algebras since $Z \leq \gamma_2 L$. As L/Z_1 is indecomposable by assumption, there must be a single $\beta \in \mathrm{Aut}_k R$ such that $L_\beta = L$. Now set $\widehat{\varphi} = \beta$. The resulting map $\widehat{\ }: \mathrm{Aut}_k(L \otimes R) \to \mathrm{Aut}_k R$ clearly has all the claimed properties. □

The argument deducing [32, Theorem 2] from [32, Lemma 1] transfers to our setting and will be used to deduce Theorem 3.8 below from Lemma 3.2. We emphasize that the ideas are due to D. Segal. Consider L as a subalgebra of $L \otimes R$ via the natural embedding $x \mapsto x \otimes 1$.

Definition 3.3. Let $\varphi \in \mathrm{Aut}_k(L \otimes R)$. We denote by $\widetilde{\varphi}$ the unique R-linear map $\widetilde{\varphi} : L \otimes R \to L \otimes R$ satisfying $\widetilde{\varphi}|_L = \varphi|_L$.

Remark 3.4. It is clear that $\widetilde{\varphi}$ is a Lie algebra endomorphism of $L \otimes R$. In fact, it is not hard to show that $\widetilde{\varphi} \in \overline{\mathfrak{Aut}\, L}(R)$, where $\overline{\mathfrak{Aut}\, L}$ is the Zariski closure of $\mathfrak{Aut}\, L$ in the algebraic endomorphism monoid $\mathfrak{End}\, L$; by [27, Lemma 1.2], $\overline{\mathfrak{Aut}\, L}$ is an algebraic submonoid. However, $\widetilde{\varphi}$ need not be an automorphism of $L \otimes R$. For instance, consider a two-dimensional abelian Lie k-algebra L with basis (x_1, x_2), and let R/k be a finite extension of fields. Viewed as a k-algebra, $L \otimes R$ is an abelian Lie algebra of dimension $2d$, where $d = [R : k]$. Thus $\mathrm{Aut}_k(L \otimes R) \simeq \mathrm{GL}_{2d}(k)$. Let $\alpha \in R \setminus k$. Since $x_1 \otimes 1$ and $x_1 \otimes \alpha$ are linearly independent over k, there exists $\varphi \in \mathrm{Aut}_k(L \otimes R)$ such that

$$\begin{aligned} \varphi(x_1 \otimes 1) &= x_1 \otimes 1 \\ \varphi(x_2 \otimes 1) &= x_1 \otimes \alpha. \end{aligned}$$

However, $\varphi(x_1 \otimes 1)$ and $\varphi(x_2 \otimes 1)$ are not linearly independent over R, so $\widetilde{\varphi}$ is not an automorphism of $L \otimes R$.

Lemma 3.5. *Suppose that L is a finite-dimensional nilpotent k-Lie algebra and that $Z \leq \gamma_2 L$ is a verbal ideal satisfying the hypotheses of Lemma 3.2. Let $\varphi \in \mathrm{Aut}_k(L \otimes R)$. Then $\widetilde{\varphi} \in \mathrm{Aut}_R(L \otimes R)$, where $\widetilde{\varphi}$ is as in Definition 3.3.*

Proof. Since $\widetilde{\varphi}$ is R-linear by construction, we need only show that $\widetilde{\varphi}$ is an automorphism. Set $\psi = \widehat{\varphi}^{-1} \circ \varphi \in \mathrm{Aut}_k(L \otimes R)$, where $\widehat{\varphi}$ is as in Lemma 3.2 and $\mathrm{Aut}_k R$ is embedded into $\mathrm{Aut}_k(L \otimes R)$ in the natural way. For any $v = \sum_i x_i \otimes \alpha_i \in L \otimes R$, we have

$$\widetilde{\varphi}(v) = \sum_i \varphi(x_i) \otimes \alpha_i \equiv \psi(v) \bmod Z \otimes R \tag{11}$$

by the definition of $\widehat{\varphi}$. As ψ is surjective, this implies $\mathrm{Im}(\widetilde{\varphi}) + Z \otimes R = L \otimes R$. However, $Z \otimes R \subseteq \gamma_2(L \otimes R)$ by assumption. Thus $\mathrm{Im}(\widetilde{\varphi})$ generates $L \otimes R$, since $L \otimes R$ is nilpotent. Hence $\widetilde{\varphi}$ is surjective. Now $\dim_k L \otimes R < \infty$, so $\widetilde{\varphi}$ is also injective. □

Corollary 3.6. *Let L be a finite-dimensional nilpotent k-Lie algebra and $Z \leq \gamma_2 L$ be a verbal ideal such that $L/[Z, L]$ is indecomposable and $\mathcal{Y}(Z)$ generates L. Let*

$$J = \ker(\mathrm{Aut}_k(L \otimes R) \to \mathrm{Aut}_k((L \otimes R)/(Z \otimes R)))$$

be the subgroup of automorphisms that are trivial modulo $Z \otimes R$. Then

$$\mathrm{Aut}_k(L \otimes R) = (J \cdot \mathrm{Aut}_R(L \otimes R)) \rtimes \mathrm{Aut}_k R.$$

Proof. Let $\varphi \in \mathrm{Aut}_k(L \otimes R)$. The hypotheses of Lemma 3.2 hold, and we set $\psi = \widehat{\varphi}^{-1} \circ \varphi$ as in the proof of Lemma 3.5. Then $\psi \circ \widetilde{\varphi}^{-1} \in J$ by (11). But $\varphi = \widehat{\varphi} \circ (\psi \circ \widetilde{\varphi}^{-1}) \circ \widetilde{\varphi}$, and so $\mathrm{Aut}_k(L \otimes R) = \mathrm{Aut}_k R \cdot J \cdot \mathrm{Aut}_R(L \otimes R)$. The splitting noted in the statement of Lemma 3.2 completes the proof. □

Definition 3.7. Let L be a Lie algebra L over a field k, and let $Z \leq L$ be a verbal ideal.

(1) The Lie algebra L is said to be *absolutely indecomposable* if $L \otimes_k E$ is indecomposable as an E-algebra for every field extension E/k.
(2) The Lie algebra L is said to be Z*-rigid* if, for any finite separable extension K/k, the following equality of algebraic groups over k holds:

$$\mathfrak{Aut}\,(L \otimes_k K) = (\mathfrak{J} \cdot \mathrm{Res}_{K/k}(\mathfrak{Aut}\,L)) \rtimes \mathfrak{Aut}\,K,$$

where $\mathfrak{J} = \ker(\mathfrak{Aut}\,(L \otimes_k K) \to \mathfrak{Aut}\,(L \otimes_k K)/(Z \otimes_k K))$.

Theorem 3.8. *Let L be a finite-dimensional nilpotent k-Lie algebra, and let $Z \leq \gamma_2 L$ be a verbal ideal such that $\mathcal{Y}(Z)$ generates L, where $\mathcal{Y}(Z)$ is the set defined in* (9)*. Suppose that L/Z_1 is absolutely indecomposable, where $Z_1 = [Z, L]$. Then L is Z-rigid.*

Proof. Let K/k be a finite separable extension, and let E/k be any extension of fields. Consider the E-Lie algebra $L_E = L \otimes E$ and its verbal ideal $Z_E = Z \otimes E \leq \gamma_2(L_E)$. Then $\mathcal{Y}(Z) \otimes E \subseteq \mathcal{Y}(Z_E)$ by Remark 3.1, so $\mathcal{Y}(Z_E)$ generates L_E. Finally, $L_E/[Z_E, L_E] = (L/Z_1) \otimes E$ is indecomposable by assumption. Thus Corollary 3.6, applied to L_E, Z_E, and the E-algebra $R = K \otimes_k E$, tells us that $(\mathfrak{Aut}\,(L \otimes_k K))(E) = \mathrm{Aut}_E((L \otimes_k K) \otimes_k E) = \mathrm{Aut}_E(L_E \otimes_E R)$ is equal to

$$(J_E \cdot \mathrm{Aut}_R(L_E \otimes_E R)) \rtimes \mathrm{Aut}_E\, R = \mathfrak{J}(E) \cdot (\mathfrak{Aut}\,L)(R) \rtimes \mathrm{Aut}_E(K \otimes_k E) = (\mathfrak{J} \cdot \mathrm{Res}_{K/k}(\mathfrak{Aut}\,L))(E) \rtimes (\mathfrak{Aut}\,K)(E),$$

where $J_E = \ker(\mathrm{Aut}_E(L_E \otimes_E R) \to \mathrm{Aut}_E(L_E \otimes_E R)/(Z_E \otimes_E R))$. Just as in the proof of [32, Theorem 2], one shows that $\mathfrak{J} \cdot \mathrm{Res}_{K/k}(\mathfrak{Aut}\,L) \cdot \mathfrak{Aut}\,K$ is a semidirect product of algebraic groups. The equality $\mathfrak{Aut}\,(L \otimes_k K) = (\mathfrak{J} \cdot \mathrm{Res}_{K/k}(\mathfrak{Aut}\,L)) \rtimes \mathfrak{Aut}\,K$ of algebraic groups over k then follows from the equality on points over the separable closure of k; cf. [26, Corollary 1.30]. ☐

We readily deduce the following corollary, which is [32, Theorem 2], from Theorem 3.8.

Corollary 3.9 (Segal)**.** *Let M and Z be verbal ideals of a finite-dimensional nilpotent k-Lie algebra L such that $Z \leq M \leq \gamma_2 L$ and $\dim_k L/M > 1$. Setting $M_1 = [M, L]$ and $Z_1 = [Z, L]$, define*

$$\begin{aligned} \mathcal{X}(M) &= \{x \in L \setminus M : C_{L/M_1}(x) = M + kx\} \\ \mathcal{Y}(M, Z) &= \{x \in L \setminus M : C_{L/Z_1}(C_{L/Z_1}(x)) = Z + kx\}. \end{aligned}$$

Assume that $\mathcal{X}(M)$ and $\mathcal{Y}(M, Z)$ each generate L. Then L is Z-rigid.

Proof. We verify the hypotheses of Theorem 3.8. Clearly, $\mathcal{Y}(M, Z) \subseteq \mathcal{Y}(Z, Z) = \mathcal{Y}(Z)$, and hence $\mathcal{Y}(Z)$ generates L. It remains to verify that L/Z_1 is absolutely indecomposable. Suppose not. Then for some field extension E/k there exist proper subalgebras $L_1, L_2 \leq L_E$ containing $(Z_1)_E$ such that $L_1/(Z_1)_E \oplus L_2/(Z_1)_E = L_E/(Z_1)_E$. In particular, $[L_1, L_2] \subseteq (Z_1)_E$. Since $M_E/(Z_1)_E$ lies in the derived subalgebra of $L_E/(Z_1)_E$, it cannot contain either of the proper direct summands $L_1/(Z_1)_E$ and $L_2/(Z_1)_E$.

Let $x \in L \setminus M$. If $x \in \mathcal{X}(M)$, then Remark 3.1 implies that $C_{L_E/(M_1)_E}(x \otimes 1) = M_E + E(x \otimes 1)$. There exist $x_1 \in L_1$ and $x_2 \in L_2$ such that $x \otimes 1 = x_1 + x_2 \bmod (Z_1)_E$. If $x_1 \in L_1 \cap M_E$, then clearly $L_1 \subseteq C_{L_E/(M_1)_E}(x \otimes 1)$. However, $L_1 \not\subseteq M_E + E(x \otimes 1) =$

$M + Ex_2$ and hence $x \notin \mathcal{X}(M)$. The case $x_2 \in L_2 \cap M_E$ is treated analogously. If neither x_1 nor x_2 lie in M_E, then $a_1x_1 + a_2x_2 \notin M_E$ for all $a_1, a_2 \in E$; indeed, M_E is a verbal ideal and is thus preserved by all endomorphisms of L_E, in particular by the projections to the components L_1 and L_2. The E-linear span of x_1 and x_2 is contained in $C_{L_E/(M_1)_E}(x \otimes 1)$, so $\dim_E C_{L_E/(M_1)_E}(x \otimes 1) \geq \dim_E M_E + 2$, and again $x \notin \mathcal{X}(M)$. Therefore $\mathcal{X}(M) = \varnothing$, contradicting the assumption that $\mathcal{X}(M)$ generates L. Thus L is Z-rigid by Theorem 3.8. □

Remark 3.10. In some of the examples considered in this paper, the hypotheses of Corollary 3.9 do not apply, whereas those of Theorem 3.8 do; see Sections 4.2 and 4.3. However, if $M = Z$, then observe that $\mathcal{X}(M) \subseteq \mathcal{Y}(M, Z) = \mathcal{Y}(Z)$. Hence, if $\mathcal{X}(M)$ generates L, then $\mathcal{Y}(Z)$ does as well. It is often easier to verify that $\mathcal{X}(M)$ generates L, when this holds, than to verify the hypotheses of Theorem 3.8 directly.

3.2. **Fixed notations and definitions.** Let $\mathcal{L}$ be a nilpotent Lie ring over $\mathbb{Z}$ such that $Z(\mathcal{L}) \leq \gamma_2\mathcal{L}$; thus $\mathcal{L}$ has no abelian direct summands. Let $L = \mathcal{L} \otimes_Z \mathbb{Q}$ be the associated $\mathbb{Q}$-Lie algebra. Fix a prime p and set $\mathcal{L}_p = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Suppose that $\mathcal{L}_p/\gamma_2\mathcal{L}_p$ is torsion-free; this holds for all but finitely many primes p.

The following notation will be used for the rest of the paper. Put $n = \dim_{\mathbb{Q}} L$ and $\overline{n} = \dim_{\mathbb{Q}} L/Z(L)$. Consider a number field K of degree $d = [K : \mathbb{Q}]$ and the semisimple $\mathbb{Q}_p$-algebra $R = K \otimes_{\mathbb{Q}} \mathbb{Q}_p$. If $p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$, where the $\mathfrak{p}_i$ are distinct prime ideals of $\mathcal{O}_K$ of inertia degree $f_i = [\mathcal{O}_K/\mathfrak{p}_i : \mathbb{F}_p]$, then $R = R_1 \times \cdots \times R_r$, where $R_i/\mathbb{Q}_p$ is a field extension of ramification index e_i and inertia degree f_i. Note that $d = \sum_{i=1}^r e_i f_i$. Write $\mathcal{O}_{R_i}$ for the ring of integers of R_i and $q_i = p^{f_i}$ for the cardinality of its residue field. For each $i \in [r]$, let $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,e_if_i})$ be a $\mathbb{Z}_p$-basis of $\mathcal{O}_{R_i}$, and let $\boldsymbol{\alpha}$ be the concatenation of $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r$. Denote the elements of $\boldsymbol{\alpha}$ by $\alpha_1, \ldots, \alpha_d$.

For every $i \in [r]$, there is a natural injection $\iota_i : R_i \hookrightarrow \mathrm{End}_{\mathbb{Q}_p} R_i$ of rings sending $\beta \in R_i$ to the multiplication-by-β map $\alpha \mapsto \alpha\beta$. Since we have fixed a $\mathbb{Q}_p$-basis $\boldsymbol{\alpha}_i$ of R_i, this induces an injection $\iota_i : R_i \hookrightarrow M_{e_if_i}(\mathbb{Q}_p)$. Moreover, since $\boldsymbol{\alpha}_i$ is an integral basis, given $\beta \in R_i$ we see that $\beta \in \mathcal{O}_{R_i}$ if and only if $\iota_i(\beta) \in M_{e_if_i}(\mathbb{Z}_p)$. It is well known that $\det \iota_i(\beta) = N_{R_i/\mathbb{Q}_p}(\beta)$ for all $\beta \in R_i$. For every $m \in \mathbb{N}$ we get a ring monomorphism $M_m(R_i) \hookrightarrow M_{me_if_i}(\mathbb{Q}_p)$; the image of $A \in M_m(R_i)$ is the matrix obtained by replacing each matrix element a_{jk} of A by the corresponding $\iota_i(a_{jk})$. Slightly abusing notation, we also denote this map by ι_i. An exercise in linear algebra shows that $|\det \iota_i(A)|_{\mathbb{Q}_p} = |N_{R_i/\mathbb{Q}_p}(\det A)|_{\mathbb{Q}_p} = |\det A|_{R_i}$ for any $A \in M_m(R_i)$; cf. [20, Theorem 1]. The basis $\boldsymbol{\alpha}_i$ determines an embedding of algebraic groups $\mathrm{Res}_{R_i/\mathbb{Q}_p}\mathbf{GL}_m \subset \mathbf{GL}_{me_if_i}$ over $\mathbb{Q}_p$; the corresponding map on $\mathbb{Q}_p$-points coincides with ι_i.

Let $\mathbf{G} = \mathfrak{Aut}\, L_p$ denote the algebraic automorphism group of L_p and define $\mathbf{J} = \ker(\mathbf{G} \to \mathfrak{Aut}\,(L_p/Z(L_p)))$. For any finite extension $F/\mathbb{Q}_p$, write $\mathbf{G}^+(F)$ for the submonoid of $\mathbf{G}(F) = \mathrm{Aut}_F(L \otimes_{\mathbb{Q}} F)$ consisting of elements that map the $\mathcal{O}_F$-lattice $\mathcal{L} \otimes_{\mathbb{Z}} \mathcal{O}_F$ into itself and $\mathbf{G}(\mathcal{O}_F)$ for the subgroup of elements inducing an automorphism of $\mathcal{L} \otimes_{\mathbb{Z}} \mathcal{O}_F$. As in Assumption 2.1, assume that $\mathbf{G}(\mathbb{Q}_p) = \mathbf{G}(\mathbb{Z}_p)\mathbf{G}^\circ(\mathbb{Q}_p)$; by [13, Lemma 4.1] this holds for almost all primes p. Replacing $\mathbf{G}$ by its connected component, we may suppose that $\mathbf{G}$ is connected. Then $\mathbf{G} = \mathbf{N} \rtimes \mathbf{H}$, where $\mathbf{N}$ is the unipotent radical of $\mathbf{G}$ and $\mathbf{H}$ is reductive. For every algebraic subgroup $\mathbf{S} \leq \mathbf{G}$ set

$\mathbf{S}^+(F) = \mathbf{S}(F) \cap \mathbf{G}^+(F)$ and $\mathbf{S}(\mathcal{O}_F) = \mathbf{S}(F) \cap \mathbf{G}(\mathcal{O}_F)$. Let $\mu_{\mathbf{S}(F)}$ denote the right Haar measure on $\mathbf{S}(F)$, normalized so that $\mu_{\mathbf{S}(F)}(\mathbf{S}(\mathcal{O}_F)) = 1$.

Since $\mathbf{H}$ is reductive, there exists a subspace $U \subset L_p$ such that $L_p = U \oplus Z(L_p)$ and such that $U \otimes_{\mathbb{Q}_p} F$ is stable under the action of $\mathbf{H}(F)$ for any field extension $F/\mathbb{Q}_p$; cf. [26, Theorem 22.138]. Moreover, since we assumed $Z(\mathcal{L}) \leq \gamma_2\mathcal{L}$, there is a subspace $U' \subset U$, possibly trivial, such that $U' \oplus Z(L_p) = \gamma_2 L_p$. By our hypotheses on $\mathcal{L}_p$, we may fix a $\mathbb{Z}_p$-basis $b_1, \dots, b_n$ of $\mathcal{L}_p$ that, viewed as a $\mathbb{Q}_p$-basis of L_p, consists of a concatenation of a lift of a basis of U/U', a basis of U', and a basis of $Z(L_p)$. This choice of basis determines an embedding $\mathbf{G} \subset \mathbf{GL}_n$. In the examples of Section 4 below, there will be a $\mathbb{Z}$-basis of $\mathcal{L}$ that, for all primes p, induces a $\mathbb{Z}_p$-basis of $\mathcal{L}_p$ satisfying our hypotheses.

3.3. **Consequences of rigidity.** Suppose, in addition to the hypotheses of the previous section, that L_p is $Z(L_p)$-rigid. A consequence of rigidity, in conjunction with the interpretation of pro-isomorphic zeta functions as p-adic integrals in Proposition 1.4, is the existence of a fine Euler decomposition. By this we mean that each $\zeta^{\wedge}_{L\otimes_{\mathbb{Z}}\mathcal{O}_K,p}(s)$ decomposes naturally into a product indexed by the primes of K dividing p. To make this precise, set $U_1^F = U \otimes_{\mathbb{Q}} F$ and $U_2^F = Z(L_p \otimes_{\mathbb{Q}_p} F)$ for any finite extension $F/\mathbb{Q}_p$. Then $L_p \otimes_{\mathbb{Q}_p} F = U_1^F \oplus U_2^F$, and each component is stable under the action of $\mathbf{H}(F)$. Note that $\mathbf{N}(F)$ need not act trivially on V^F/U_2^F, where V^F is the underlying F-vector space of $L_p \otimes_{\mathbb{Q}_p} F$, so we are not necessarily in the precise setup of Section 2.2. Nevertheless, the map $(\psi_2')^F : \mathbf{G}(F) \to \mathrm{Aut}(V^F/U_2^F)$ can be defined as in Section 2.2; as a consequence of the assumption that $Z(L_p) \leq \gamma_2 L_p$, every element of $\ker(\psi_2')^F$ is unipotent. Hence $N_2^F = \mathbf{N}(F) \cap \ker(\psi_2')^F = \mathbf{J}(F)$ and, as in Section 2.2, we obtain a map $\psi_2^F : \mathbf{G}(F)/N_2^F \to \mathrm{Aut}_F(V^F/U_2^F) \subset \mathrm{GL}_{\overline{n}}(F)$, where the final embedding is determined by our choice of basis of L.

Define $\overline{N}^F = \mathbf{N}(F)/\mathbf{J}(F)$, and let $\mu_{\overline{N}^F}$ be the right Haar measure on $\overline{N}^F$ normalized so that $\mu_{\overline{N}^F}((\psi_2^F)^{-1}(\psi_2^F(\overline{N}^F) \cap M_{\overline{n}}(\mathcal{O}_F))) = 1$. Finally, define a function $\widetilde{\theta}^F : \mathbf{H}(F) \to \mathbb{R}$ as follows. If $h \in \mathbf{H}(F)$, then we set

$$\widetilde{\theta}^F(h) = \mu_{\overline{N}^F}(\{\nu \in \overline{N}^F : \psi_2^F(\nu h) \in M_{\overline{n}}(\mathcal{O}_F)\}).$$

Remark 3.11. It is easy to see that if $\widetilde{\theta}^F(h) \neq 0$, then necessarily $h \in \mathbf{H}^+(F)$. Moreover, if $\mathbf{N}(F)$ acts trivially on U_1^F, then $\widetilde{\theta}^F(h) = 1$ for all $h \in \mathbf{H}^+(F)$.

Any $h \in \mathbf{H}(F)$ maps the center of $L_p \otimes_{\mathbb{Q}_p} F$ onto itself and thus induces an element $\varepsilon^F(h) \in \mathrm{Aut}_F Z(L_p \otimes_{\mathbb{Q}_p} F) \simeq \mathrm{GL}_{n-\overline{n}}(F)$. For every $i \in [r]$, write ε_i for the map ε^{R_i}.

Proposition 3.12. *Suppose that $\mathcal{L}$ is a Lie ring such that $Z(\mathcal{L}) \leq \gamma_2\mathcal{L}$ and that p is a prime such that $\mathcal{L}_p/\gamma_2\mathcal{L}_p$ is torsion-free and L_p is $Z(L_p)$-rigid. Here $\mathcal{L}_p = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $L_p = \mathcal{L} \otimes_Z \mathbb{Q}_p$. Let K be a number field. Set $d = [K : \mathbb{Q}]$ and $n' = \dim_{\mathbb{Q}} L/\gamma_2 L$. Then*

$$\zeta^{\wedge}_{\mathcal{L}\otimes_{\mathbb{Z}}\mathcal{O}_K,p}(s) = \prod_{i=1}^{r} \int_{\mathbf{H}^+(R_i)} \widetilde{\theta}^{R_i}(h_i) |\det \varepsilon_i(h_i)|_{R_i}^{-dn'} |\det h_i|_{R_i}^s d\mu_{\mathbf{H}(R_i)}(h_i).$$

Proof. For brevity, set $\mathbf{\Gamma} = (\mathfrak{Aut}\,(L_p \otimes_{\mathbb{Q}_p} R))^\circ$, where $L_p \otimes_{\mathbb{Q}_p} R$ is viewed as a $\mathbb{Q}_p$-Lie algebra. Let $\Gamma = \mathbf{\Gamma}(\mathbb{Q}_p)$, and let $\Gamma^+ \subset \Gamma$ be the submonoid consisting of $\mathbb{Q}_p$-automorphisms of $L_p \otimes_{\mathbb{Q}_p} R$ that preserve the lattice $\mathcal{L}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_K$. Let $\mathbf{\Gamma} = \mathbf{N}_{\mathbf{\Gamma}} \rtimes \mathbf{H}_{\mathbf{\Gamma}}$, where

$\mathbf{N_\Gamma}$ is the unipotent radical of $\mathbf{\Gamma}$ and $\mathbf{H_\Gamma}$ is reductive. This induces a decomposition $\Gamma = N_\Gamma \rtimes H_\Gamma$, where $N_\Gamma = \mathbf{N_\Gamma}(\mathbb{Q}_p)$ and $H_\Gamma = \mathbf{H_\Gamma}(\mathbb{Q}_p)$. We have an embedding $\Gamma \subseteq \mathrm{GL}_{dn}(\mathbb{Q}_p)$ determined by the $\mathbb{Q}_p$-basis

$$(\alpha_1 b_1, \dots, \alpha_d b_1, \alpha_1 b_2, \dots, \alpha_d b_2, \dots, \alpha_1 b_n, \dots, \alpha_d b_n)$$

of $L_p \otimes_{\mathbb{Q}_p} R$. By Proposition 1.4,

$$\zeta^{\wedge}_{\mathcal{L}\otimes_{\mathbb{Z}}\mathcal{O}_K,p}(s) = \int_\Gamma \chi_{\Gamma^+}(g)|\det g|^{-s}_{\mathbb{Q}_p} d\mu_\Gamma(g) = \int_{H_\Gamma} f(h)|\det h|^{-s}_{\mathbb{Q}_p} d\mu_{H_\Gamma}(h), \tag{12}$$

where $f(h) = \mu_{N_\Gamma}(\{\nu \in N_\Gamma : \nu h \in \Gamma^+\})$, while μ_{H_Γ} and μ_{N_Γ} are right Haar measures as above. Observe that $f(h) = 0$ whenever $h \notin H_\Gamma \cap \Gamma^+$. Let $J_\Gamma \leq N_\Gamma$ be the kernel of the map

$$\psi'_\Gamma : \Gamma \to \mathrm{Aut}_{\mathbb{Q}_p}((L_p \otimes_{\mathbb{Q}_p} R)/(Z(L_p) \otimes_{\mathbb{Q}_p} R)) \subseteq \mathrm{GL}_{d\overline{n}}(\mathbb{Q}_p).$$

This induces a map $\psi_\Gamma : \Gamma/J_\Gamma \to \mathrm{GL}_{d\overline{n}}(\mathbb{Q}_p)$. Let μ_{J_Γ} and μ_{N_Γ/J_Γ} be right Haar measures on J_Γ and N_Γ/J_Γ, respectively, normalized so that $\mu_{J_\Gamma}(J_\Gamma \cap M_{dn}(\mathbb{Z}_p)) = 1$ and $\mu_{N_\Gamma/J_\Gamma}(\psi_\Gamma^{-1}(\psi_\Gamma(N_\Gamma/J_\Gamma) \cap M_{d\overline{n}}(\mathbb{Z}_p))) = 1$. Clearly $\mu_{N_\Gamma} = \mu_{N_\Gamma/J_\Gamma} \cdot \mu_{J_\Gamma}$.

Every element of J_Γ acts trivially on $\gamma_2(L_p \otimes_{\mathbb{Q}_p} R)$ and therefore

$$J_\Gamma = \left\{ \begin{pmatrix} I_{dn'} & 0 & B \\ 0 & I_{d(\overline{n}-n')} & 0 \\ 0 & 0 & I_{d(n-\overline{n})} \end{pmatrix} : B \in M_{dn', d(n-\overline{n})}(\mathbb{Q}_p) \right\}. \tag{13}$$

Fix elements $h \in H_\Gamma$ and $\nu \in N_\Gamma$. For any $\gamma \in \Gamma$, let $\varepsilon(\gamma)$ be the automorphism of $Z(L_p \otimes_{\mathbb{Q}_p} R)$ induced by γ. It follows from the $Z(L_p)$-rigidity of L_p that we may take $H_\Gamma = \prod_{i=1}^r \iota_i(\mathbf{H}(R_i))$. In particular, h corresponds to an r-tuple $h = (\iota_1(h_1), \dots, \iota_r(h_r))$, where $h_i \in \mathbf{H}(R_i)$ for every $i \in [r]$. It is easy to see from (13) that, for $j \in J_\Gamma$, the condition $j\nu h \in \Gamma^+$ amounts to dn' independent conditions on the rows of B. Hence

$$\mu_{J_\Gamma}(\{j \in J_\Gamma : j\nu h \in \Gamma^+\}) = |\det \varepsilon(\nu h)|^{-dn'}_{\mathbb{Q}_p} = |\det \varepsilon(h)|^{-dn'}_{\mathbb{Q}_p} = \prod_{i=1}^r |\det \varepsilon^{R_i}(h_i)|^{-dn'}_{R_i}. \tag{14}$$

Note that this quantity depends only on h. Again by rigidity, we have $N_\Gamma/J_\Gamma = \prod_{i=1}^r \overline{N}^{R_i}$. Moreover, since we chose each $\boldsymbol{\alpha}_i$ to be an integral basis of R_i, it follows that for any $g_i \in \mathbf{G}(R_i)$ we have $\iota_i(g_i) \in M_{e_i f_i n}(\mathbb{Z}_p)$ if and only if $g_i \in \mathbf{G}(\mathcal{O}_{R_i})$. Thus μ_{N_Γ/J_Γ} is the product of the measures $\mu_{\overline{N}^{R_i}}$ defined earlier. Recall that $\mu_{N_\Gamma} = \mu_{N_\Gamma/J_\Gamma} \cdot \mu_{J_\Gamma}$. Consequently,

$$f(h) = \left(\prod_{i=1}^r |\det \varepsilon_i(h_i)|^{-dn'}_{R_i} \right) \mu_{N_\Gamma/J_\Gamma}(\{\overline{\nu} \in N_\Gamma/J_\Gamma : \psi_\Gamma(\overline{\nu}h) \in M_{d\overline{n}}(\mathbb{Z}_p)\}) =$$

$$\prod_{i=1}^r |\det \varepsilon_i(h_i)|^{-dn'}_{R_i} \mu_{\overline{N}^{R_i}}\{\overline{\nu}_i \in \overline{N}^{R_i} : \psi_2^{R_i}(\overline{\nu}_i h_i \in M_{\overline{n}}(\mathcal{O}_{R_i}))\} = \prod_{i=1}^r |\det \varepsilon_i(h_i)|^{-dn'}_{R_i} \widetilde{\theta}^{R_i}(h_i).$$

From this it is obvious that the rightmost integral of (12) splits into a product as claimed, noting that the integrand is supported on $H_\Gamma \cap \Gamma^+ = \prod_{i=1}^r \iota_i(\mathbf{H}^+(R_i))$. □

Suppose further that Assumption 2.2 holds for the action of $\mathbf{G}(\mathbb{Q}_p)$ on the underlying $\mathbb{Q}_p$-vector space V of L_p and for some decomposition $V = U_1 \oplus \cdots \oplus U_t$ compatible with the basis $(b_1, \dots, b_n)$. Moreover, suppose that $V_t = U_t = Z(L_p)$ and there exists some $t' \le t$ such that $V_{t'} = \gamma_2 L_p$. Since we have assumed L_p to be $Z(L_p)$-rigid, it follows that, for any finite extension $F/\mathbb{Q}_p$, the action of $\mathbf{G}(F)$ on

$$L_p \otimes_{\mathbb{Q}_p} F = (U_1 \otimes_{\mathbb{Q}_p} F) \oplus \cdots \oplus (U_t \otimes_{\mathbb{Q}_p} F)$$

satisfies Assumption 2.2.

Remark 3.13. Let $\mathcal{L}$ be a Lie ring over $\mathbb{Z}$ of rank n and let $\mathbf{G} = \mathfrak{Aut}\, L$. By the proof of [13, Lemma 4.1], $\mathbf{G}(F)$ satisfies Assumption 2.1 for almost all primes p and any finite extension $F/\mathbb{Q}_p$. Similarly, for almost all primes p, Assumption 2.2 holds for L_p; if in addition L_p is $Z(L_p)$-rigid, Assumption 2.2 holds for any base extension $L_p \otimes_{\mathbb{Q}_p} F$, where $F/\mathbb{Q}_p$ is finite. Indeed, it is easy to see using Lemmas 4.2 and 4.3 of [13] that one can choose a $\mathbb{Z}$-basis $(b_1, \dots, b_n)$ of $\mathcal{L}$ such that, for almost all primes p, the induced basis of $\mathcal{L}_p$ satisfies the hypotheses in the previous paragraph. Note that the crucial step in the proof of [13, Lemma 4.3] is the claim that the action of $\mathfrak{N}(\mathbb{Q})$ on L has a non-zero fixed point, where $\mathfrak{N}$ is the unipotent radical of $\mathfrak{Aut}\, L$; this is true because $\mathfrak{N}$ is trigonalizable over $\mathbb{Q}$ [7, Theorem 15.4(ii)], which in turn is a consequence of the Lie-Kolchin theorem.

By contrast, and contrary to [13, Corollary 4.5], there may be no decomposition of L_p affording Assumption 2.3 at any prime; see [2, p. 6] for an example.

We require a stronger version of Assumption 2.3. Informally, we assume that the lifting condition for the action of $\mathbf{G}(\mathbb{Q}_p)$ on L_p is realized by a polynomial map defined over $\mathbb{Z}_p$; this condition ensures that Assumption 2.3 holds not only for L_p, but for the base extension $L_p \otimes_{\mathbb{Q}_p} F$, where $F/\mathbb{Q}_p$ is any finite extension. Recall the maps ψ_i and the notation $d_i = \dim_{\mathbb{Q}_p}(U_1 \oplus \cdots \oplus U_i)$ from Section 2.2. Observe that if $t' \le 3$, then ψ_i is injective for all $i \in [2, t-1]$. If $i \in [2, t-1]$ and $\kappa_i : \mathbf{GL}_{d_i} \to \mathbf{GL}_n$ is a morphism of algebraic varieties, defined over $\mathbb{Z}_p$, write

$$\kappa_i^{\mathbb{Q}_p} : \mathrm{Aut}_{\mathbb{Q}_p}(V/V_{i+1}) \simeq \mathrm{GL}_{d_i}(\mathbb{Q}_p) \to \mathrm{GL}_n(\mathbb{Q}_p) \simeq \mathrm{Aut}_{\mathbb{Q}_p}(V)$$

for the induced map on $\mathbb{Q}_p$-points.

Assumption 3.14. We assume for all $i \in [2, t-1]$ that ψ_i is injective and that there is a morphism $\kappa_i : \mathbf{GL}_{d_i} \to \mathbf{GL}_n$, defined over $\mathbb{Z}_p$, such that if $\overline{g} \in NH^+/N_i \cap (G/N_i)^+$, then the image $\gamma = \kappa_i^{\mathbb{Q}_p}(\psi_i(\overline{g})) \in \mathrm{GL}_n(\mathbb{Q}_p)$ lies in $\mathbf{G}^+(\mathbb{Q}_p)$ and satisfies $\overline{g} = \gamma N_i$.

Assumption 3.14 implies, for any finite extension $F/\mathbb{Q}_p$, that the map κ_i^F on F-points realizes Assumption 2.3 for the action of $\mathbf{G}(F)$ on $L_p \otimes_{\mathbb{Q}_p} F$. Hence, under the preceding series of assumptions, the setup of Section 2.2 applies to the action of $\mathrm{Aut}_F((L_p \otimes_{\mathbb{Q}_p} F)/Z(L_p \otimes_{\mathbb{Q}_p} F))$ on $(L_p \otimes_{\mathbb{Q}_p} F)/Z(L_p \otimes_{\mathbb{Q}_p} F)$. By Proposition 2.4,

$$\widetilde{\theta}^F(h) = \prod_{i=1}^{t-2} \theta_i^F(h) \tag{15}$$

for all $h \in \overline{\mathbf{H}}(F) \simeq \mathbf{H}(F)$, where $\overline{\mathbf{H}}$ is the reductive part of $\mathfrak{Aut}\, L_p/Z(L_p)$. Many of the examples considered in Section 4 rely on the following formula. Its meaning is that, under the hypotheses above, computing $\zeta^{\wedge}_{\mathcal{L} \otimes_{\mathbb{Z}} \mathcal{O}_K, p}(s)$ is "no harder" than finding $\zeta^{\wedge}_{\mathcal{L}, p}(s)$.

Corollary 3.15. *Suppose that $Z(\mathcal{L}) \leq \gamma_2\mathcal{L}$, that $\mathcal{L}/\gamma_2\mathcal{L}$ is torsion-free, and that $L_p = L \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is $Z(L_p)$-rigid. Suppose that Assumptions 2.1, 2.2, and 3.14 hold. Let K be a number field of degree d, and put $n' = \dim_{\mathbb{Q}} L/\gamma_2 L$. Then*

$$\zeta^{\wedge}_{\mathcal{L}\otimes_{\mathbb{Z}}\mathcal{O}_K,p}(s) = \prod_{i=1}^{r} \int_{\mathbf{H}^+(R_i)} \left(\prod_{j=1}^{t-2} \theta_j^{R_i}(h_i) \right) |\det \varepsilon_i(h_i)|_{R_i}^{-dn'} |\det h_i|_{R_i}^{s} d\mu_{\mathbf{H}(R_i)}(h_i) =$$
$$\prod_{i=1}^{r} \int_{\mathbf{H}^+(R_i)} \left(\prod_{j=1}^{t-2} \theta_j^{R_i}(h_i) \right) (\theta_{t-1}^{R_i}(h_i))^d |\det h_i|_{R_i}^{s} d\mu_{\mathbf{H}(R_i)}(h_i).$$

Proof. The first equality is immediate from Proposition 3.12 and (15). We prove the second by showing that $\theta_{t-1}^F(h) = |\det \varepsilon^F(h)|_F^{-n'}$ for any finite extension $F/\mathbb{Q}_p$ and any $h \in \mathbf{H}^+(F)$. In the notation of (6), we have $N_{t-1} = \mathbf{J}(F)$ and $N_t = (0)$. Thus,

$$\theta_{t-1}^F(h) = \mu_{\mathbf{J}(F)}(\{j \in \mathbf{J}(F) : \tau(h)(j) \in M_{\overline{n},n-\overline{n}}(\mathcal{O}_F)\}) =$$
$$\mu_{\mathbf{J}(F)}(\{j \in \mathbf{J}(F) : jh \in \mathbf{G}^+(F)\}) = |\det \varepsilon^F(h)|_F^{-n'}.$$

Here the second equality holds because $h \in \mathbf{H}^+(F)$ and the third is analogous to (14) in the proof of Proposition 3.12. □

Remark 3.16. Our arguments in this section follow the path laid out by du Sautoy and Lubotzky [13]. Assumptions 2.1 and 2.2 appear in [13], as does Assumption 3.14 implicitly. In [13, §6] the authors anticipate our line of inquiry by considering integrals of the form (12) where the reductive group $\mathbf{H}$ over the number field E can be identified with the restriction of scalars of an algebraic matrix group defined over a finite extension E'/E. If $\mathcal{L}$ is a nilpotent Lie ring such that $Z(\mathcal{L}) \leq \gamma_2\mathcal{L}$ and $L = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Q}$ is $Z(L)$-rigid, and if K is a number field, then the situation of [13, §6] is obtained, with $E'/E = K/\mathbb{Q}$, in the computation of $\zeta^{\wedge}_{\mathcal{L}\otimes_{\mathbb{Z}}\mathcal{O}_K,p}$ at almost all p. Under further hypotheses (for instance, that the integrand of (12) is a character) it is shown [13, Theorem 6.9] that the local pro-isomorphic zeta functions $\zeta^{\wedge}_{\mathcal{L}\otimes_{\mathbb{Z}}\mathcal{O}_K,p}$ satisfy functional equations for almost all p.

By emphasizing the notion of rigidity and proving the sufficient condition of Theorem 3.8, we provide a way of identifying Lie rings $\mathcal{L}$ to which the framework of [13, §6] applies. We show how the local zeta functions $\zeta^{\wedge}_{\mathcal{L}\otimes_{\mathbb{Z}}\mathcal{O}_K,p}(s)$ may be determined by a uniform calculation that depends only mildly on the number field K. We do not make further assumptions about the integrand of (12), and indeed our method applies to Lie rings $\mathcal{L}$ such that $\zeta^{\wedge}_{\mathcal{L}\otimes_{\mathbb{Z}}\mathcal{O}_K,p}(s)$ does not satisfy a functional equation for any prime p; see Section 4.6.

3.4. **A non-rigid example.** In this section we exhibit a family of indecomposable class-two nilpotent Lie algebras that are not rigid over their centers. This is the first explicit such example, so far as we are aware.

Let k be any field, and let $\alpha \in k$ be such that the polynomial $x^2 - \alpha$ has no roots in k. Consider the k-Lie algebra L of nilpotency class two, defined as follows:

$$L = \langle x_1, x_2, x_3, x_4, z_1, z_2 \rangle_k, \tag{16}$$

where $\gamma_2 L = \langle z_1, z_2\rangle_k$ is the center, and the remaining generators satisfy the relations

$$\begin{array}{lll} [x_1,x_2]=z_1 & [x_1,x_3]=z_2 & [x_1,x_4]=0 \\ [x_3,x_4]=\alpha z_1 & [x_2,x_4]=z_2 & [x_2,x_3]=0. \end{array}$$

This family generalizes Scheuneman's construction [31, Proposition 1] of Lie algebras that are non-isomorphic over $\mathbb{Q}$ but become isomorphic over a suitable quadratic number field; we are grateful to Boris Kunyavskii for directing our attention to [31].

We will mention two other descriptions of L. Let $H = \langle x, y, z\rangle_k$ be the Heisenberg Lie algebra over k, in which $[x, y] = z$ and $Z(H) = \langle z\rangle_k$; see Remark 4.12. Put $K = k(\sqrt{\alpha})$. Then the map $\varphi : L \to H \otimes_k K$ given by

$$(x_1,x_2,x_3,x_4,z_1,z_2) \mapsto (x\otimes 1, y\otimes 1, y\otimes -\sqrt{(\alpha)}, x\otimes\sqrt{\alpha}, z\otimes 1, z\otimes -\sqrt{\alpha})$$

is an isomorphism of k-algebras. Furthermore, if $k = \mathbb{Q}$ then $L \simeq \mathcal{L}\otimes_{\mathbb{Z}}\mathbb{Q}$, where $\mathcal{L}$ is the Lie ring $\mathcal{L}(G)$ associated, via the correspondence of (4), to the nilpotent group G arising from the irreducible polynomial $x^2 - \alpha \in \mathbb{Z}[x]$ via the construction of [15, Theorem 6.3]; see also [4].

Lemma 3.17. *Let $y \in L \setminus Z(L)$. Then* $\dim_k C_L(y) = 4$.

Proof. It suffices to verify the claim for elements of the form $y = b_1x_1+b_2x_2+b_3x_3+b_4x_4$, with the b_i not all zero. With respect to the basis of (16) and the basis (z_1, z_2) of $\gamma_2 L$, the k-linear operator $\operatorname{Ad} y : L \to \gamma_2 L$ corresponds to the matrix

$$\begin{pmatrix} b_2 & -b_1 & \alpha b_4 & -\alpha b_3 & 0 & 0 \\ b_3 & b_4 & -b_1 & -b_2 & 0 & 0 \end{pmatrix}^t.$$

Since we have assumed that k does not contain a square root of α, it is readily seen that this matrix has rank two. Thus $C_L(y) = \ker \operatorname{Ad} y$ is four-dimensional over k. □

Corollary 3.18. *The k-Lie algebra L is indecomposable.*

Proof. Suppose that there is a non-trivial decomposition $L = L_1 \oplus L_2$. The derived subalgebra of L is equal to its center, so L has no abelian direct summands. Since $\dim_k L = 6$, while L_1 and L_2 are nilpotent and non-abelian, we have $\dim_k L_1 = \dim_k L_2 = 3$. Let $y \in L_1 \setminus Z(L_1)$. Then L_2, $Z(L_1)$, and y all commute with y. Thus $\dim_k C_L(y) \geq 5$, contradicting Lemma 3.17. □

Proposition 3.19. *Suppose that* $\operatorname{char} k \neq 2$. *The k-algebra L is not $Z(L)$-rigid.*

Proof. Let $K = k(\sqrt{\alpha})$ as above and define the following K-subalgebras of $L \otimes_k K$:

$$H_1 = \langle \sqrt{\alpha}x_1 + x_4, \sqrt{\alpha}x_2 - x_3, \sqrt{\alpha}z_1 - z_2\rangle_K$$
$$H_2 = \langle \sqrt{\alpha}x_1 - x_4, \sqrt{\alpha}x_2 + x_3, \sqrt{\alpha}z_1 + z_2\rangle_K$$

One easily verifies that $L \otimes_k K = H_1 \oplus H_2$ and that H_1 and H_2 are each isomorphic to $H \otimes_k K$ as K-algebras. Let ψ be a K-linear automorphism of $H \otimes_k K$, and let τ be the non-trivial element of $\operatorname{Gal}(K/k)$. Let $\varphi \in \operatorname{Aut}_k(L \otimes_k K)$ be the automorphism acting as ψ on H_1 and as $\tau \circ \psi$ on H_2. More explicitly,

$$\begin{array}{lll} \varphi(x_1 \otimes 1) = x_1 \otimes 1 & \varphi(x_3 \otimes 1) = x_2 \otimes -\sqrt{\alpha} & \varphi(z_1 \otimes 1) = z_1 \otimes 1 \\ \varphi(x_1 \otimes \sqrt{\alpha}) = x_4 \otimes 1 & \varphi(x_3 \otimes \sqrt{\alpha}) = x_3 \otimes \sqrt{\alpha} & \varphi(z_1 \otimes \sqrt{\alpha}) = z_2 \otimes -1 \\ \varphi(x_2 \otimes 1) = x_2 \otimes 1 & \varphi(x_4 \otimes 1) = x_1 \otimes \sqrt{\alpha} & \varphi(z_2 \otimes 1) = z_1 \otimes -\sqrt{\alpha} \\ \varphi(x_2 \otimes \sqrt{\alpha}) = x_3 \otimes -1 & \varphi(x_4 \otimes \sqrt{\alpha}) = x_4 \otimes \sqrt{\alpha} & \varphi(z_2 \otimes \sqrt{\alpha}) = z_2 \otimes \sqrt{\alpha}. \end{array}$$

The automorphism of $L/Z(L)$ induced by φ is neither K-linear nor τ-semilinear; indeed, $\varphi(x_4 \otimes 1)$ is not a K-scalar multiple of $x_4 \otimes 1$. Thus L is not $Z(L)$-rigid. □

4. Calculations for base extensions of Lie rings

4.1. **Free nilpotent Lie rings.** Let $\mathcal{F}_{c,g}$ be the free nilpotent Lie ring over $\mathbb{Z}$ of nilpotency class c generated by g elements. The pro-isomorphic zeta functions of $\mathcal{F}_{c,g}$ and its base extensions were determined by Grunewald, Segal, and Smith [16, Theorem 7.1]. We compute them here as a first illustration of our method, noting that our argument is essentially equivalent to that of [16], although expressed in somewhat different terms.

Fix a decomposition $F_{c,g} = \mathcal{F}_{c,g} \otimes_{\mathbb{Z}} \mathbb{Q} = W_1 \oplus \cdots \oplus W_c$, in the notation of Section 2.2, such that $W_i \oplus \cdots \oplus W_c = \gamma_i F_{c,g}$ for all $i \in [c]$; see [17, §5] for constructions. In particular, our chosen basis of W_1 is a collection $x_1, \ldots, x_g$ of elements of $\mathcal{F}_{c,g}$ that generate it as a Lie ring. For every $i \in [c]$, let m_i denote the $\mathbb{Z}$-rank of $\gamma_i \mathcal{F}_{c,g}/\gamma_{i+1}\mathcal{F}_{c,g}$. By a result of Witt [17, Theorem 5.7], we have

$$m_i = \frac{1}{i} \sum_{j|i} \mu(j) g^{i/j},$$

where μ is the Möbius function. Note that $g|m_i$ for all $i \in [c]$. Let $E/\mathbb{Q}$ be any extension of fields. By freeness, any E-linear map $\varphi : \langle x_1, \ldots, x_g \rangle_E \to F_{c,g} \otimes_{\mathbb{Q}} E$ inducing an isomorphism of vector spaces $\langle x_1, \ldots, x_g \rangle_E \simeq (F_{c,g} \otimes_{\mathbb{Q}} E)/\gamma_2(F_{c,g} \otimes_{\mathbb{Q}} E)$ can be extended to an E-automorphism $\varphi \in (\mathfrak{Aut}\, F_{c,g})(E)$. Hence, fixing bases of the subspaces W_i, we get an embedding $\mathfrak{Aut}\, F_{c,g} \hookrightarrow \mathrm{GL}_{\sum_{i=1}^c m_i}$ whose image consists of block upper-triangular matrices of the form

$$\begin{pmatrix} A & B_2 & \cdots & B_c \\ & A^{(2)} & * & * \\ & & \ddots & \vdots \\ & & & A^{(c)} \end{pmatrix},$$

where $A \in \mathrm{GL}_g$ and $B_i \in M_{g,m_i}$ for $i \in [2, c]$ are arbitrary and determine all the other blocks. Every diagonal block $A^{(i)} \in \mathrm{GL}_{m_i}$ corresponds to an automorphism of $\gamma_i F_{c,g}/\gamma_{i+1} F_{c,g}$ and depends only on A, whereas the off-diagonal blocks depend also on $B_2, \ldots, B_c$. Observe that $\det A^{(i)} = (\det A)^{im_i/g}$ for all $i \in [2, c]$. Indeed, the map $A \mapsto \det A^{(i)}$ is a character of GL_g and thus is some power of the determinant. To find the power, consider a diagonal matrix $A = \mathrm{diag}(t_1, \ldots, t_g)$. Now $\gamma_i F_{c,g}/\gamma_{i+1} F_{c,g}$ is spanned by projections of elements of the form $[x_{j_1}, [x_{j_2}, \cdots, [x_{j_{i-1}}, x_{j_i}] \cdots]]$. With respect to such a basis, the induced automorphism of $\gamma_i F/\gamma_{i+1} F$ is represented by a diagonal matrix with products $t_{j_1} \cdots t_{j_i}$ on the diagonal. Thus its determinant is a product of im_i diagonal elements of A, and our claim follows.

Set $Z = Z(F_{c,g}) = \gamma_c F_{c,g}$. Then $F_{c,g}$ is Z-rigid by [32, Theorem 1], which is proved by means of Segal's rigidity criterion (quoted here as Corollary 3.9); indeed, [32] appears to have been motivated by this example. It is clear from the structure of $\mathfrak{Aut}\, F_{c,g}$ that, for any prime p, the assumptions of Section 2.2 are satisfied for the decomposition of $L_p = \mathcal{F}_{c,g} \otimes_{\mathbb{Z}} \mathbb{Q}_p$ given by $U_i = W_i \otimes_{\mathbb{Q}} \mathbb{Q}_p$. The lifting condition holds because, for any $i \in [2, c-1]$, specifying a class $\overline{g}$ in the notation of Assumption 2.3 amounts to choosing $A, B_2, \ldots, B_i$. Thus we may take γ to be the element of $(\mathfrak{Aut}\, F_{c,g})(\mathbb{Q}_p)$ determined by these data and arbitrary matrices $B_{i+1}, \ldots, B_c$ with elements in $\mathbb{Z}_p$. In particular we may take $B_{i+1}, \ldots, B_c$ to be zero matrices. Hence the hypotheses of Corollary 3.15 hold. We are now in a position to recover [16, Theorem 7.1].

Theorem 4.1 (Grunewald-Segal-Smith)**.** *Let K be a number field and p a prime. Then*

$$\zeta^{\wedge}_{\mathcal{F}_{c,g}\otimes\mathcal{O}_K,p}(s) = \prod_{i=1}^{r} \frac{1}{\prod_{j=0}^{g-1}(1 - q_i^{\beta+j-\alpha s})}, \tag{17}$$

where $\alpha = \frac{1}{g}\sum_{j=1}^{g} jm_j$ and $\beta = 2m_2 + \cdots + (c-1)m_{c-1} + dcm_c$. In particular, the global pro-isomorphic zeta function is

$$\zeta^{\wedge}_{\mathcal{F}_{c,g}\otimes\mathcal{O}_K}(s) = \prod_{j=0}^{g-1} \zeta_K(\alpha s - \beta - j),$$

where $\zeta_K(s)$ is the Dedekind zeta function of K; its abscissa of convergence is $\alpha^{\wedge}_{\mathcal{F}_{c,g}\otimes\mathcal{O}_K} = \frac{\beta+g}{\alpha}$. The local zeta factors satisfy the functional equation

$$\zeta^{\wedge}_{\mathcal{F}_{c,g}\otimes\mathcal{O}_K,p}(s)|_{p\to p^{-1}} = (-1)^g \left(\prod_{i=1}^{r} q_i\right)^{g\beta+\binom{g}{2}-g\alpha s} \zeta^{\wedge}_{\mathcal{F}_{c,g}\otimes\mathcal{O}_K,p}(s).$$

Proof. Clearly an isomorphism $\mathbf{H} \simeq \mathrm{GL}_n$ of algebraic groups is afforded by the map

$$h = \mathrm{diag}(A, A^{(2)}, \ldots, A^{(c)}) \mapsto A.$$

Observe that $\det h = (\det A)^{\alpha}$. Moreover, for any finite extension $F/\mathbb{Q}_p$ and any $i \in [c-1]$, it is easy to see that if $h \in \mathbf{H}^+(F)$, then (6) amounts to

$$\theta_i^F(h) = \mu(\{B_{i+1} \in M_{g,m_{i+1}}(F) : B_{i+1}A^{(i+1)} \in M_{g,m_{i+1}}(\mathcal{O}_F)\}),$$

where $\mu(M_{g,m_{i+1}}(\mathcal{O}_F)) = 1$. As there are g independent conditions on each of the rows of B_{i+1}, we get $\theta_i^F(h) = |\det A^{(i+1)}|_F^{-g} = |\det A|_F^{-(i+1)m_{i+1}}$. We have all the ingredients to apply Corollary 3.15:

$$\zeta^{\wedge}_{\mathcal{F}_{c,g}\otimes\mathcal{O}_K,p}(s) = \prod_{i=1}^{r} \int_{\mathrm{GL}_g^+(R_i)} |\det A|_{R_i}^{\alpha s-\beta} d\mu_{\mathrm{GL}_g(R_i)}(A).$$

Now (17) follows from Example 2.6, and the functional equation is clear. The abscissa of convergence of $\zeta_K(s)$ is 1, and the claim regarding $\alpha^{\wedge}_{\mathcal{F}_{c,g}\otimes\mathcal{O}_K}$ follows. □

Remark 4.2. Observe that $\mathrm{rk}_{\mathbb{Z}}\, \gamma_i(\mathcal{F}_{c,g} \otimes_{\mathbb{Z}} \mathcal{O}_K) = d(m_i + m_{i+1} + \cdots + m_c)$ for every $i \in [c]$. Thus, $\sum_{i=1}^{g} \mathrm{rk}_{\mathbb{Z}}\, \gamma_i(\mathcal{F}_{c,g} \otimes_{\mathbb{Z}} \mathcal{O}_K) = dg\alpha$. Note that $\prod_{i=1}^{r} q_i = p^{f_1 + \cdots + f_r}$, and this is equal to p^d if and only if p is unramified in K. The Lie rings $\mathcal{F}_{c,g} \otimes_{\mathbb{Z}} \mathcal{O}_K$ are clearly graded in the sense of Conjecture 1.3, and the conjecture holds. The finitely many excluded primes are precisely the ones that ramify in K. One verifies similarly that Conjecture 1.3 holds for the graded Lie rings considered in Sections 4.2, 4.3, and 4.4; in each case the excluded primes are the ones that ramify in K.

For comparison, we briefly survey what is known about the ideal and subring zeta functions of the Lie rings $\mathcal{F}_{c,g} \otimes_{\mathbb{Z}} \mathcal{O}_K$. In contrast to the pro-isomorphic situation, where *all* local factors of $\zeta^{\wedge}_{\mathcal{F}_{c,g} \otimes \mathcal{O}_K}$ have been computed, for the "sibling" zeta functions there are often finitely many local factors that cannot currently be treated. We will see the same phenomenon for other Lie rings considered in this section.

If K is an arbitrary number field, p is unramified in K, and g is arbitrary, then an explicit expression for $\zeta^{\triangleleft}_{\mathcal{F}_{2,g} \otimes \mathcal{O}_K, p}(s)$ is indicated in [9, Section 5.2] by Carnevale, the second author, and Voll; all these local factors satisfy the generic functional equation of [37, Theorem C]. Theorems 2.7 and 2.9 of [14] provide $\zeta^{\triangleleft}_{\mathcal{F}_{2,2} \otimes \mathcal{O}_K, p}(s)$ explicitly when $d \in \{2, 3\}$ and p is arbitrary; they are originally due to Grunewald, Segal, and Smith [16] and to G. Taylor [35]. The only other published result about ramified primes is [30, Theorem 3.8], where the functions $\zeta^{\triangleleft}_{\mathcal{F}_{2,2} \otimes \mathcal{O}_K, p}(s)$ are computed for p non-split in K. These are found to satisfy a modified functional equation; see also [29, Conjecture 1.4], where a functional equation for $\zeta^{\triangleleft}_{\mathcal{F}_{2,2} \otimes \mathcal{O}_K, p}(s)$ is conjectured at every prime p. The subring zeta factors $\zeta^{\leq}_{\mathcal{F}_{2,g} \otimes \mathcal{O}_K, p}(s)$ are known if $g = 2$ and $d \in \{1, 2\}$, provided p splits in K, or if $g = 3$ and $d = 1$; see Theorems 2.3, 2.4, and 2.16 of [14].

The functions $\zeta^{\triangleleft}_{\mathcal{F}_{3,2}, p}(s)$ and $\zeta^{\leq}_{\mathcal{F}_{3,2}, p}(s)$ were computed by Woodward; see [14, Theorem 2.35]. Given an arbitrary pair (c, g), functional equations for almost all local ideal zeta factors $\zeta^{\triangleleft}_{\mathcal{F}_{c,g}, p}(s)$ were recently proved by Voll [38, Theorem 4.4]. If K is a number field, then functional equations for almost all local subring zeta factors $\zeta^{\leq}_{\mathcal{F}_{c,g} \otimes \mathcal{O}_K, p}(s)$ follow from the general result of Voll [37, Corollary 1.1] mentioned in the introduction.

4.2. **The higher Heisenberg Lie rings.** Let $m \geq 1$ and consider the Lie ring $\mathcal{H}_m$ with presentation

$$\langle x_1, \ldots, x_m, y_1, \ldots, y_m, z | [x_i, y_i] = z, 1 \leq i \leq m \rangle_{\mathbb{Z}},$$

with the usual convention that all other pairs of generators commute. This is a centrally amalgamated product of m Heisenberg Lie rings. Note that $Z(\mathcal{H}_m) = [\mathcal{H}_m, \mathcal{H}_m] = \langle z \rangle$. Write H_m for the $\mathbb{Q}$-Lie algebra $\mathcal{H}_m \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proposition 4.3. *Let K be any field. The K-Lie algebra $(H_m)_K = \mathcal{H}_m \otimes_{\mathbb{Z}} K$ is $Z((H_m)_K)$-rigid.*

Proof. It is easy to check that $x_i, y_i \in \mathcal{Y}(Z((H_m)_K))$ for all $i \in [m]$. Thus $\mathcal{Y}(Z((H_m)_K))$ generates $(H_m)_K$ as a K-Lie algebra. Furthermore, $(H_m)_K$ is absolutely indecomposable. Indeed, for any field E, let $H_m \otimes_{\mathbb{Z}} E = M_1 \oplus M_2$ be a decomposition. At least one of the components, say M_1, must be non-abelian. The derived subalgebra of $H_m \otimes E$ is one-dimensional and so contained in M_1. Thus M_2 is abelian. Hence $M_2 \subseteq Z(H_m \otimes E) \subseteq M_1$, and we conclude that $M_2 = 0$. Then $(H_m)_K$ is $Z((H_m)_K)$-rigid by Theorem 3.8. □

Before proceeding we note that if $m \geq 2$, then the results of [32] (recalled in this paper as Corollary 3.9) are insufficient to establish the rigidity of $(H_m)_K$; we genuinely need the weakened hypotheses of Theorem 3.8. It is an exercise to show that there is no verbal ideal (or, indeed, ideal) $M \leq \gamma_2(H_m)_K$ such that $\mathcal{X}(M)$ generates $(H_m)_K$.

It is useful to consider an equivalent presentation of H_m, following du Sautoy and Lubotzky [13, Section 3.3]. Consider the $\mathbb{Q}$-vector space $V_m = \mathbb{Q}^{2m} \times \mathbb{Q}$ with Lie bracket given by

$$[(v_1, w_1), (v_2, w_2)] = (0, v_1 \Omega v_2^T)$$

for all $v_1, v_2 \in \mathbb{Q}^{2m}$ and $w_1, w_2 \in \mathbb{Q}$, where Ω is the symplectic form

$$\Omega = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

Here I_m denotes the $m \times m$ identity matrix. The map $V_m \to H_m$ of vector spaces that sends the standard bases of $\mathbb{Q}^{2m}$ and $\mathbb{Q}$ to $(x_1, \ldots, x_m, y_1, \ldots, y_m)$ and z, respectively, is an isomorphism of Lie algebras. Fixing the basis $(x_1, \ldots, x_m, y_1, \ldots, y_m, z)$ provides an embedding $\mathfrak{Aut}\, H_m \leq \mathbf{GL}_{2m+1}$. As in [13, Lemma 3.12], we find that $\mathfrak{Aut}\, H_m$ is the algebraic subgroup of $\mathbf{GL}_{2m+1}$ consisting of the matrices of the form

$$\begin{pmatrix} A & B \\ 0 & \lambda \end{pmatrix}, \tag{18}$$

where $\lambda \in \mathbf{GL}_1$, the matrix $A \in \mathbf{GL}_{2m}$ satisfies $A\Omega A^T = \lambda\Omega$, and B is an arbitrary $2m \times 1$ matrix. Comparing the determinants of the two sides of the relation $A\Omega A^T = \lambda\Omega$ (which is equivalent to $A \in \mathbf{GSp}_{2m}$), we observe that $(\det A)^2 = \lambda^{2m}$. Given a number field K of degree $d = [K : \mathbb{Q}]$ and a prime p, we invoke the notation of Section 3.2. For every $i \in [r]$, set $\mathrm{GSp}^+_{2m}(R_i) = \mathrm{GSp}_{2m}(R_i) \cap M_{2m}(\mathcal{O}_{R_i})$, and consider the right Haar measure on $\mathrm{GSp}_{2m}(R_i)$ normalized so that $\mu_{\mathrm{GSp}_{2m}(R_i)}(\mathrm{GSp}_{2m}(\mathcal{O}_{R_i})) = 1$.

Proposition 4.4. *Let $m \geq 1$. Let K be a number field and p be any prime. Then*

$$\zeta^{\wedge}_{(\mathcal{H}_m \otimes_{\mathbb{Z}} \mathcal{O}_K),p}(s) = \prod_{i=1}^{r} \int_{\mathrm{GSp}^+_{2m}(R_i)} |\det A_i|_{R_i}^{(1+1/m)s-2d} d\mu_{\mathrm{GSp}_{2m}(R_i)}(A_i).$$

Proof. By the rigidity established in Proposition 4.3, we may use Proposition 3.12. From the description of $\mathfrak{Aut}\, H_m$ given above, we see, for all $i \in [r]$, that $\mathbf{H}(R_i) \simeq \mathrm{GSp}_{2m}(R_i)$ consists of matrices of the form

$$h_i = \begin{pmatrix} A_i & 0 \\ 0 & \lambda \end{pmatrix},$$

where $A_i \in \mathrm{GSp}_{2m}(R_i)$. Since $\mathbf{N}(R_i)$ acts trivially on $\langle x_1, \ldots, x_m, y_1, \ldots, y_m \rangle_{R_i}$, we have $\widetilde{\theta}_i(h_i) = 1$ for all $h_i \in \mathbf{H}^+(R_i)$ by Remark 3.11. Finally, $|\det h_i|_{R_i} = |\det A_i|_{R_i}^{1+1/m}$ and $|\det \varepsilon_i(h_i)|_{R_i} = |\lambda|_{R_i} = |\det A_i|_{R_i}^{1/m}$. Since $\overline{n} = 2m$, our claim now follows from Proposition 3.12. □

Note that in the case $d = 1$ this result matches the formula obtained in [1, Proposition 3.14] and differs from the one obtained in [13, Lemma 3.14] because of an error in the computation of the function $\theta(h)$ appearing in [13].

The integral of Proposition 4.4 may be computed using Proposition 2.5. We recall some basic facts about the root system C_n of the reductive group $\mathbf{GSp}_{2m}$ over a field F. The subset of diagonal elements of $\mathbf{GSp}_{2m}$ is

$$\mathbf{T} = \left\{\mathrm{diag}(\nu t_1^{-1}, \nu t_2^{-1} \ldots, \nu t_m^{-1}, t_1, t_2, \ldots, t_m) : \nu, t_1, \ldots t_m \in \mathbf{G}_m\right\}$$

and is a split maximal torus of rank $m+1$. Consider the cocharacters

$$\xi_k(t) = \begin{cases} \mathrm{diag}(\underbrace{1,\ldots,1}_{k}, \underbrace{t,\ldots,t}_{m-k}, \underbrace{1,\ldots,1}_{k}, \underbrace{t^{-1},\ldots,t^{-1}}_{m-k}) & : k \in [m-1]_0 \\ \mathrm{diag}(\underbrace{t,\ldots,t}_{m}, \underbrace{1,\ldots,1}_{m}) & : k = m. \end{cases}$$

They constitute a $\mathbb{Z}$-basis of $\Xi = \mathrm{Hom}(\mathbf{G}_m, \mathbf{T})$. Indeed, any element $\xi \in \Xi$ has the form

$$\xi(t) = \mathrm{diag}(t^{\lambda - a_1}, t^{\lambda - a_2}, \ldots, t^{\lambda - a_m}, t^{a_1}, t^{a_2}, \ldots, t^{a_m}), \tag{19}$$

and one readily verifies that

$$\xi = \xi_0^{-a_1} \xi_1^{a_1 - a_2} \cdots \xi_{m-2}^{a_{m-2} - a_{m-1}} \xi_{m-1}^{a_{m-1} - a_m} \xi_m^{\lambda}. \tag{20}$$

Thus $\{\xi_0, \ldots, \xi_m\}$ spans the $\mathbb{Z}$-module Ξ of rank $m+1$.

Let $\mathrm{Sym}_m \subset M_m$ be the set of symmetric $m \times m$ matrices, and let $\mathbf{U}_m \subset \mathbf{GL}_m$ be the group of upper triangular matrices. It is well known that $\mathbf{Sp}_{2m}$ has the same root system as $\mathbf{GSp}_{2m}$, and that

$$(\mathbf{T} \cap \mathbf{Sp}_{2m}) \ltimes \left\{ \begin{pmatrix} (U^T)^{-1} & MU \\ 0 & U \end{pmatrix} : U \in \mathbf{U}_m, M \in \mathrm{Sym}_n \right\}$$

is a Borel subgroup of $\mathbf{Sp}_{2m}$. Let Φ^+ be the set of m^2 positive roots given by this choice of Borel subgroup. These consist of the $\binom{m}{2}$ roots $e_i - e_j$, for $1 \le i < j \le m$, associated to the root subgroups whose F-points are $I_{2m} + F E_{m+i,m+j}$, and $\binom{m+1}{2}$ roots $-e_i - e_j$, for $1 \le i \le j \le m$, associated to the subgroups $I_{2m} + F(E_{i,m+j} + E_{j,m+i})$ when $i \neq j$, and $I_{2m} + F E_{i,m+i}$ when $i = j$. Here $E_{i,j}$ is the usual notation for elementary matrices. We fix the collection $\Delta = \{\alpha_0, \alpha_1, \ldots, \alpha_{m-1}\}$ of simple roots, where $\alpha_0 = -2e_1$ and $\alpha_i = e_i - e_{i+1}$ for $i \in [m-1]$. For $\xi \in \Xi$ as in (19), one verifies that $\langle e_i - e_j, \xi \rangle = a_i - a_j$ and $\langle -e_i - e_j, \xi \rangle = \lambda - (a_i + a_j)$, for i, j in the suitable ranges as above. In particular,

$$\langle \prod_{\beta \in \Phi^+} \beta, \xi_k \rangle = \begin{cases} 2\left(\binom{m+1}{2} - \binom{k+1}{2}\right) & : k \in [m-1]_0 \\ \binom{m+1}{2} & : k = m. \end{cases} \tag{21}$$

Let $F/\mathbb{Q}_p$ be a finite extension, and fix a uniformizer $\pi \in \mathcal{O}_F$. Then the condition that $\xi \in \Xi^+$ and $\alpha(\xi(\pi)) \in \mathcal{O}_K$ for all $\alpha \in \Delta$ is equivalent to the conditions $\lambda \ge 2a_1$ and $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$.

The Weyl group in this setting is naturally identified with the hyperoctahedral group B_m of signed permutations; see [6, Section 8.1] for an introduction to this group. Elements $w \in B_m$ may be viewed as permutations of the set $\{1, \ldots, m\} \cup \{-1, \ldots, -m\}$ satisfying $w(-j) = -w(j)$ for all $j \in [m]$. We write $w \in B_m$ in the "window notation"

$w = [w(1), \dots, w(m)]$. Our choice of simple roots gives rise to the Coxeter generating set $S = \{s_0, s_1, \dots, s_{m-1}\}$ of B_m, where s_0 transposes 1 and -1 and s_i transposes i and $i+1$ for $i \in [m-1]$. Note these are the Coxeter generators considered in [6]. Let ℓ be the associated length function. For every $w \in B_m$ we have the descent set

$$\mathrm{Des}(w) = \{i \in [m-1]_0 : \ell(ws_i) < \ell(w)\}.$$

To compute $\mathrm{Des}(w)$ in practice, observe that if $i \in [m-1]$, then $i \in \mathrm{Des}(w)$ if and only if $w(i) > w(i+1)$. Also, $0 \in \mathrm{Des}(w)$ if and only if $w(1) < 0$. For $w \in B_m$, set

$$\mathrm{Inv}(w) = \{(i,j) \in [m]^2 : i < j, w(i) > w(j)\}$$
$$\mathrm{Npr}(w) = \{(i,j) \in [m]^2 : i \leq j, w(i) + w(j) < 0\}$$

and denote $\mathrm{inv}(w) = |\mathrm{Inv}(w)|$ and $\mathrm{npr}(w) = |\mathrm{Npr}(w)|$. We refer to the elements of $\mathrm{Inv}(w)$ and of $\mathrm{Npr}(w)$ as *inversions* and *negative pairs* of w, respectively. The length of w is given by $\ell(w) = \mathrm{inv}(w) + \mathrm{npr}(w)$; cf. [6, Proposition 8.1.1]. Write $\mathrm{des}(w) = |\mathrm{Des}(w)|$ for the descent number of w. Set $\varepsilon_1(w) = 1$ if $w(1) < 0$ and $\varepsilon_1(w) = 0$ otherwise. Define the following two statistics for $w \in B_m$:

$$\sigma_B(w) = \sum_{i \in \mathrm{Des}(w)} (m^2 - i^2) \qquad \mathrm{rmaj}(w) = \sum_{i \in \mathrm{Des}(w)} (m - i).$$

Remark 4.5. We identify S_m with the subgroup of B_m consisting of signed permutations w such that $w(i) > 0$ for all $i \in [m]$. The Coxeter length function on S_m with respect to the generators $\{s_1, \dots, s_{m-1}\}$ coincides with ℓ, and the usual descent set $\{i \in [m-1] : \sigma(i) > \sigma(i+1)\}$ of a permutation $\sigma \in S_m$ coincides with $\mathrm{Des}(\sigma)$ as defined above.

We will also need two statistics for permutations $\sigma \in S_m$:

$$\sigma_A(\sigma) = \sum_{i \in \mathrm{Des}(\sigma)} i(m-i) \qquad \mathrm{rbin}(\sigma) = \sum_{i \in \mathrm{Des}(\sigma)} \binom{m-i+1}{2}.$$

It is easily seen that σ_A and σ_B are special cases of the statistic σ on Weyl groups defined in [33], whereas rbin is a variant of the statistic bin introduced in [8].

Recalling the notation of (19), we put $r_0 = \lambda - 2a_1$ and $r_i = a_i - a_{i+1}$ for $i \in [m-1]$; in addition, we set $r_m = a_m$. It is easy to see that the set $w\Xi_w^+$ defined in (8) is parametrized by the conditions $r_i \geq 1$ if $i \in \mathrm{Des}(w)$ and $r_i \geq 0$ otherwise. Moreover, given $w \in B_n$ and $\xi \in \Xi$, it follows from (20) and (21) that

$$q^{\langle \prod_{\beta \in \Phi^+} \beta, \xi \rangle} |\det \xi(\pi)|_F^s = \left(q^{\binom{m+1}{2} - ms}\right)^{r_0} \prod_{i=1}^{m} \left(q^{2\left(\binom{m+1}{2} - \binom{i+1}{2}\right) - 2ms}\right)^{r_i}.$$

Hence by Proposition 2.5 we have

$$\int_{\mathrm{GSp}_{2m}^+(F)} |\det A|_F^s d\mu(A) = \frac{\sum_{w \in B_m} q^{-\ell(w)} \prod_{i \in \mathrm{Des}(w)} \widetilde{X}_i}{\prod_{i=0}^{m} (1 - \widetilde{X}_i)}, \tag{22}$$

where $\widetilde{X}_0 = q^{\binom{m+1}{2} - ms}$ and $\widetilde{X}_i = q^{2\left(\binom{m+1}{2} - \binom{i+1}{2}\right) - 2ms}$ for $i \in [m]$.

Remark 4.6. The integral of (22) was studied by Satake, who computed it explicitly in the case $m = 2$; see (21) of [28, Appendix I]. This computation, in terms of spherical functions, was completed by Macdonald [25, V.(5.4)] for arbitrary m. In contrast to the formula of [25], the right-hand side of (22) provides an expression from which one may readily deduce a functional equation; see Remark 4.11 below.

It turns out that cancellation occurs in the rational function in the right-hand side of (22), so that the sum over elements of the hyperoctahedral group B_m in its numerator reduces to a sum over the symmetric group S_m. This will be a special case of the following identity. We refer the reader to [10, Proposition 3.4], where a similar phenomenon occurs.

Lemma 4.7. *The following identity of rational functions holds in* $\mathbb{Q}(X,Y)$*:*

$$\sum_{w\in B_m} X^{(\sigma_B-\ell+\mathrm{rmaj}-\binom{m+1}{2}\varepsilon_1)(w)}Y^{(2\,\mathrm{des}-\varepsilon_1)(w)} = \left(\prod_{j=1}^{m}(1+X^{\binom{m+1}{2}-\binom{j+1}{2}}Y)\right)\sum_{\sigma\in S_m} X^{(\sigma_A-\ell+\mathrm{rbin})(\sigma)}Y^{\mathrm{des}(\sigma)}.$$

Proof. We start by defining, for every $j\in[m]$, an involution $\eta_j : B_m\to B_m$. For every $w\in B_m$, consider the set $S_j(w)=\{|w(k)| : k\in[j]\}\subseteq[m]$. Let $c_1<\cdots<c_j$ be the elements of $S_j(w)$, arranged in increasing order, and consider the signed permutation $w_j\in B_m$ determined by

$$w_j(x)=\begin{cases}-c_{j+1-k} & : x=c_k \text{ for some } k\in[j]\\ x & : x\in[m]\setminus S_j(w).\end{cases}$$

Now set $\eta_j(w)=w_j\circ w$. Informally, to construct $\eta_j(w)$ we take the j leftmost entries in the window notation for w, ignoring their signs, and replace the largest of these with the smallest one, the second largest with the second smallest, and so forth. Then we arrange signs so that $(\eta_j(w))(k)$ has the opposite sign from $w(k)$, for all $k\in[j]$. For example, if $w=[3,-5,-1,6,2,7,-4]\in B_7$ in window notation, then $\eta_5(w)=[-3,2,6,-1,-5,7,-4]$.

It is easy to see that $\eta_j(\eta_j(w))=w$ and that $\eta_j\circ\eta_i=\eta_i\circ\eta_j$ for any $i,j\in[m]$. Thus we obtain an action on the set B_m of the group $\Gamma\simeq(\mathbb{Z}/2\mathbb{Z})^m$ generated by $\eta_1,\dots,\eta_m$. Henceforth write $\eta_j w$ instead of $\eta_j(w)$ for brevity. Consider a subset $J\subseteq[m]$ and put

$$J'=\{j\in[m-1] : j\notin J, j+1\in J\}\cup\{j\in[m] : j\in J, j+1\notin J\},$$

where $m+1\notin J$ for all J. Setting $\gamma=\prod_{j\in J'}\eta_j\in\Gamma$, we observe, for every $w\in B_m$ and every $j\in[m]$, that $(\gamma w)(j)$ and $w(j)$ have opposite signs if $j\in J$ and the same sign otherwise. Hence every orbit of our action contains exactly $|\Gamma|=2^m$ elements, each with a different arrangement of signs. In particular, every orbit contains a unique element of S_m. Let $\mathcal{C}_\sigma\in\Gamma\backslash B_m$ denote the orbit containing $\sigma\in S_m$. To establish our claim, it clearly suffices to prove the following for every $\sigma\in S_m$:

$$\sum_{w\in\mathcal{C}_\sigma} X^{(\sigma_B-\ell+\mathrm{rmaj}-\binom{m+1}{2}\varepsilon_1)(w)}Y^{(2\,\mathrm{des}-\varepsilon_1)(w)} = \left(\prod_{j=1}^{m}(1+X^{\binom{m+1}{2}-\binom{j+1}{2}}Y)\right)X^{(\sigma_A-\ell+\mathrm{rbin})(\sigma)}Y^{\mathrm{des}(\sigma)}. \tag{23}$$

If $a, b, c \in Z$, then we say that c lies between a and b if either $a < c < b$ or $b < c < a$. The remainder of the argument makes use of the following definition.

Definition 4.8. Let $j \in [m-1]$. We say that $w \in B_m$ satisfies property (P_j) if the following statement is true:

$$w(j+1) \textit{ lies between } w(j) \textit{ and } \eta_j w(j) \textit{ if and only if } w(j) < 0.$$

We say that w satisfies property (P_m) if $w(m) > 0$.

It is clear, for every $j \in [m]$ and $w \in B_m$, that exactly one member of the pair $\{w, \eta_j w\}$ satisfies property (P_j). The following statement, whose proof is very technical, is crucial to our argument.

Sublemma 4.9. *Suppose that $j \in [m]$ and that $w \in B_m$ satisfies property (P_j). Then*

$$X^{(\sigma_B - \ell + \mathrm{rmaj} - \binom{m+1}{2}\varepsilon_1)(\eta_j w)} Y^{(2\,\mathrm{des} - \varepsilon_1)(\eta_j w)} = \left(X^{\binom{m+1}{2} - \binom{j+1}{2}} Y \right) X^{(\sigma_B - \ell + \mathrm{rmaj} - \binom{m+1}{2}\varepsilon_1)(w)} Y^{(2\,\mathrm{des} - \varepsilon_1)(w)}.$$

We assume Sublemma 4.9 for the moment and show how to deduce Lemma 4.7. It is obvious from the definitions that the property (P_j), as well as its negation, are preserved by η_k for all $k < j$. This implies, for every $\sigma \in S_m$, that the orbit $\mathcal{C}_\sigma$ contains an element w_σ having the property (P_j) for all $j \in [m]$. Indeed, we choose an arbitrary $w \in \mathcal{C}_\sigma$ and construct w_σ recursively as follows: $w_\sigma = \eta_1^{\delta_1} \cdots \eta_m^{\delta_m} w$, where $\delta_m = 0$ if w has property (P_m) and $\delta_m = 1$ otherwise. Similarly, for $k \geq 1$, we take $\delta_{m-k} = 0$ if $\left(\prod_{i=0}^{k-1} \eta_{m-i}^{\delta_{m-i}} \right) w$ has property (P_{m-k}) and $\delta_{m-k} = 1$ otherwise. Once we know that such an element w_σ exists, it is readily seen that for every $J \subseteq [m]$ there is a unique element of $\mathcal{C}_\sigma$ satisfying (P_j) exactly when $j \in J$, namely $(\prod_{j \notin J} \eta_j) w_\sigma$.

Let $\gamma \in \Gamma$. Then $\gamma = \prod_{j \in J} \eta_j$ for some $J \subseteq [m]$. Applying Sublemma 4.9 to the elements of J in increasing order, we find that

$$X^{(\sigma_B - \ell + \mathrm{rmaj} - \binom{m+1}{2}\varepsilon_1)(\gamma w_\sigma)} Y^{(2\,\mathrm{des} - \varepsilon_1)(\gamma w_\sigma)} = \left(\prod_{j \in J} X^{\binom{m+1}{2} - \binom{j+1}{2}} Y \right) X^{(\sigma_B - \ell + \mathrm{rmaj} - \binom{m+1}{2}\varepsilon_1)(w_\sigma)} Y^{(2\,\mathrm{des} - \varepsilon_1)(w_\sigma)}. \tag{24}$$

Hence the left-hand side of (23) is equal to

$$\left(\prod_{j=1}^{m} (1 + X^{\binom{m+1}{2} - \binom{j+1}{2}} Y) \right) X^{(\sigma_B - \ell + \mathrm{rmaj} - \binom{m+1}{2}\varepsilon_1)(w_\sigma)} Y^{(2\,\mathrm{des} - \varepsilon_1)(w_\sigma)}. \tag{25}$$

Thus, to establish (23) it suffices to prove, for every $\sigma \in S_m$, that

$$X^{(\sigma_B - \ell + \mathrm{rmaj} - \binom{m+1}{2}\varepsilon_1)(w_\sigma)} Y^{(2\mathrm{des} - \varepsilon_1)(w_\sigma)} = X^{(\sigma_A - \ell + \mathrm{rbin})(\sigma)} Y^{\mathrm{des}(\sigma)}. \tag{26}$$

The permutation $\sigma \in S_m$, viewed as an element of B_m, clearly satisfies property (P_m). If $j \in [m-1]$, then σ fails to satisfy (P_j) if and only if $\sigma(j+1)$ lies between $\sigma(j)$ and

$\eta_j\sigma(j) < 0$, which is equivalent to $j \in \mathrm{Des}(\sigma)$. Hence $\sigma = \left(\prod_{j\in\mathrm{Des}(\sigma)} \eta_j\right)(w_\sigma)$. Noting that $\varepsilon_1(\sigma) = 0$, we obtain

$$X^{(\sigma_B-\ell+\mathrm{rmaj}-\binom{m+1}{2}\varepsilon_1)(w_\sigma)}Y^{(2\mathrm{des}-\varepsilon_1)(w_\sigma)} =$$

$$X^{(\sigma_B-\ell+\mathrm{rmaj})(\sigma)}Y^{2\mathrm{des}(\sigma)}\left(\prod_{j\in\mathrm{Des}(\sigma)} X^{-\left(\binom{m+1}{2}-\binom{j+1}{2}\right)}Y^{-1}\right) = X^{(\sigma_A-\ell+\mathrm{rbin})(\sigma)}Y^{\mathrm{des}(\sigma)},$$

where the first equality is immediate from (24) and the second arises from a simple calculation. This verifies (26), and Lemma 4.7 follows.

It remains to prove Sublemma 4.9, which is done by a computation. Let $j \in [m]$ and assume that w satisfies property (P_j). We first claim that

$$\ell(\eta_j w) = \ell(w) + \sum_{\substack{i\in[j]\\ w(i)>0}} i - \sum_{\substack{i\in[j]\\ w(i)<0}} i. \tag{27}$$

Consider pairs $(i_1, i_2) \in [j]^2$ with $i_1 \le i_2$. Observe that $|\eta_j w(i_1)| > |\eta_j w(i_2)|$ if and only if $|w(i_1)| < |w(i_2)|$. Moreover, $\eta_j w(i)$ and $w(i)$ have opposite sign for all $i \in [j]$. It follows that if $w(i_1)$ and $w(i_2)$ have the same sign, then (i_1, i_2) is an inversion of $\eta_j w$ if and only if it was already an inversion of w. Similarly, if $w(i_1)$ and $w(i_2)$ have opposite sign, then (i_1, i_2) is a negative pair of $\eta_j w$ if and only if it already was one of w. However:

- If $w(i_1) < 0$ and $w(i_2) > 0$, then (i_1, i_2) is an inversion of $\eta_j w$ but not of w.
- If $w(i_1) > 0$ and $w(i_2) > 0$, then (i_1, i_2) is a negative pair of $\eta_j w$ but not of w.
- If $w(i_1) > 0$ and $w(i_2) < 0$, then (i_1, i_2) is an inversion of w but not of $\eta_j w$.
- If $w(i_1) < 0$ and $w(i_2) < 0$, then (i_1, i_2) is a negative pair of w but not of $\eta_j w$.

It follows that the contribution to $\ell(\eta_j w) - \ell(w) = \mathrm{inv}(\eta_j w) + \mathrm{npr}(\eta_j w) - \mathrm{inv}(w) - \mathrm{npr}(w)$ arising from pairs $(i_1, i_2) \in [j]^2$ is

$$|\{(i_1,i_2) \in [j]^2 : i_1 \le i_2, w(i_2) > 0\}| - |\{(i_1,i_2) \in [j]^2 : i_1 \le i_2, w(i_2) < 0\}| = \sum_{\substack{i\in[j]\\ w(i)>0}} i - \sum_{\substack{i\in[j]\\ w(i)<0}} i.$$

Next we consider pairs (i_1, i_2) such that $i_1 \le j$ and $i_2 > j$. For every $i \in [j]$ we denote by i' the unique element of $[j]$ such that $|\eta_j w(i')| = |w(i)|$. If $|w(i_2)| < |w(i_1)|$ and if both of the conditions $w(i_1) < 0$ and $-w(i_1) \in \eta_j w([j])$ hold, then (i_1, i_2) is not an inversion of w yet gives rise to an inversion (i'_1, i_2) of $\eta_j w$. On the other hand, (i_1, i_2) is a negative pair of w, while (i'_1, i_2) is not a negative pair of $\eta_j w$. Analogously, if $w(i_1) > 0$ and $|w(i_2)| < |w(i_1)|$, then (i_1, i_2) is an inversion and not a negative pair of w, while (i'_1, i_2) is a negative pair and not an inversion of $\eta_j w$. In all other cases, (i'_1, i_2) is an inversion of $\eta_j w$ if and only if (i_1, i_2) is an inversion of w, and the same is true for negative pairs. Thus, the pairs (i_1, i_2) such that $i_1 \le j$ and $i_2 > j$ make no contribution to $\ell(\eta_j w) - \ell(w)$. Since the pairs (i_1, i_2) such that $j < i_1 \le i_2$ are obviously unaffected by η_j, we have established (27).

Now consider the descent sets of $\eta_j w$ and of w. It follows from the observations above that:

- If $i \in D_- = \{i \in [j-1] : w(i) > 0, w(i+1) < 0\}$, then $i \in \mathrm{Des}(w)$ and $i \notin \mathrm{Des}(\eta_j w)$.
- If $i \in D_+ = \{i \in [j-1] : w(i) < 0, w(i+1) > 0\}$, then $i \notin \mathrm{Des}(w)$ and $i \in \mathrm{Des}(\eta_j w)$.
- If $i \in [j-1] \setminus (D_+ \cup D_-)$ or $i > j$, then $i \in \mathrm{Des}(w)$ if and only if $i \in \mathrm{Des}(\eta_j w)$.

The remaining case $i = j$ will be treated below. Let $\beta_1 < \cdots < \beta_s$ be the elements of D_- and $\gamma_1 < \cdots < \gamma_t$ the elements of D_+.

Suppose first that $w(1) > 0$. In this case, 0 is a descent of $\eta_j w$ but not of w. If $w(j) > 0$, then $s = t$ and we have the arrangement $1 \le \beta_1 < \gamma_1 < \cdots < \beta_s < \gamma_s \le j-1$. By property (P_j) of Definition 4.8, observe that if $j \in [m-1]$, then $j \le m-1$ and $w(j+1)$ does not lie between $w(j)$ and $\eta_j w(j)$. Hence $j \in \mathrm{Des}(w)$ if and only if $j \in \mathrm{Des}(\eta_j w)$. Letting $\Sigma[a,b]$ denote the sum of the elements of the set $[a,b]$, it follows from (27) that

$$\ell(\eta_j w) - \ell(w) = \sum_{\substack{i\in[j]\\ w(i)>0}} i - \sum_{\substack{i\in[j]\\ w(i)<0}} i =$$
$$\Sigma[1,\beta_1] - \Sigma[\beta_1+1,\gamma_1] + \cdots - \Sigma[\beta_s+1,\gamma_s] + \Sigma[\gamma_s+1,j] =$$
$$\binom{\beta_1+1}{2} - \left(\binom{\gamma_1+1}{2} - \binom{\beta_1+1}{2}\right) + \cdots + \left(\binom{j+1}{2} - \binom{\gamma_s+1}{2}\right) =$$
$$\binom{j+1}{2} - \sigma_B(w) + \sigma_B(\eta_j w) - m^2 - \mathrm{rmaj}(w) + \mathrm{rmaj}(\eta_j w) - m.$$

Sublemma 4.9 follows for this case by a simple calculation. If $w(j) < 0$, then $t = s-1$ and $1 \le \beta_1 < \gamma_1 < \cdots < \gamma_{s-1} < \beta_s \le j-1$. By property (P_j), we see that $j \in [m-1]$ and that j is a descent of $\eta_j w$ but not of w. In this case,

$$\ell(\eta_j w) - \ell(w) = \sum_{\substack{i\in[j]\\ w(i)>0}} i - \sum_{\substack{i\in[j]\\ w(i)<0}} i = -\binom{j+1}{2} + 2\sum_{i=1}^{s}\binom{\beta_i+1}{2} - 2\sum_{i=1}^{s-1}\binom{\gamma_i+1}{2} =$$
$$\binom{j+1}{2} - \sigma_B(w) + \sigma_B(\eta_j w) - m^2 - \mathrm{rmaj}(w) + \mathrm{rmaj}(\eta_j w) - m.$$

Again Sublemma 4.9 follows. The case $w(1) < 0$ is treated analogously, completing the proof of Sublemma 4.9 and thus of Lemma 4.7. □

After this combinatorial digression, we return to the computation of the pro-isomorphic zeta function $\zeta^{\wedge}_{(\mathcal{H}_m \otimes_{\mathbb{Z}} \mathcal{O}_K),p}(s)$. Recall the notation introduced in Section 3.2.

Theorem 4.10. *Let K be a number field of degree $d = [K : \mathbb{Q}]$, let p be a prime, and let $m \ge 1$. Then*

$$\zeta^{\wedge}_{(\mathcal{H}_m \otimes_{\mathbb{Z}} \mathcal{O}_K),p}(s) = \prod_{i=1}^{r} W(q_i, q_i^{-s}),$$

where

$$W(X,Y) = \frac{\sum_{\sigma\in S_m} X^{-\ell(\sigma)} \prod_{j\in \mathrm{Des}(\sigma)} Z_j}{\prod_{j=0}^{m}(1-Z_j)} \in \mathbb{Q}(X,Y)$$

and $Z_j = X^{\binom{m+1}{2}-\binom{j+1}{2}+2md}Y^{m+1}$ *for all* $j \in [m]_0$*. The following functional equation holds:*

$$\zeta^{\wedge}_{(\mathcal{H}_m\otimes_{\mathbb{Z}}\mathcal{O}_K),p}(s)|_{p\to p^{-1}} = (-1)^{(m+1)r}\left(\prod_{i=1}^{r} q_i\right)^{m^2+4md-2(m+1)s} \zeta^{\wedge}_{(\mathcal{H}_m\otimes_{\mathbb{Z}}\mathcal{O}_K),p}(s).$$

The abscissa of convergence is $\alpha^{\wedge}_{\mathcal{H}_m\otimes_{\mathbb{Z}}\mathcal{O}_K} = \frac{m}{2} + \frac{2md+1}{m+1}$.

Proof. Let $F/\mathbb{Q}_p$ be a finite extension with residue field of cardinality q, and set $X_j = q^{\binom{m+1}{2}-\binom{j+1}{2}-ms}$ for any $j \in [m]_0$. Recall the quantities $\widetilde{X}_j$ defined in (22) for $j \in [m]_0$. Since $\widetilde{X}_0 = X_0$ and $\widetilde{X}_j = X_j^2$ for $j \in [m]$, it follows from (22) and Lemma 4.7 that

$$\int_{\mathrm{GSp}^+_{2m}(F)} |\det A|^s_F d\mu(A) = \frac{\sum_{w\in S_m} q^{-\ell(w)} \prod_{j\in \mathrm{Des}(w)} X_j}{\prod_{j=0}^{m}(1-X_j)}. \tag{28}$$

Indeed, the equality of the right-hand sides of (22) and (28) is verified by substituting $(X,Y) = (q, q^{-ms})$ into the statement of Lemma 4.7 and performing elementary calculations. Substituting $\frac{m+1}{m}s - 2d$ for s in (28), we obtain the first part of our claim by Proposition 4.4. Let $\sigma_0 \in S_m$ be the longest word, and recall that $\ell(\sigma_0\sigma) = \binom{m}{2} - \ell(\sigma)$ and that $\mathrm{Des}(\sigma_0\sigma) = [m-1]\setminus \mathrm{Des}(\sigma)$ for all $\sigma \in S_m$. Hence,

$$W(X^{-1},Y^{-1}) = \frac{\sum_{\sigma\in S_m} X^{\ell(\sigma)} \prod_{j\in\mathrm{Des}(\sigma)} Z_j^{-1}}{\prod_{j=0}^{m}(1-Z_j^{-1})} =$$

$$(-1)^{m+1}Z_0Z_m \frac{\sum_{\sigma} X^{\binom{m}{2}-\ell(\sigma_0\sigma)}\prod_{j\in\mathrm{Des}(\sigma_0\sigma)} Z_j}{\prod_{j=0}^{m}(1-Z_j)} = (-1)^{m+1}X^{m^2+4md}Y^{2(m+1)}W(X,Y).$$

The functional equation follows immediately. Finally, by Lemma 4.13 below we have

$$\alpha^{\wedge}_{\mathcal{H}_m\otimes_{\mathbb{Z}}\mathcal{O}_K} = \max_{j\in[m]_0}\left\{\frac{\binom{m+1}{2}-\binom{j+1}{2}+2md+1}{m+1}\right\} = \frac{\binom{m+1}{2}+2md+1}{m+1}. \qquad \square$$

The functions $\zeta^{\wedge}_{(\mathcal{H}_m\otimes_{\mathbb{Z}}\mathcal{O}_K),p}(s)$, in the case $d=1$ and $m \in \{2,3\}$, were computed explicitly by du Sautoy and Lubotzky [13]. Otherwise, Theorem 4.10 is new even for $d = 1$. For comparison we mention that, for arbitrary m and K, the ideal zeta factors $\zeta^{\triangleleft}_{(\mathcal{H}_m\otimes_{\mathbb{Z}}\mathcal{O}_K),p}(s)$ are known when p is unramified in K by work of Carnevale, the second author, and Voll [9, Section 5.4].

Remark 4.11. Observe that the functional equation of Theorem 4.10 could have been derived directly from (22) using a similar multiplication by the longest element of B_m. The proof of the functional equation is essentially that of [36, Theorem 4]; see [19, p. 707] and the proofs of [13, Theorem 5.9] and [2, Corollary 2.7] for analogous arguments. It is easily verified that $W(X,Y) = \frac{1}{1-Z_0}I_m(X^{-1}; Z_1,\dots,Z_m)$, where I_m is the "Igusa function" of [29, Definition 2.5]; the functional equation for these functions is stated as [29, Proposition 4.2] and follows directly from [36, Theorem 4]. Igusa functions and their generalizations also play central roles in explicit computations of ideal zeta functions of Lie rings [9, 30].

Remark 4.12. Consider the Heisenberg Lie ring $\mathcal{H} = \langle x, y, z\rangle_{\mathbb{Z}}$ with $[x, y] = z$ and $Z(\mathcal{H}) = \langle z\rangle$; this is the smallest non-abelian nilpotent Lie ring. Since $\mathcal{H} = \mathcal{H}_1 = \mathcal{F}_{2,2}$, for any number field K we obtain

$$\zeta^{\wedge}_{\mathcal{H}\otimes_{\mathbb{Z}}\mathcal{O}_K}(s) = \zeta_K(2s - 2d)\zeta_K(2s - 2d - 1)$$

as a special case of either Theorem 4.1 or Theorem 4.10.

The abscissa of convergence computed in Theorem 4.10 is given by the following claim. It is well-known and follows from the proof of [39, Corollary 3.1].

Lemma 4.13. *Let $m \in \mathbb{N}$. Let $W(X, Y) \in \mathbb{Q}(X, Y)$ be a rational function of the form*

$$W(X,Y) = \frac{\sum_{\sigma\in S_m} X^{-\ell(\sigma)} \prod_{j\in \mathrm{Des}(\sigma)} X^{a_j} Y^{b_j}}{\prod_{j=1}^{m-1}(1 - X^{a_j} Y^{b_j})},$$

where $a_j \in \mathbb{N} \cup \{0\}$ and $b_j \in \mathbb{N}$ for all $j \in [m]$, while $\ell(\sigma)$ and $\mathrm{Des}(\sigma)$ are as in Remark 4.5. Let K be a number field and V_K the set of finite places of K. For every $\mathfrak{p} \in V_K$, let $q_{\mathfrak{p}}$ be the cardinality of the residue field $\mathcal{O}_K/\mathfrak{p}$. Then the abscissa of convergence of the function $F(s) = \prod_{\mathfrak{p}\in V_K} W(q_{\mathfrak{p}}, q_{\mathfrak{p}}^{-s})$ is $\max_{j\in[m]} \left\{\frac{a_j+1}{b_j}\right\}$.

4.3. **The Lie rings $\mathcal{L}_{m,n}$.** We recall the family of Lie rings introduced by the first author, Klopsch, and Onn in [5, Definition 2.1]. Let $m, n \in \mathbb{N}$, with $n \geq 2$, and consider the sets

$$\begin{aligned} \mathbf{E} &= \{\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{N}_0^n : e_1 + \cdots + e_n = m - 1\} \\ \mathbf{F} &= \{\mathbf{f} = (f_1, \ldots, f_n) \in \mathbb{N}_0^n : f_1 + \cdots + f_n = m\}. \end{aligned}$$

Let $(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be the standard generators of the additive monoid $\mathbb{N}_0^n$, so that $\mathbf{b}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 in the i-th position. Then $\mathcal{L}_{m,n}$ is the Lie ring with $\mathbb{Z}$-basis

$$\{x_{\mathbf{e}} : \mathbf{e} \in \mathbf{E}\} \cup \{y_{\mathbf{f}} : \mathbf{f} \in \mathbf{F}\} \cup \{z_j : j \in [n]\},$$

and with the Lie bracket defined by the relations

$$[x_{\mathbf{e}}, y_{\mathbf{f}}] = \begin{cases} z_i & : \mathbf{f} - \mathbf{e} = \mathbf{b}_i \\ 0 & : \mathbf{f} - \mathbf{e} \notin \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \end{cases}$$

and where all other pairs of elements of the $\mathbb{Z}$-basis above commute. Then $\mathcal{L}_{m,n}$ is nilpotent of class two, and $Z(\mathcal{L}_{m,n}) = [\mathcal{L}_{m,n}, \mathcal{L}_{m,n}] = \langle z_1, \ldots, z_n\rangle$. Let $L_{m,n} = \mathcal{L}_{m,n} \otimes_{\mathbb{Z}} \mathbb{Q}$ be the associated Lie algebra. The Lie rings $\mathcal{L}_{m,n}$ provide a common generalization of two well-known families of Lie rings:

- The Lie rings $\mathcal{L}_{1,n}$, for $n \geq 2$, are the Grenham Lie rings

  $$G_n = \langle x_0, x_1, \ldots, x_n, z_1, \ldots, z_n\rangle,$$

  with relations $[x_0, x_i] = z_i$ for $i \in [n]$; all other pairs of generators commute. Their pro-isomorphic zeta functions were computed by the first author [1, §3.3.2].
- The Lie rings $\mathcal{L}_{m,2}$ are associated via the correspondence of (4) to the D^* groups of odd Hirsch length defined in [5, §1.1]. These represent commensurability classes introduced by Grunewald and Segal [15, §6] in the course of their classification of torsion-free nilpotent radicable groups of class two with finite rank and center of rank two.

Lemma 4.14. *Let K be any field. The K-Lie algebra $L_{m,n} \otimes_{\mathbb{Q}} K$ is indecomposable.*

Proof. Write L_K for $L_{m,n} \otimes_{\mathbb{Q}} K$. By slight abuse of notation, we denote the elements of the natural K-basis of L_K by $x_{\mathbf{e}}, y_{\mathbf{f}}, z_i$. Let

$$v = \sum_{\mathbf{e} \in \mathbf{E}} a_{\mathbf{e}} x_{\mathbf{e}} + \sum_{\mathbf{f} \in \mathbf{F}} c_{\mathbf{f}} y_{\mathbf{f}} + \sum_{i=1}^{n} d_i z_i \in L_K,$$

where the coefficients lie in K. Suppose that $a_{\mathbf{e}} \neq 0$ for some $\mathbf{e} \in \mathbf{E}$. We claim that $\dim_K [L_K, v] = n$. Indeed, consider the lexicographical total ordering on $\mathbf{E}$, for which $\mathbf{e} \leq \mathbf{e}'$ if there exists some $i \in [n]$ such that $e_i \leq e'_i$ and $e_j = e'_j$ for all $j < i$. Let $\widehat{\mathbf{e}}$ be maximal, with respect to this ordering, among all $\mathbf{e} \in \mathbf{E}$ such that $a_{\mathbf{e}} \neq 0$. It is easy to see that $[v, y_{\widehat{\mathbf{e}}+\mathbf{b}_1}] = a_{\widehat{\mathbf{e}}} z_1$, since $a_{\widehat{\mathbf{e}}+\mathbf{b}_1-\mathbf{b}_j} = 0$ for all $j > 1$ by the maximality of $\widehat{\mathbf{e}}$.

Consider the map $\xi : \mathbf{E} \to \mathbf{E}$ given by $\xi(\mathbf{e}) = (e_2, e_3, \ldots, e_n, e_1)$. Similarly to the above, we see that for all $j \in [n]$, if $\widehat{\mathbf{e}}_j$ is such that $\xi^{j-1}(\widehat{\mathbf{e}}_j)$ is maximal in the set $\{\xi^{j-1}(\mathbf{e}) : a_{\mathbf{e}} \neq 0\}$, then $[v, y_{\widehat{\mathbf{e}}_j+\mathbf{b}_j}] = a_{\widehat{\mathbf{e}}_j} z_j$. Thus $\dim_K [v, L_K] = \dim_K Z(L_K) = n$.

Suppose that $L_K = L_1 \oplus L_2$ is a direct sum of non-trivial K-Lie subalgebras. If $[L_2, L_2] = [L_K, L_K] = Z(L_K)$, then L_1 must be abelian. But then $L_1 \subseteq Z(L_K) \subseteq L_2$, which is impossible since we assumed $L_1 \neq 0$. Hence $\dim_K [L_2, L_2] < n$, and similarly $\dim_K [L_1, L_1] < n$. If $w \in L_1$, then $\dim_K [w, L_K] = \dim_K [w, L_1] \leq \dim_K [L_1, L_1] < n$. Similarly, $\dim_K [w, L_K] < n$ for all $w \in L_2$. Thus L_K is spanned by the set $\{v \in L_K : \dim_K [v, L_K] < n\}$. However, we just showed that this set is contained in the proper subspace spanned by $\{y_{\mathbf{f}} : \mathbf{f} \in \mathbf{F}\} \cup \{z_i : i \in [n]\}$, giving rise to a contradiction. □

As in the case of the higher Heisenberg Lie algebras considered in Section 4.2, there is no ideal $M \leq \gamma_2 L_K$ such that $\mathcal{X}(M)$ generates L_K. Thus the $Z(L_K)$-rigidity of L_K cannot be shown using Corollary 3.9.

Proposition 4.15. *Let K be a field of characteristic zero. Then the K-Lie algebra $L_K = L_{m,n} \otimes_{\mathbb{Q}} K$ is $Z(L_K)$-rigid.*

Proof. We verify the hypotheses of Theorem 3.8. Since $Z(L_K)$ is central, the absolute indecomposability of $L_K/[Z(L_K), L_K]$ is given by Lemma 4.14. Thus it suffices to show that $\mathcal{Y}(Z(L_K))$ generates L_K. We first check that $x_{\mathbf{e}} \in \mathcal{Y}(Z(L_K))$ for all $\mathbf{e} \in \mathbf{E}$. Indeed, let $v \in C_{L_K}(C_{L_K}(x_{\mathbf{e}}))$. Since the linear span of $\{x_{\mathbf{e}'} : \mathbf{e}' \in \mathbf{E}\}$ is contained in $C_{L_K}(x_{\mathbf{e}})$, it is clear that $v \equiv \sum_{\mathbf{e}' \in \mathbf{E}} a_{\mathbf{e}'} x_{\mathbf{e}'} \bmod Z(L_K)$. Suppose that $v \notin K x_{\mathbf{e}} + Z(L_K)$. Then $a_{\mathbf{e}'} \neq 0$ for some $\mathbf{e}' \neq \mathbf{e}$. Recall the order on $\mathbf{E}$ and the map $\xi : \mathbf{E} \to \mathbf{E}$ defined in the proof of Lemma 4.14. There is some $i \in [n]$ such that $e_i < e'_i$. Let $\widehat{\mathbf{e}} \in \mathbf{E}$ be such that $\xi^{i-1}(\widehat{\mathbf{e}}) = \max\{\xi^{i-1}(\mathbf{e}') : a_{\mathbf{e}'} \neq 0\}$. Then $\widehat{\mathbf{e}} \neq \mathbf{e}$ and v does not commute with $y_{\widehat{\mathbf{e}}+\mathbf{b}_i} \in C_{L_K}(x_{\mathbf{e}})$, contradicting the assumption $v \in C_{L_K}(C_{L_K}(x_{\mathbf{e}}))$. It follows that $x_{\mathbf{e}} \in \mathcal{Y}(Z(L_K))$ as claimed.

However, it is not true that $y_{\mathbf{f}} \in \mathcal{Y}(Z(L_K))$ for all $\mathbf{f} \in \mathbf{F}$. Indeed, $C_{L_K}(y_{(m-1,1,0,\ldots,0)})$ is the K-linear span of

$$\{x_{\mathbf{e}} : \mathbf{e} \in \mathbf{E} \setminus \{(m-2, 1, 0, \ldots, 0), (m-1, 0, \ldots, 0)\}\} \cup \{y_{\mathbf{f}} : \mathbf{f} \in \mathbf{F}\} \cup \{z_i : i \in [n]\}.$$

Therefore $y_{(m,0,\dots,0)} \in C_{L_K}(C_{L_K}(y_{(m-1,1,0,\dots,0)}))$, whence $y_{(m-1,1,0,\dots,0)} \notin \mathcal{Y}(Z(L_K))$. Instead, fix an arbitrary vector $\mathbf{c} = (c_1, \dots, c_{n-1}) \in K^{n-1}$ and set

$$v_{\mathbf{c}} = \sum_{\mathbf{f} \in \mathbf{F}} c_1^{f_1} c_2^{f_2} \cdots c_{n-1}^{f_{n-1}} y_{\mathbf{f}}.$$

First we show that $v_{\mathbf{c}} \in \mathcal{Y}(Z(L_K))$. Obviously, the linear span of $\{y_{\mathbf{f}} : \mathbf{f} \in \mathbf{F}\} \cup Z(L_K)$ centralizes $v_{\mathbf{c}}$. Consequently, $C_{L_K}(C_{L_K}(v_{\mathbf{c}}))$ is contained in the span of $\{y_{\mathbf{f}} : \mathbf{f} \in \mathbf{F}\} \cup Z(L_K)$. Set $\boldsymbol{\delta}_j = \mathbf{b}_j - \mathbf{b}_n$ for $j \in [n-1]$. Then

$$[x_{\mathbf{e}}, v_{\mathbf{c}}] = c_1^{e_1} \cdots c_{n-1}^{e_{n-1}} (c_1 z_1 + \cdots + c_{n-1} z_{n-1} + z_n)$$

for all $\mathbf{e} \in \mathbf{E}$, from which it is clear that $x_{\mathbf{e}+\boldsymbol{\delta}_j} - c_j x_{\mathbf{e}} \in C_{L_K}(v_{\mathbf{c}})$ for all $j \in [n-1]$ and all $\mathbf{e} \in \mathbf{E}$ such that $e_n > 0$. Hence if

$$v = \sum_{\mathbf{f} \in \mathbf{F}} a_{\mathbf{f}} y_{\mathbf{f}} \in C_{L_K}(C_{L_K}(v_{\mathbf{c}})),$$

then $a_{\mathbf{f}+\boldsymbol{\delta}_j} = c_j a_{\mathbf{f}}$ for all $j \in [n-1]$ and all $\mathbf{f} \in \mathbf{F}$ such that $f_n > 0$. Since $(f_1, \dots, f_n) = (0, \dots, 0, m) + \sum_1^{j-1} f_j \boldsymbol{\delta}_j$ and the coefficient of $y_{(0,\dots,0,m)}$ in $v_{\mathbf{c}}$ is 1, it is easy to see that v is necessarily a K-scalar multiple of $v_{\mathbf{c}}$. It follows that $v_{\mathbf{c}} \in \mathcal{Y}(Z(L_K))$ for all $\mathbf{c} \in K^{n-1}$. Thus it remains only to show that the K-linear span of $\{y_{\mathbf{f}} : \mathbf{f} \in \mathbf{F}\}$ is spanned by elements of the form $v_{\mathbf{c}}$.

Fix a natural number $N > m$. If $\mathbf{f} = (f_1, \dots, f_n) \in \mathbf{F}$, set

$$\sigma(\mathbf{f}) = f_1 + f_2 N + \cdots + f_{n-1} N^{n-2}.$$

Since $\mathbf{f}$ is determined by its first $n-1$ coordinates, the $\sigma(\mathbf{f})$ are all distinct. Note that $\sigma((0, \dots, 0, m)) = 0$. Given $\lambda \in K$, set $\mathbf{c}(\lambda) = (\lambda, \lambda^N, \lambda^{N^2}, \dots, \lambda^{N^{n-2}}) \in K^{n-1}$, so that $v_{\mathbf{c}(\lambda)} = \sum_{\mathbf{f} \in \mathbf{F}} \lambda^{\sigma(\mathbf{f})} y_{\mathbf{f}}$. Let $\lambda_1, \dots, \lambda_{|\mathbf{F}|}$ be distinct elements of K; these exist as we assumed K to be infinite. We order the elements of $\mathbf{F} = \{\mathbf{f}_1, \dots, \mathbf{f}_{|\mathbf{F}|}\}$ so that the sequence $\sigma(\mathbf{f}_i)$ is decreasing. The $|\mathbf{F}| \times |\mathbf{F}|$ matrix whose rows are $v_{\mathbf{c}(\lambda_1)}, \dots, v_{\mathbf{c}(\lambda_{|\mathbf{F}|})}$, with respect to the basis $(y_{\mathbf{f}_1}, \dots, y_{\mathbf{f}_{|\mathbf{F}|}})$, has the form

$$\begin{pmatrix} \lambda_1^{\sigma(\mathbf{f}_1)} & \lambda_1^{\sigma(\mathbf{f}_2)} & \cdots & \lambda_1^{\sigma(\mathbf{f}_{|\mathbf{F}|})} \\ \lambda_2^{\sigma(\mathbf{f}_1)} & \lambda_2^{\sigma(\mathbf{f}_2)} & \cdots & \lambda_2^{\sigma(\mathbf{f}_{|\mathbf{F}|})} \\ \ddots & \ddots & & \ddots \\ \lambda_{|\mathbf{F}|}^{\sigma(\mathbf{f}_1)} & \lambda_{|\mathbf{F}|}^{\sigma(\mathbf{f}_2)} & \cdots & \lambda_{|\mathbf{F}|}^{\sigma(\mathbf{f}_{|\mathbf{F}|})} \end{pmatrix}.$$

This is a generalized Vandermonde matrix whose determinant is, by definition,

$$S_\nu(\lambda_1, \dots, \lambda_{|\mathbf{F}|}) \prod_{1 \le i < j \le |\mathbf{F}|} (\lambda_i - \lambda_j),$$

where S_ν is the Schur polynomial associated to the partition $\nu = (\nu_1, \dots, \nu_{|\mathbf{F}|})$ whose parts are the non-negative integers $\{\sigma(\mathbf{f}_i) - |\mathbf{F}| + i : i \in [|\mathbf{F}|]\}$. Since S_ν is well-known to be a non-zero symmetric polynomial with integer coefficients (see, for instance, [25, Section I.3]) and the field K is infinite, we may choose the distinct λ_i so that $S(\lambda_1, \dots, \lambda_{|\mathbf{F}|}) \neq 0$, in which case the above matrix is invertible. For any $\mathbf{f} \in \mathbf{F}$ it follows that $y_{\mathbf{f}}$ lies in the K-linear span of the $v_{\mathbf{c}(\lambda_i)}$, and we conclude that L_K is indeed generated by $\mathcal{Y}(Z(L_K))$. □

The remaining hypotheses of Corollary 3.15 are verified in [5]. Moreover, the functions θ_1^F and θ_2^F, for arbitrary finite extensions $F/\mathbb{Q}_p$, are computed in (4.8) and (4.9) of [5]. As in that paper, set $r_1 = \binom{m+n-2}{m-1}$ and $r_2 = \binom{m+n-1}{m}$. Since the computation of the integral resulting from Corollary 3.15 is completely analogous to the case $d = 1$, which relies on Proposition 2.5 and is performed at the end of [5, §4], we omit it and only record the final result.

Theorem 4.16. *Let $m, n \in \mathbb{N}$, with $n \geq 2$. Let K be a number field and p a prime. Then*

$$\zeta^{\wedge}_{\mathcal{L}_{m,n}\otimes\mathcal{O}_K,p}(s) = \prod_{i=1}^{r} W(q_i, q_i^{-s}),$$

where $W(X,Y) \in \mathbb{Q}(X,Y)$ is the rational function

$$W(X,Y) = \frac{\sum_{w\in S_n} X^{-\ell(w)} \prod_{i\in\mathrm{Des}(w)} X_i}{\prod_{i=0}^{n}(1-X_i)}.$$

The monomials $X_i = X^{\beta_i}Y^{\gamma_i}$ are given by

$$\beta_i = \begin{cases} i(n-i) + d(r_1+r_2)((m-1)n+i)+ & \\ \displaystyle\sum_{j=1}^{i}\left(1+\frac{(m-1)(i-j+1)}{n-j+1}\right)\binom{m+j-2}{m-1}\binom{m+n-j-1}{m-1} & : i \in [n-1] \\ dn(r_1+r_2) & : i = 0 \\ dn(r_1+r_2) + \binom{2m+n-2}{2m-1} & : i = n \end{cases}$$

$$\gamma_i = \begin{cases} (1+r_1)((m-1)n+i) - m(m-1)r_1 & : i \in [n-1] \\ r_1 + n & : i = 0 \\ r_2 + n & : i = n. \end{cases}$$

Corollary 4.17. *The local pro-isomorphic zeta function $\zeta^{\wedge}_{\mathcal{L}_{m,n}\otimes\mathcal{O}_K,p}(s)$ satisfies a functional equation with symmetry factor*

$$(-1)^{(n-1)r}\left(\prod_{i=1}^{r} q_i\right)^{\binom{n}{2}+\binom{2m+n-2}{2m-1}+2dn(r_1+r_2)-(r_1+r_2+2n)s}.$$

The abscissa of convergence is $\alpha^{\wedge}_{\mathcal{L}_{m,n}\otimes_{\mathbb{Z}}\mathcal{O}_K} = \max_{i\in[n]_0}\left\{\frac{\beta_i+1}{\gamma_i}\right\}$.

Proof. The functional equation is found as in the proof of Theorem 4.10. The abscissa of convergence follows from Theorem 4.16 and Lemma 4.13. Determining in general which of these fractions is maximal is laborious already when $K = \mathbb{Q}$; see [5, §5]. □

The ideal zeta functions $\zeta^{\triangleleft}_{\mathcal{L}_{m,n},p}(s)$ were determined by Voll [39, Theorem 1.1] for arbitrary (m,n) and all primes p, when $K = \mathbb{Q}$. More is known for the Grenham Lie rings $\mathcal{L}_{1,n}$: for any number field K and any unramified prime p, the functions $\zeta^{\triangleleft}_{\mathcal{L}_{1,n}\otimes\mathcal{O}_K,p}(s)$ were computed by Carnevale, the second author, and Voll [9, Proposition 5.8].

4.4. **The maximal class Lie rings.** Let $c \geq 3$ and let $\mathcal{M}_c$ be the Lie ring over $\mathbb{Z}$ with the following presentation:

$$\mathcal{M}_c = \langle z, x_1, \ldots, x_c | [z, x_i] = x_{i+1}, i \in [c-1] \rangle.$$

Here, as always, we follow the convention that all pairs of generators not explicitly mentioned commute. Then $\mathcal{M}_c$ is a nilpotent Lie ring of class c, which is the maximal possible nilpotency class of a Lie ring of rank $c+1$. Let $M_c = \mathcal{M}_c \otimes_{\mathbb{Z}} \mathbb{Q}$ denote the associated $\mathbb{Q}$-Lie algebra. For any field K of characteristic zero, write $M_{c,K} = M_c \otimes_{\mathbb{Q}} K$.

We claim that $M_{c,K}$ is $Z(M_{c,K})$-rigid. Indeed, it is easily verified that z and $z+x_1$ each belong to $\mathcal{X}(Z(M_{c,K}))$, and hence $\mathcal{X}(Z(M_{c,K}))$ generates $M_{c,K}$ as a Lie algebra. Then rigidity follows by Segal's criterion (Corollary 3.9; note also Remark 3.10). Consider the decomposition $M_{c,K} = U_1 \oplus \cdots \oplus U_{c+1}$, where $U_1 = \langle z \rangle_K$ and $U_i = \langle x_{i-1} \rangle_K$ for all $i \in [2, c+1]$. The subspace $V_i = U_i \oplus \cdots \oplus U_{c+1}$, for any $i \in [c+1]$, is preserved by any K-automorphism of $M_{c,K}$; indeed, V_2 is the unique maximal abelian subalgebra (since $c \geq 3$), whereas $V_i = \gamma_{i-1} M_{c,K}$ for all $i \in [3, c+1]$. Let $y = \lambda z + \sum_{i=1}^{c} a_i x_i \in M_{c,K}$ and $y' = \mu x_1 + \sum_{i=2}^{c} b_i x_i \in V_2$ be arbitrary, where $\lambda, \mu \in K^\times$ and $a_i, b_i \in K$. The matrix

$$\begin{pmatrix} \lambda & a_1 & a_2 & a_3 & \cdots & a_c \\ & \mu & b_2 & b_3 & \cdots & b_c \\ & & \lambda\mu & \lambda b_2 & \cdots & \lambda b_{c-1} \\ & & & \lambda^2 \mu & \cdots & \lambda^2 b_{c-2} \\ & & & & \ddots & \vdots \\ & & & & & \lambda^{c-1}\mu \end{pmatrix}, \tag{29}$$

with respect to the basis $(z, x_1, \ldots, x_c)$, corresponds to $\varphi \in \mathrm{Aut}_K M_{c,K}$ such that $\varphi(z) = y$ and $\varphi(x_1) = y'$; it is the unique automorphism with this property since z and x_1 generate $M_{c,K}$. We have thus determined the structure of the algebraic automorphism group $\mathfrak{Aut}\, M_c$, and it is clear that the three assumptions of Section 2.2 are satisfied.

Recall the notation of Section 3.2; for instance, $\mathbf{H}$ is the reductive subgroup of $\mathfrak{Aut}\, M_c$ corresponding to diagonal matrices in (29). If $F/\mathbb{Q}_p$ is a finite extension and $h = \mathrm{diag}(\lambda, \mu, \lambda\mu, \ldots, \lambda^{c-1}\mu) \in \mathbf{H}^+(F)$, then

$$\theta_j^F(h) = \begin{cases} |\mu|_F^{-1} & : j = 1 \\ |\lambda^{j-1}\mu|_F^{-2} & : 2 \leq j \leq c. \end{cases}$$

Indeed, $\theta_j^F(h)$ measures the size of the set of elements $a_1 \in F$, if $j = 1$, or of pairs $(a_j, b_j) \in F^2$ if $j > 1$, such that $\lambda^{j-1}\mu a_j \in \mathcal{O}_F$ and $\lambda^{j-1}\mu b_j \in \mathcal{O}_F$. Note that $\det h = \lambda^{\binom{c}{2}+1}\mu^c$. Moreover, Assumption 2.3 is realized by a polynomial map in the sense of Section 3.2: the class $\overline{g} \in NH^+/N_i \cap (G/N_i)^+$, in the notation of that section, determines $\lambda, \mu, a_1, \ldots, a_{i-1}, b_2, \ldots, b_{i-1} \in \mathcal{O}_F$ in (29), and one may take the lifting $g \in G^+$ to correspond to the matrix with $a_j = b_j = 0$ for all $j \in [i, c]$.

The remaining hypotheses of Corollary 3.15 obviously hold, enabling us to conclude, for any prime p and any number field K of degree $d = [K : \mathbb{Q}]$, that

$$\zeta^{\wedge}_{\mathcal{M}_c \otimes \mathcal{O}_K, p}(s) = \prod_{i=1}^{r} \int_{\mathrm{GL}_1^+(R_i)^2} |\lambda_i|_{R_i}^{\left(\binom{c}{2}+1\right)s - (c-1)(2d+c-2)} |\mu_i|_{R_i}^{cs - (2d+2c-3)} d\mu_{\mathrm{GL}_1(R_i)^2}(\lambda_i, \mu_i).$$

Using Example 2.6 to evaluate the integral, we arrive at the following statement.

Theorem 4.18. *Let K be a number field of degree $d = [K : \mathbb{Q}]$, and let p be a prime. Then*

$$\zeta^{\wedge}_{\mathcal{M}_c \otimes \mathcal{O}_K, p}(s) = \prod_{i=1}^{r} \left(\zeta_{q_i} \left(\left(\binom{c}{2} + 1 \right) s - (c-1)(2d+c-2) \right) \zeta_{q_i} \left(cs - (2d+2c-3) \right) \right),$$

giving the global pro-isomorphic zeta function

$$\zeta^{\wedge}_{\mathcal{M}_c \otimes \mathcal{O}_K}(s) = \zeta_K \left(\left(\binom{c}{2} + 1 \right) s - (c-1)(2d+c-2) \right) \zeta_K \left(cs - (2d+2c-3) \right)$$

with abscissa of convergence

$$\alpha^{\wedge}_{\mathcal{M}_c \otimes \mathcal{O}_K} = \begin{cases} 2 & : d = 1 \\ \frac{(c-1)(2d+c-2)+1}{\binom{c}{2}+1} & : d \geq 2. \end{cases}$$

The local pro-isomorphic zeta functions satisfy the following functional equation:

$$\zeta^{\wedge}_{\mathcal{M}_c \otimes \mathcal{O}_K, p}(s)|_{p \mapsto p^{-1}} = \left(\prod_{i=1}^{r} q_i \right)^{c(2d+c-2)-1-\left(\binom{c+1}{2}+1\right)s} \zeta^{\wedge}_{\mathcal{M}_c \otimes \mathcal{O}_K, p}(s).$$

Proof. We only discuss the abscissa of convergence, as the rest of the statement is clear. From the properties of the Dedekind zeta function,

$$\alpha^{\wedge}_{\mathcal{M}_c \otimes \mathcal{O}_K} = \max \left\{ \frac{2(d+c-1)}{c}, \frac{(c-1)(2d+c-2)+1}{\binom{c}{2}+1} \right\}.$$

Computing the difference between these two fractions and observing that its numerator is a quadratic polynomial in c, an elementary analysis allows us to determine its sign. Observe that $\frac{2(d+c-1)}{c} = 2$ if $d = 1$. □

In the case $d = 1$, this result was obtained in the first author's thesis [1, Section 3.3.1]. The functions $\zeta^{\triangleleft}_{\mathcal{M}_c}(s)$ and $\zeta^{\leq}_{\mathcal{M}_c}(s)$ are known for $c \in \{3, 4\}$ by work of Taylor and Woodward, as well as the local factors of $\zeta^{\triangleleft}_{\mathcal{M}_3 \otimes \mathcal{O}_K}(s)$ for quadratic number fields K at split primes p; see Theorems 2.26, 2.29, and 2.37 of [14]. We are not aware of explicit computations of ideal or subring zeta functions for any nilpotent Lie ring of class greater than 4. However, functional equations for almost all local factors of $\zeta^{\triangleleft}_{\mathcal{M}_c}(s)$, for arbitrary c, were proved by Voll [38, Theorem 4.8]; the analogous statement for any $\zeta^{\leq}_{\mathcal{M}_c \otimes \mathcal{O}_K}(s)$ is a special case of [37, Corollary 1.1].

4.5. **A non-graded example.** A Lie ring of $\mathbb{Z}$-rank $c+1$ is called *filiform* if it is nilpotent of class c, which is the maximal possible. The maximal class Lie rings $\mathcal{M}_c$ considered in the previous section are filiform, but they are not the only ones. Consider the Lie ring $\mathcal{FIL}_4$ of class 4 with the following presentation:

$$\mathcal{FIL}_4 = \langle z, x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_4, [z, x_i] = x_{i+1} \text{ for all } i \in [3] \rangle,$$

with the usual convention that all other pairs of generators commute. Let $\mathrm{Fil}_4 = \mathcal{FIL}_4 \otimes_{\mathbb{Z}} \mathbb{Q}$, and let K be any field of characteristic zero. Observe that the underlying $\mathbb{Z}$-module of $\mathcal{FIL}_4$ is the same as that of $\mathcal{M}_4$, and consider the decomposition

$\mathrm{Fil}_4 \otimes_{\mathbb{Q}} K = U_1 \oplus U_2 \oplus U_3 \oplus U_4 \oplus U_5$ that was defined in the previous section for $M_{4,K}$. For every $i \in [5]$, the subspace $V_i = U_i \oplus \cdots \oplus U_5$ is preserved by any $\varphi \in \mathrm{Aut}_K(\mathrm{Fil}_4 \otimes_{\mathbb{Q}} K)$; indeed, $V_2 = \{v \in \mathrm{Fil}_4 \otimes_{\mathbb{Q}} K : \dim_K C_{\mathrm{Fil}_4 \otimes K}(v) \geq 3\}$, whereas $V_i = \gamma_{i-1}(\mathrm{Fil}_4 \otimes_{\mathbb{Q}} K)$ for $i \in \{3, 4, 5\}$. Thus φ corresponds to an upper triangular matrix with respect to the basis (z, x_1, x_2, x_3, x_4), with the same first two rows and diagonal elements as in (29). The relation $[x_1, x_2] = x_4$ implies that $\lambda^3 \mu x_4 = \varphi(x_4) = [\varphi(x_1), \varphi(x_2)] = \lambda \mu^2 x_4$. Since $\lambda, \mu \in K^\times$, we obtain $\mu = \lambda^2$. One checks that $\mathrm{Aut}_K(\mathrm{Fil}_4 \otimes_{\mathbb{Q}} K)$ is exactly the following:

$$\left\{ \left(\begin{array}{ccccc} \lambda & a_1 & a_2 & a_3 & a_4 \\ 0 & \lambda^2 & b_2 & b_3 & b_4 \\ 0 & 0 & \lambda^3 & \lambda b_2 & \lambda b_3 + a_1 b_2 - \mu a_2 \\ 0 & 0 & 0 & \lambda^4 & \lambda^2 b_2 + a_1 \lambda^3 \\ 0 & 0 & 0 & 0 & \lambda^5 \end{array} \right) \middle| \begin{array}{rcl} \lambda & \in & K^\times \\ a_1, a_2, a_3, a_4 & \in & K \\ b_2, b_3, b_4 & \in & K \end{array} \right\}. \tag{30}$$

It is readily verified that z and $z + x_1$ are contained in $\mathcal{X}(Z(\mathrm{Fil}_4 \otimes_{\mathbb{Q}} K))$ and hence that $\mathrm{Fil}_4 \otimes_{\mathbb{Q}} K$ is $Z(\mathrm{Fil}_4 \otimes_{\mathbb{Q}} K)$-rigid by Corollary 3.9 and Remark 3.10. For any prime p, the decomposition of Fil_4 considered above satisfies the hypotheses of Corollary 3.15. For any finite extension $F/\mathbb{Q}_p$ with residue field of cardinality q, we read off from (30) that the elements $h \in \mathbf{H}^+(F)$ are those of the form $h = \mathrm{diag}(\lambda, \lambda^2, \lambda^3, \lambda^4, \lambda^5)$ for some $\lambda \in \mathrm{GL}_1^+(F) = \mathcal{O}_F \setminus \{0\}$. By essentially the same computation as for $\mathcal{M}_4$, keeping in mind that $\mu = \lambda^2$, we obtain that

$$\theta_j^F(h) = \begin{cases} |\lambda|_F^{-2} & : j = 1 \\ |\lambda|_F^{-2(j+1)} & : 2 \leq j \leq 4. \end{cases}$$

Since $\det h = \lambda^{15}$, it follows from Corollary 3.15 that for any number field K of degree d and any prime p, we have

$$\zeta^{\wedge}_{\mathcal{FIL}_4 \otimes \mathcal{O}_K, p}(s) = \prod_{i=1}^{r} \int_{\mathrm{GL}_1^+(R_i)} |\lambda|^{15s - 16 - 10d} d\mu_{R_i^\times}(\lambda) = \prod_{i=1}^{r} \zeta_{q_i}(15s - 16 - 10d).$$

We have thus established the following.

Theorem 4.19. *Let $\mathcal{FIL}_4$ be the filiform Lie ring defined above, let K be a number field of degree $d = [K : \mathbb{Q}]$, and let p be a prime. Then*

$$\zeta^{\wedge}_{\mathcal{FIL}_4 \otimes \mathcal{O}_K, p}(s) = \prod_{i=1}^{r} \zeta_{q_i}(15s - 16 - 10d),$$

and this pro-isomorphic local zeta factor satisfies the functional equation

$$\zeta^{\wedge}_{\mathcal{FIL}_4 \otimes \mathcal{O}_K, p}(s)|_{p \mapsto p^{-1}} = (-1)^r \left(\prod_{i=1}^{r} q_i \right)^{16 + 10d - 15s} \zeta^{\wedge}_{\mathcal{FIL}_4 \otimes \mathcal{O}_K, p}(s).$$

The abscissa of convergence of $\zeta^{\wedge}_{\mathcal{FIL}_4 \otimes \mathcal{O}_K}(s) = \zeta_K(15s - 16 - 10d)$ is $\frac{17+10d}{15}$.

Observe that $\sum_{i=1}^{4} \mathrm{rk}_{\mathbb{Z}} \gamma_i(\mathcal{FIL}_4 \otimes_{\mathbb{Z}} \mathcal{O}_K) = d(5 + 3 + 2 + 1) = 11d$. For all primes p that are unramified in K, the symmetry factor is $(-1)^r p^{d(16+10d) - 15ds}$, and thus the statement of Conjecture 1.3 fails. It is easy to see that $\mathcal{FIL}_4$ is not a graded Lie ring; this example illustrates why Conjecture 1.3 applies only to graded Lie rings.

Theorem 4.19 generalizes a computation of the first author [1, §3.3.11] for the case $K = \mathbb{Q}$. The ideal zeta function $\zeta^{\triangleleft}_{\mathcal{FIL}_4}(s)$ was computed by Woodward [14, Theorem 2.39]. Its local factors do not satisfy functional equations. The failure of $\mathcal{FIL}_4$ to be a graded Lie ring implies that it does not satisfy the homogeneity condition of [21, Condition 1.5]. This condition guarantees the existence of functional equations for the local factors of the ideal zeta function by [21, Theorem 1.7].

4.6. **A family of Lie rings lacking functional equations.** The first author and Klopsch [3] have constructed a nilpotent Lie ring $\mathcal{L}$, which they denote as Λ, none of whose local pro-isomorphic zeta functions satisfy functional equations in the sense defined in the introduction to the present paper. We show that this property is retained by the base extensions $\mathcal{L} \otimes_{\mathbb{Z}} \mathcal{O}_K$, for any number field K, by computing their pro-isomorphic zeta functions explicitly.

As in Section 4.1, let $\mathcal{F}_{4,3}$ be the free nilpotent Lie ring of nilpotency class four on three generators X, Y, Z. For brevity we use the left-normed simple product notation for the Lie bracket, so that, for instance, we write XYZ for $[[X, Y], Z]$. Let $\mathcal{I} \leq \mathcal{F}_{4,3}$ be the ideal generated by $YXXX - YZY$ and $ZXXX - ZYZ$, and set $\mathcal{L} = \mathcal{F}_{4,3}/\mathcal{I}$. Write x, y, z for the projections to $\mathcal{L}$ of X, Y, Z, respectively. In [3, (3.4)] it is determined that $\mathcal{L}$ is a free $\mathbb{Z}$-module of rank 25 and that a basis is given by

$$\begin{aligned}(b_1, \ldots, b_{25}) \quad = \quad & (x, y, z, xy, xz, yz, xyy, xzz, xyz, xzy, xyx, xzx, xyyy, xzzz, xyxx, xzxx, \\ & xyxy, xzxz, xyxz, xzxy, xyzx, xyzz, xzyy, xyzy, xzyz).\end{aligned}$$

As usual, write $L = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $L_p = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Q}_p$ for any prime p; it is easy to verify that $Z(L_p) = \gamma_4 L_p$. By explicit computations involving repeated application of the identities (3.2) and (3.3) of [3], we find that $x, y, z \in \mathcal{X}(Z(L_p))$. Hence L_p is $Z(L_p)$-rigid by Corollary 3.9 and Remark 3.10. The algebraic automorphism group $\mathfrak{Aut}\, L$ was determined in [3, Theorem 4.2]. It follows from the description of $\mathfrak{Aut}\, L$ given there that the decomposition $L_p = U_1 \oplus U_2 \oplus U_3 \oplus U_4$, where

$$U_1 = \langle b_1, b_2, b_3 \rangle_{\mathbb{Q}_p}, \quad U_2 = \langle b_4, b_5, b_6 \rangle_{\mathbb{Q}_p}, \quad U_3 = \langle b_7, \ldots, b_{12} \rangle_{\mathbb{Q}_p}, \quad U_4 = \langle b_{13}, \ldots, b_{25} \rangle_{\mathbb{Q}_p},$$

satisfies Assumptions 2.1 and 2.2; note that $V_i = \gamma_i L_p$ for all $i \in [4]$. The argument on [3, p. 505] shows that Assumption 3.14 holds. Thus Corollary 3.15 is applicable.

As in [3, (5.1)], we see that $\mathbf{H}(F)$, for any field $F/\mathbb{Q}_p$ with residue field of cardinality q, consists of the diagonal matrices of the form

$$\begin{aligned} h = \operatorname{diag}(& a, b, c, ab, ac, bc, ab^2, ac^2, abc, abc, a^2b, a^2c, ab^3, ac^3, a^3b, a^3c, \\ & a^2b^2, a^2c^2, a^2bc, a^2bc, a^2bc, abc^2, ab^2c, ab^2c, abc^2), \end{aligned} \tag{31}$$

where $a, b, c \in F^\times$ satisfy $a^3 = bc$. Analogously to the computations of [3, Section 5], where the case $F = \mathbb{Q}_p$ is treated, we determine that, for $h \in \mathbf{H}^+(F)$ as in (31), we have

$$\begin{aligned} \theta_1^F(h) &= |a^3 b^4 c^4|_F^{-1} \min\{|b|_F^{-1}, |c|_F^{-1}\} \\ \theta_2^F(h) &= |a^{24} b^{15} c^{15}|_F^{-1} \\ \theta_3^F(h) &= |a^{66} b^{45} c^{45}|_F^{-1}. \end{aligned}$$

It is clear from (31) that $|\det h|_F = |a^{33}b^{23}c^{23}|_F$. Thus, for any number field K of degree d and any prime p, by Corollary 3.15 we find that $\zeta^{\wedge}_{\mathcal{L}\otimes\mathcal{O}_K,p}(s)$ is given by:

$$\prod_{i=1}^{r}\int_{\mathbf{H}^+(R_i)}|a|_{R_i}^{33s-(27+66d)}|b|_{R_i}^{23s-(19+45d)}|c|_{R_i}^{23s-(19+45d)}\min\{|b|_{R_i}^{-1},|c|_{R_i}^{-1}\}d\mu_{\mathbf{H}(R_i)}(h).$$

Since the computation of this integral is very similar to the one performed in [3], we omit the details and proceed to state the final result.

Theorem 4.20. *Let K be a number field of degree $d = [K:\mathbb{Q}]$, and let p be prime. Then*

$$\zeta^{\wedge}_{\mathcal{L}\otimes\mathcal{O}_K,p}(s) = \prod_{i=1}^{r}\frac{1+q_i^{84+201d-102s}+2q_i^{85+201d-102s}+2q_i^{170+402d-204s}}{(1-q_i^{84+201d-102s})(1-q_i^{171+402d-204s})}.$$

None of the local pro-isomorphic zeta factors of $\mathcal{L}\otimes\mathcal{O}_K$ satisfy a functional equation.

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Department of Mathematics, Ort Braude College, P.O. Box 78, Snunit St. 51, Karmiel 2161002, Israel
E-mail address: berman@braude.ac.il

Department of Mathematics, Bar-Ilan University, Ramat Gan 5290002, Israel
E-mail address: mschein@math.biu.ac.il