A FAMILY OF IRREDUCIBLE SUPERSINGULAR REPRESENTATIONS OF $\mathrm{GL}_2(F)$ FOR SOME RAMIFIED p-ADIC FIELDS

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Abstract. We construct infinite families of irreducible supersingular mod p representations of $\mathrm{GL}_2(F)$ with $\mathrm{GL}_2(\mathcal{O}_F)$-socle compatible with Serre's modularity conjecture, where $F/\mathbb{Q}_p$ is any finite extension with residue field $\mathbb{F}_{p^2}$ and ramification degree $e \leq (p-1)/2$.

The irreducible smooth mod p representations of $\mathrm{GL}_2(F)$ admitting a central character, where $F/\mathbb{Q}_p$ is a finite extension, were classified by Barthel and Livné [2, Theorems 33-34], except for the *supersingular* representations. The irreducible supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ were determined by Breuil [4, Théorème 1.1]. By contrast, little is known about the irreducible supersingular representations of $\mathrm{GL}_2(F)$ when $F \neq \mathbb{Q}_p$. Wu [17, Theorem 1.1] has shown that they do not have finite presentation; this was proved earlier by Schraen [16, Théorème 2.23] in the case $[F : \mathbb{Q}_p] = 2$. Thus a direct construction of these representations appears to be difficult. However, the method of diagrams introduced by Paškūnas [12] can be used to show the existence of supersingular representations with certain properties; indeed, such representations of $\mathrm{GL}_2(F)$, for arbitrary F, were produced in [12, Theorem 6.25].

For applications to the mod p local Langlands correspondence, one is most interested in supersingular representations of $\mathrm{GL}_2(F)$ whose $\mathrm{GL}_2(\mathcal{O}_F)$-socle is compatible with Serre's modularity conjecture; see condition (1) of [6]. We shall refer to such representations as having *good socle*. While the supersingular representations constructed in [12] only have good socle when $F = \mathbb{Q}_p$, Breuil and Paškūnas constructed infinite families of diagrams giving rise to supersingular irreducible representations of $\mathrm{GL}_2(F)$ with good socle for all unramified $F/\mathbb{Q}_p$. These families are parametrized by choices of a collection of isomorphisms between one-dimensional $\overline{\mathbb{F}}_p$-vector spaces. While the construction of [6] is not exhaustive, the supersingular representations considered there play an important role in ongoing investigations towards the local Langlands correspondence; see [5] and references therein.

In this note we construct families of diagrams giving rise to supersingular representations of $\mathrm{GL}_2(F)$ with good socle for arbitrary finite extensions $F/\mathbb{Q}_p$ with residue field $\mathbb{F}_{p^2}$ and ramification degree $e \leq (p-1)/2$. Our family specializes to that of [6] when F is unramified. In general, our family is parametrized by choices of isomorphisms analogous to those in [6], as well as a choice of an element in a large finite set, namely the set of Hamiltonian walks on an $e \times e$ square lattice. The number of such Hamiltonian walks grows exponentially in e^2; see [3, §9] and numerical computations in [10, §XD]. To the

Date: May 19, 2021.

author's knowledge this is the first construction of supersingular represenations of good socle for ramified p-adic fields.

1. Preliminaries

Let $F/\mathbb{Q}_p$ be a finite extension, and let F_0 be the maximal unramified subextension. Let $\mathcal{O}$ be the ring of integers of F, fix a uniformizer $\pi \in \mathcal{O}$, and let $k = \mathcal{O}/(\pi)$ denote the residue field. Let $q = p^f$ be the cardinality of k. A Serre weight is an irreducible $\overline{\mathbb{F}}_p$-representation of the finite group $\Gamma = \mathrm{GL}_2(k)$, which can be viewed as a representation of $K = \mathrm{GL}_2(\mathcal{O})$ or of $\mathrm{GL}_2(\mathcal{O}_{F_0})$ by inflation. Every irreducible smooth $\overline{\mathbb{F}}_p$-representation of these two profinite groups arises from a Serre weight in this way. Let $B \leq \Gamma$ be the subgroup of upper triangular matrices, and let $U \leq B$ be the subgroup of upper triangular matrices all of whose eigenvalues are 1. Let $K(1)$ be the kernel of the reduction map $K \twoheadrightarrow \Gamma$, and let I and $I(1)$ be the preimages of B and U, respectively.

Given a Serre weight σ, write $\chi(\sigma)$ for the character by which the diagonal torus $H \leq B$ acts on σ^U; we also view $\chi(\sigma)$ as a character of B by inflation. For $v \in \sigma$ and $g \in G = \mathrm{GL}_2(F)$, we denote by $g \otimes v$ the element of the compact induction $\mathrm{ind}_{KZ}^G \sigma$ supported on the right coset KZg^{-1} and sending g^{-1} to v. Set the notations $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$ and $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and denote $\Pi = \alpha w = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$. Let Z be the center of G and Id_2 the identity matrix. The normalizer $N = N_G(I(1))$ is generated by IZ and Π. Write $\chi^w : H \to \overline{\mathbb{F}}_p^\times$ for the character $\chi^w(h) = \chi(whw)$; this is denoted χ^s in [6]. Let $\sigma^{[w]}$ be the unique Serre weight distinct from σ such that $\chi(\sigma^{[w]}) = \chi(\sigma)^w$. We view Serre weights as representations of KZ by letting $\pi\mathrm{Id}_2$ act trivially.

Recall that a basic 0-diagram is a triple (D_0, D_1, ι), where D_0 and D_1 are smooth $\overline{\mathbb{F}}_p$-representations of KZ and of N, respectively, such that $\pi\mathrm{Id}_2$ acts trivially, and $\iota : D_1 \hookrightarrow D_0$ is an injective map inducing an IZ-equivariant isomorphism $D_1 \simeq D_0^{I(1)}$. For instance, if V is a smooth representation of G, then $(V^{K(1)}, V^{I(1)}, \mathrm{can})$ is a basic 0-diagram, where $\mathrm{can} : V^{I(1)} \hookrightarrow V^{K(1)}$ is the canonical inclusion. This is a special case of diagrams as defined in [12, Definition 5.14]; in this note we work only with 0-basic diagrams.

Fix an embedding $\varepsilon_0 : k \hookrightarrow \overline{\mathbb{F}}_p$ and define embeddings ε_i for every $i \in \mathbb{N}$ by means of the recursion $\varepsilon_i = \varepsilon_{i-1}^p$. Then every Serre weight has the form

$$\sigma = \bigotimes_{i=0}^{f-1} \left(\mathrm{Sym}^{r_i} k^2 \otimes_{k,\varepsilon_i} \overline{\mathbb{F}}_p\right) \otimes \eta,$$

where $0 \leq r_i \leq p-1$ for every i and $\eta : k^\times \to \overline{\mathbb{F}}_p^\times$ is a character. We say that a Serre weight σ is regular if $1 \leq r_i \leq p-2$ for every i. Observe that a regular Serre weight σ as above is determined by the f-tuple $(r_0, \dots, r_{f-1})$ and its central character $\xi : Z(\Gamma) \simeq k^\times \to \overline{\mathbb{F}}_p^\times$. If ξ has been fixed, then we will write $\sigma = (r_0, \dots, r_{f-1})$ for short; then $\sigma^{[w]} = (p-1-r_0, \dots, p-1-r_{f-1})$ An irreducible $\overline{\mathbb{F}}_p$-representation V of G is called *supersingular* if $\lambda = 0$ for every surjective map (equivalently, for one such map) of the form $\mathrm{ind}_{KZ}^G \sigma/(T-\lambda)\,\mathrm{ind}_{KZ}^G \sigma \to V$, where $\lambda \in \overline{\mathbb{F}}_p$ and $T \in \mathrm{End}_G(\mathrm{ind}_{KZ}^G \sigma)$ is the operator of [2, Proposition 8].

Lemma 1. *Let σ be a Serre weight and let $0 \neq v \in \sigma^{I(1)}$. Then $\langle \alpha \otimes wv \rangle_K \subset (\operatorname{ind}_{KZ}^G \sigma)^{K(1)}$. Moreover, $\langle \alpha \otimes wv \rangle_K \simeq \operatorname{Ind}_I^K \chi^w$ and $\operatorname{soc}_K(\langle \alpha \otimes wv \rangle_K) = T(\langle \mathrm{id} \otimes v \rangle_K)$.*

Proof. Simple computations using [6, Lemma 2.7] and an explicit description of the operator T; see, for instance, [15, Lemma 2.1]. □

Let G_F be the absolute Galois group of F and $I_F \leq G_F$ the inertia subgroup. Let k' be the quadratic extension of k, let F^{nr} be the maximal unramified extension of F, and let $L'/L/F^{\mathrm{nr}}$ be a tower of totally ramified extensions such that $\mathrm{Gal}(L/F^{\mathrm{nr}}) \simeq k^\times$ and $\mathrm{Gal}(L'/F^{\mathrm{nr}}) \simeq (k')^\times$. Consider the natural projections $\nu : I_F \to \mathrm{Gal}(L/F^{\mathrm{nr}})$ and $\nu' : I_F \to \mathrm{Gal}(L'/F^{\mathrm{nr}})$. Let $\omega_{2f} = \varepsilon_0' \circ \nu' : I_F \to \overline{\mathbb{F}}_p^\times$ be a fundamental character of level $2f$ corresponding to an embedding $\varepsilon_0' : k' \hookrightarrow \overline{\mathbb{F}}_p$ that restricts to ε_0 on k. We also denote the analogous character of I_{F_0} by ω_{2f}. Let $\rho' : G_{F_0} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible Galois representation such that $\rho'_{|I_{F_0}}$ is isomorphic to a twist of

$$\omega_{2f}^{\sum_{i=0}^{f-1} p^i(r_i+1)} \oplus \omega_{2f}^{q\sum_{i=0}^{f-1} p^i(r_i+1)} \tag{1}$$

and $\det \rho' = \xi \circ \nu$. Let $\mathcal{D}(r_0, \ldots, r_{f-1})$ denote the set of modular Serre weights associated to ρ' in [7, §3.1]; see [6, §11] for an explicit description. If $1 \leq r_0 \leq p-2$ and $0 \leq r_i \leq p-3$ for $i > 0$, then $\mathcal{D}(r_0, \ldots, r_{f-1})$ consists of 2^f distinct Serre weights.

Let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible mod p Galois representation and let $\mathcal{D}(\rho)$ be the set of Serre weights associated to it in [13, §2]; see [8, §7] for a more modern perspective on the weight part of Serre's modularity conjecture. It was observed in [14, Proposition 3.1] that $\mathcal{D}(\rho)$ is a union of sets of Serre weights associated to irreducible representations of G_{F_0}.

Assume from now on that ρ is generic, namely that $\rho_{|I_F}$ is isomorphic to a twist of (1) with $2e-1 \leq r_0 \leq p-2$ and $2e-2 \leq r_i \leq p-3$ for $i > 0$. Suppose that $\det \rho = \xi \circ \nu$. Consider the set $\Delta = \{(\delta_0, \ldots, \delta_{f-1}) : \forall i,\ 0 \leq \delta_i \leq e-1\}$. Then

$$\mathcal{D}(\rho) = \coprod_{\underline{\delta} \in \Delta} \mathcal{D}(r_0 - 2\delta_0, \ldots, r_{f-1} - 2\delta_{f-1}).$$

We associate to each element σ of this disjoint union a pair $(\underline{\delta}_\sigma, J_\sigma)$, where $\underline{\delta}_\sigma = (\delta_0, \ldots, \delta_{f-1})$ is determined by $\sigma \in \mathcal{D}(r_0 - 2\delta_0, \ldots, r_{f-1} - 2\delta_{f-1})$, and $J_\sigma \subseteq \{0, \ldots f-1\}$ is the set associated to σ in [6, §11] as an element of $\mathcal{D}(r_0 - 2\delta_0, \ldots, r_{f-1} - 2\delta_{f-1})$.

For the rest of this note, assume $f = 2$.

Lemma 2. *Let $\sigma = (r_0, r_1)$ be a regular Serre weight with central character ξ. The socle filtration of $\operatorname{Ind}_B^\Gamma \chi(\sigma)^w$ is as follows, reading from left to right:*

$$(r_0, r_1) \rule[0.5ex]{2em}{0.4pt} (p-2-r_0, r_1-1) \oplus (r_0-1, p-2-r_1) \rule[0.5ex]{2em}{0.4pt} (p-1-r_0, p-1-r_1).$$

Proof. This is a special case of [6, Theorem 2.4], which itself was originally established by Bardoe and Sin [1, Theorem C]. □

Given a Serre weight $\sigma = (r_0, r_1)$ as above, define $Q_{\{0\}}(\sigma)$ and $Q_{\{1\}}(\sigma)$ to be the quotients of $\operatorname{Ind}_B^\Gamma \chi(p-2-r_0, r_1+1)$ and $\operatorname{Ind}_B^\Gamma \chi(r_0+1, p-2-r_1)$, respectively, having socle σ. It is clear from Lemma 2 that these are Γ-modules of length two, and it is easy to see that $\dim_{\overline{\mathbb{F}}_p} Q_{\{0\}}(\sigma)^U = \dim_{\overline{\mathbb{F}}_p} Q_{\{1\}}(\sigma)^U = 2$. We view $Q_{\{0\}}(\sigma)$ and $Q_{\{1\}}(\sigma)$ as K-modules by inflation.

Let $(\delta_0, \delta_1) \in \Delta$, and let $\widetilde{D}_0(\delta_0, \delta_1)$ be the K-module generated by the $I(1)$-invariants of the K-module $D_0(\rho')$ associated to $\mathcal{D}(r_0 - 2\delta_0, r_1 - 2\delta_1)$ in [6, §13]. This is a direct sum of the following four K-modules of length two that factor through Γ:

$$\begin{array}{rcl} (r_0 - 2\delta_0, r_1 - 2\delta_1) & \_\_\_ & (r_0 - 2\delta_0 + 1, p - r_1 + 2\delta_1 - 2) \\ (r_0 - 2\delta_0 - 1, p - r_1 + 2\delta_1 - 2) & \_\_\_ & (p - r_0 + 2\delta_1 - 1, p - r_1 + 2\delta_1 - 1) \\ (p - r_0 + 2\delta_0 - 2, r_1 - 2\delta_1 + 1) & \_\_\_ & (r_0 - 2\delta_0, r_1 - 2\delta_1 + 2) \\ (p - r_0 + 2\delta_0 - 1, p - r_1 + 2\delta_1 - 3) & \_\_\_ & (p - r_0 + 2\delta_0, r_1 - 2\delta_1 + 1). \end{array} \tag{2}$$

Observe that $\widetilde{D}_0(\delta_0, \delta_1) = \bigoplus_{\sigma \in \mathcal{D}(r_0 - 2\delta_0, r_1 - 2\delta_1)} \widetilde{D}_{0,\sigma}$, where $\widetilde{D}_{0,\sigma} = Q_{\{0\}}(\sigma)$ if $J_\sigma = \{0\}$ or $J_\sigma = \{1\}$, and $\widetilde{D}_{0,\sigma} = Q_{\{1\}}(\sigma)$ otherwise.

2. Construction of diagrams

We would now like to follow the spirit, although not the actual technique, of the constructions of non-admissible irreducible $\overline{\mathbb{F}}_p$-representations of G in [11, 9] and obtain an irreducible diagram from $\widetilde{D}_0(\rho) = \bigoplus_{(\delta_0, \delta_1) \in \Delta} \widetilde{D}_0(\delta_0, \delta_1)$ by defining an action of Π on $\widetilde{D}_0(\rho)^{I(1)}$ that interweaves the direct summands $\widetilde{D}_0(\delta_0, \delta_1)$. In fact, we first replace $\widetilde{D}_0(\rho)$ with a new K-module $D_0(\rho)$ by modifying some of the components $\widetilde{D}_{0,\sigma}$.

If $\sigma \in \mathcal{D}(\rho)$, let $\kappa(\sigma)$ denote the K-cosocle of $\widetilde{D}_{0,\sigma}$. Note that $\kappa(\sigma)$ is a Serre weight. For generic ρ the following is obvious by inspection.

Lemma 3. *Let $\sigma, \tau \in \mathcal{D}(\rho)$. Then $\kappa(\sigma) \simeq \kappa(\tau)^{[w]}$ if and only if one of the following holds:*

1. *The pairs associated to σ and τ are $((\delta_0, \delta_1), \{0\})$ and $((\delta_0, \delta_1 + 1), \{1\})$ for some $(\delta_0, \delta_1) \in \Delta$.*
2. *The pairs associated to σ and τ are $((\delta_0, \delta_1), \{0, 1\})$ and $((\delta_0 + 1, \delta_1), \varnothing)$ for some $(\delta_0, \delta_1) \in \Delta$.*

Consider the graph with vertex set Δ, where two vertices (δ_0, δ_1) and (δ_0', δ_1') are adjacent if $(\delta_0', \delta_1') \in \{(\delta_0 \pm 1, \delta_1), (\delta_0, \delta_1 \pm 1)\}$; this is an $e \times e$ square lattice. Fix a Hamiltonian walk γ in this graph, namely a path that traverses each vertex exactly once, with no restriction on the starting and ending vertices. It is clear that such paths exist. We say that two adjacent elements of Δ are γ-adjacent if γ contains the edge connecting them.

Let $\sigma \in \mathcal{D}(\rho)$ be associated to the pair $((\delta_0, \delta_1), J)$. Define a Γ-module $D_{0,\sigma}$ as follows:

$$D_{0,\sigma} = \begin{cases} Q_{\{1\}}(\sigma) & : (\delta_0, \delta_1) \text{ is } \gamma\text{-adjacent to } (\delta_0, \delta_1 + 1) \text{ and } J = \{0\} \\ Q_{\{1\}}(\sigma) & : (\delta_0, \delta_1) \text{ is } \gamma\text{-adjacent to } (\delta_0, \delta_1 - 1) \text{ and } J = \{1\} \\ Q_{\{0\}}(\sigma) & : (\delta_0, \delta_1) \text{ is } \gamma\text{-adjacent to } (\delta_0 + 1, \delta_1) \text{ and } J = \{0, 1\} \\ Q_{\{0\}}(\sigma) & : (\delta_0, \delta_1) \text{ is } \gamma\text{-adjacent to } (\delta_0 - 1, \delta_1) \text{ and } J = \varnothing \\ \widetilde{D}_{0,\sigma} & : \text{ otherwise.} \end{cases}$$

Now set $D_0(\rho) = \bigoplus_{\sigma \in \mathcal{D}(\rho)} D_{0,\sigma}$. Informally, if we view $\widetilde{D}_0(\rho)$ schematically as in (2), then to obtain $D_0(\rho)$ we switch one Serre weight appearing in the cosocle of $\widetilde{D}_0(\delta_0, \delta_1)$ with a Serre weight appearing in the cosocle of $\widetilde{D}_0(\delta_0', \delta_1')$ whenever (δ_0, δ_1) and (δ_0', δ_1')

are γ-adjacent. If $e = 1$, then Δ has only one vertex and the $D_0(\rho)$ constructed here is the Γ-submodule of the Γ-module $D_0(\rho)$ constructed in [6] generated by its U-invariants. In all cases, we have the following analogue of [6, Corollary 13.6].

Lemma 4. *There is a unique partition of the B-eigencharacters of $D_0(\rho)^U$ into pairs $\{\chi, \chi^w\}$, with $\chi \neq \chi^w$, such that one character of each pair arises from the socle of $D_0(\rho)$ and the other from the cosocle.*

Proof. The claim is true by inspection for the B-eigencharacters of $\widetilde{D}_0(\delta_0, \delta_1)^U$ for every $(\delta_0, \delta_1) \in \Delta$. Since the sets $\mathcal{D}(\delta_0, \delta_1)$ are disjoint, the claim remains true for the B-eigencharacters of $\widetilde{D}_0(\rho)^U$. Now the sets of B-eigencharacters arising from the socle and cosocle of $D_0(\rho)$ are the same as for $\widetilde{D}_0(\rho)$, completing the proof.

Note that if $e > 1$, then $\mathcal{D}(\rho)$ contains pairs $\{\sigma, \sigma^{[w]}\}$ of Serre weights. Thus the uniqueness in the claim fails without the requirement that one character of each pair $\{\chi, \chi^w\}$ come from the socle and one from the cosocle. □

View $D_0(\rho)$ as a KZ-module, with $\pi \mathrm{Id}_2$ acting trivially. The partition of Lemma 4 gives rise to a family of basic 0-diagrams $(D_0(\rho), \{\,\})$ in the sense of [6, Definition 13.7]. Now we show that this family is indecomposable.

Proposition 5. *The family $(D_0(\rho), \{\,\})$ cannot be written as a direct sum of two non-zero families of diagrams.*

Proof. For every $\sigma \in \mathcal{D}(\rho)$, let $\beta(\sigma) \in \mathcal{D}(\rho)$ be the Serre weight such that $\chi(\sigma)^w$ arises in the cosocle of $D_{0,\beta(\sigma)}$. Note that $\beta(\sigma)$ is well-defined by Lemma 4 and that β is a bijection of $\mathcal{D}(\rho)$ onto itself. It suffices to show that the action of $\mathbb{Z} \simeq \langle \beta \rangle$ on the set $\mathcal{D}(\rho)$ is transitive. Choose a direction of the path γ, which amounts to fixing a numbering $(\gamma_1, \ldots, \gamma_{e^2})$ of the elements of Δ such that γ_i and γ_{i+1} are γ-adjacent for every $0 \leq i \leq e^2 - 1$. For every $\widetilde{D}_0(\delta_0, \delta_1)$, the analogously defined $\langle \beta \rangle$-action is transitive; this follows by inspection of (2) or by [6, Theorem 15.4], noting that the pairing $\{\,\}$ there matches characters arising from the socle with characters arising from the cosocle. Since γ_1 is γ-adjacent to only one other element of Δ, there is a unique Serre weight $\tau \in \mathcal{D}(\gamma_1)$ such that $D_{0,\tau} \not\simeq \widetilde{D}_{0,\tau}$. The previous observation implies that $\{\tau, \beta(\tau), \beta^2(\tau), \beta^3(\tau)\} = \mathcal{D}(\gamma_1)$.

We proceed by induction. Suppose it is known that all elements of $\mathcal{D}(\gamma_{k-1})$ lie in the same orbit as τ. The same holds for $\mathcal{D}(\gamma_k)$ by an easy but tedious analysis of cases. For instance, if $\gamma_k = \gamma_{k-1} + (0,1)$ and $\gamma_{k+1} = \gamma_k + (1,0)$, then $\beta(\gamma_{k-1}, \varnothing) = (\gamma_k, \{1\})$ and $\beta(\gamma_k, \{1\}) = (\gamma_k, \varnothing)$ and $\beta(\gamma_k, \varnothing) = (\gamma_k, \{0\})$, whereas $\beta^{-1}(\gamma_{k-1}, \{0\}) = (\gamma_k, \{0,1\})$; the other cases, including the case where γ_k is the terminal vertex of γ, are treated similarly. □

Remark 6. The choice of Hamiltonian walk γ in the definition of $D_0(\rho)$ really is necessary. It would be more canonical to start with $\widetilde{D}_0(\rho)$ and switch Serre weights in the cosocle for every pair of adjacent elements of Δ. However, the family of diagrams obtained in this way is easily seen to be decomposable in general.

Theorem 7. *Let $(D_0(\rho), D_1(\rho), r)$ be a basic 0-diagram arising from the family constructed above. Let V be a smooth admissible representation of $G = \mathrm{GL}_2(F)$ satisfying the following conditions:*

(1) $\mathrm{soc}_K(V) = \bigoplus_{\sigma \in \mathcal{D}(\rho)} \sigma$.
(2) $(D_0(\rho), D_1(\rho), r) \hookrightarrow (V^{K(1)}, V^{I(1)}, \mathrm{can})$.
(3) *V is generated by $D_0(\rho)$.*

Then V is irreducible and supersingular.

Proof. Note first that smooth admissible representations of G satisfying our hypotheses exist, as in [6, Theorem 19.8]. Let $U \subseteq V$ be a non-zero G-submodule, let σ be a Serre weight contained in $\mathrm{soc}_K(U)$, and let $0 \neq v \in \sigma^{I(1)}$. By Frobenius reciprocity, the inclusion $\varphi : \sigma \hookrightarrow U_{|K}$ corresponds to a non-zero map $\psi : \mathrm{ind}_{KZ}^G \sigma \to U$ of G-modules with $\psi(\mathrm{id} \otimes v) = \varphi(v)$. Hence $\psi(\alpha \otimes wv) = \Pi(\varphi(v))$ generates $D_{0,\beta(\sigma)}$ and in particular $\beta(\sigma) \subseteq \mathrm{soc}_K(U)$. Proposition 5 ensures that by iterating this procedure we obtain $D_{0,\sigma} \subset U$ for all $\sigma \in \mathcal{D}(\rho)$. Since $D_0(\rho)$ generates V, this implies $U = V$. Thus V is irreducible.

Moreover, the restriction to $\langle \alpha \otimes wv \rangle_K$ of the map ψ above has image $\langle \Pi(\varphi(v)) \rangle_K = D_{0,\beta(\sigma)}$, which is a K-module of length two by construction. We have $\langle \alpha \otimes wv \rangle_K \simeq \mathrm{Ind}_I^K \chi^w$ by Lemma 1, which is a K-module of length $2^f = 4$ by Lemma 2. Hence $T(\mathrm{id} \otimes v) \in \mathrm{soc}_K(\langle \alpha \otimes wv \rangle_K) \subset \ker \psi$, and ψ factors through $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$. Moreover, ψ is surjective since V is irreducible. Hence V is supersingular. $\square$

Note that different choices of Hamiltonian walks in the construction of $D_0(\rho)$ give disjoint families of irreducible supersingular representations.

As an example, we write out $D_0(\rho)$ in the case $e = 2, f = 2$, where we have taken the Hamiltonian walk in Δ consisting of solid edges in the diagram below.

$$\begin{tikzcd} (0,0) \arrow[r, dash] \arrow[d, dash, dotted] & (1,0) \arrow[d, dash] \\ (0,1) \arrow[r, dash] & (1,1) \end{tikzcd}$$

The superscript n of each Serre weight $\sigma \in \mathcal{D}(\rho)$ is such that $\sigma = \beta^n((0,0), \varnothing)$, illustrating Proposition 5. The superscripts on the cosocles indicate the matching of Lemma 4.

$((0,0), \varnothing)$	$(r_0, r_1)^0$	—	$(r_0 + 1, p - 2 - r_1)^{15}$
$((0,0), \{1\})$	$(r_0 - 1, p - 2 - r_1)^1$	—	$(p - 1 - r_0, p - 1 - r_1)^0$
$((0,0), \{0\})$	$(p - 2 - r_0, r_1 + 1)^{15}$	—	$(r_0, r_1 + 2)^{14}$
$((0,0)\{0,1\})$	$(p - 1 - r_0, p - 3 - r_1)^{14}$	—	$(r_0 - 1, p - 2 - r_1)^{13}$
$((0,1), \varnothing)$	$(r_0, r_1 - 2)^6$	—	$(r_0 + 1, p - r_1)^5$
$((0,1), \{1\})$	$(r_0 - 1, p - r_1)^7$	—	$(p - 1 - r_0, p + 1 - r_1)^6$
$((0,1), \{0\})$	$(p - 2 - r_0, r_1 - 1)^5$	—	$(r_0, r_1)^4$
$((0,1), \{0,1\})$	$(p - 1 - r_0, p - 1 - r_1)^4$	—	$(r_0 - 1, p - r_1)^3$
$((1,0), \varnothing)$	$(r_0 - 2, r_1)^2$	—	$(p - r_0, r_1 + 1)^1$
$((1,0), \{1\})$	$(r_0 - 3, p - 2 - r_1)^{11}$	—	$(r_0 - 2, r_1)^{10}$
$((1,0), \{0\})$	$(p - r_0, r_1 + 1)^{13}$	—	$(r_0 - 2, r_1 + 2)^{12}$
$((1,0), \{0,1\})$	$(p + 1 - r_0, p - 3 - r_1)^{12}$	—	$(p + 2 - r_0, r_1 + 1)^{11}$
$((1,1), \varnothing)$	$(r_0 - 2, r_1 - 2)^8$	—	$(p - r_0, r_1 - 1)^7$

$((1,1),\{1\})$	$(r_0-3,p-r_1)^9$	——	$(p+1-r_0,p+1-r_1)^8$
$((1,1),\{0\})$	$(p-r_0,r_1-1)^3$	——	$(p+1-r_0,p-1-r_1)^2$
$((1,1),\{0,1\})$	$(p+1-r_0,p-1-r_1)^{10}$	——	$(p+2-r_0,r_1-1)^9$

Acknowledgements. The author is grateful to Eknath Ghate, Nathan Keller, and Mihir Sheth for helpful correspondence and conversations.

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