AN IRREDUCIBILITY CRITERION FOR SUPERSINGULAR mod p REPRESENTATIONS OF $\mathrm{GL}_2(F)$ FOR TOTALLY RAMIFIED EXTENSIONS F OF $\mathbb{Q}_p$

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Abstract. Let F be a totally ramified extension of $\mathbb{Q}_p$. We consider supersingular representations of $\mathrm{GL}_2(F)$ whose socles as $\mathrm{GL}_2(\mathcal{O}_F)$-modules are of a certain form that is expected to appear in the mod p local Langlands correspondence and establish a condition under which they are irreducible.

1. Introduction

Let F be a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}$. Choose a uniformizer $\pi \in \mathcal{O}$ and denote the residue field by $k = \mathcal{O}/(\pi)$. A question of immediate relevance to the emerging mod p local Langlands correspondence is to construct smooth mod p representations of the group $G = \mathrm{GL}_2(F)$. If $K = \mathrm{GL}_2(\mathcal{O})$ and Z is the center of G, then any irreducible $\overline{\mathbb{F}}_p$-representation σ of the finite group $\mathrm{GL}_2(k)$ may be viewed naturally as a representation of KZ. We may then consider the compact induction $\mathrm{ind}_{KZ}^G \sigma$; a precise definition is given below. Barthel and Livné proved ([BL], Prop. 8) that the endomorphism algebra $\mathrm{End}_G(\mathrm{ind}_{KZ}^G \sigma)$ is isomorphic to a polynomial ring $\overline{\mathbb{F}}_p[T]$ for an explicitly defined generator T. Moreover, they showed ([BL], Theorem 33) that any irreducible mod p representation V of G is, up to twist by an unramified character, a quotient of $\mathrm{ind}_{KZ}^G \sigma/(T-\lambda)\mathrm{ind}_{KZ}^G \sigma$ for some σ as above and some $\lambda \in \overline{\mathbb{F}}_p$. If $\lambda \neq 0$, then Barthel and Livné classified these quotients completely. On the other hand, quotients of $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ are called supersingular and are still very poorly understood. In this paper we prove an irreducibility criterion for certain quotients of $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ when $F/\mathbb{Q}_p$ is totally ramified.

Given a tamely ramified continuous irreducible Galois representation $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, for any finite extension $F/\mathbb{Q}_p$, Serre's weight conjecture and its generalizations associate to ρ a set $\mathcal{D}(\rho)$ of irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(k)$; these are called the modular weights of ρ. These conjectures were formulated by Serre for $F = \mathbb{Q}_p$, by Buzzard, Diamond, and Jarvis [BDJ] for F unramified over $\mathbb{Q}_p$, and by the author [Sch1] in general; the reader is referred to those articles and to the beginning of the last section of this paper for more details. These conjectures may be seen as describing the socle of the smooth representation $\pi(\rho)$ of $\mathrm{GL}_2(F)$ associated to ρ by the mod p local Langlands correspondence: generically, one expects $\mathrm{soc}_K \pi(\rho) = \oplus_{\sigma \in \mathcal{D}(\rho)} \sigma$. In particular, this implies that a surjection $\mathrm{ind}_{KZ}^G \sigma \twoheadrightarrow \pi(\rho)$ exists if and only if $\sigma \in \mathcal{D}(\rho)$.

Let $F/\mathbb{Q}_p$ be totally ramified of degree e. Consider the $\overline{\mathbb{F}}_p$-representation $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ of $\mathrm{GL}_2(\mathbb{F}_p)$, where $0 < r < p-2$. Let $f_\sigma \in \mathrm{ind}_{KZ}^G \sigma$ be a non-zero function supported on the single coset KZ that satisfies $f_\sigma(\mathrm{id}) \in \sigma^{I(1)}$. Here $I(1) \subset K$ is the upper triangular pro-p-Iwahori subgroup. Observe that f_σ generates an irreducible K-submodule isomorphic to σ. The following lemma is proved by computation.

Lemma 1.1. *Let $0 < r < p-2$ and let σ and f_σ be as above.*

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(a) *The image of* $\begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix} f_\sigma$ *in* $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ *is invariant under the action of* $I(1)$ *and generates a* K*-submodule that is irreducible and isomorphic to* $\det^{w+r} \otimes \mathrm{Sym}^{p-r-1}\overline{\mathbb{F}}_p^2$.

(b) *The image of* $\sum_{\mu_0,\mu_1 \in \mathbb{F}_p} \mu_1^{r+1} \begin{pmatrix} \pi^2 & [\mu_0]+\pi[\mu_1] \\ 0 & 1 \end{pmatrix} f_\sigma$ *in* $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ *is invariant under the action of* $I(1)$ *and generates a* K*-submodule that is irreducible and isomorphic to* $\det^{w+r+1} \otimes \mathrm{Sym}^{p-r-3}\overline{\mathbb{F}}_p^2$. *Here* $[\mu] \in \mathcal{O}$ *is the canonical (Teichmüller) lift of* $\mu \in \mathbb{F}_p$.

Proof. The first statement is Lemma 3.6. The second follows from the case $n = 1$ of Lemma 3.1 and Proposition 3.3. □

Now let $0 < r \leq p - 2e - 1$, and consider the set $\mathcal{D} = \{\sigma_0, \dots, \sigma_{e-1}\} \cup \{\sigma_0', \dots, \sigma_{e-1}'\}$ of $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(\mathbb{F}_p)$, where

$$\begin{aligned} \sigma_i &= \det^{-i} \otimes \mathrm{Sym}^{r+2i}\overline{\mathbb{F}}_p^2, \\ \sigma_i' &= \det^{r+i} \otimes \mathrm{Sym}^{p-r-1-2i}\overline{\mathbb{F}}_p^2 \end{aligned} \qquad (1)$$

This $\mathcal{D}$ arises as $\mathcal{D}(\rho)$ for a suitable Galois representation ρ, and it consists of $2e$ distinct regular weights. Let $\beta = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$. We now define e explicit elements of $\mathrm{ind}_{KZ}^G \sigma_0 / T(\mathrm{ind}_{KZ}^G \sigma_0)$ as follows. For $1 \leq i \leq e-1$ define

$$\begin{aligned} f_i &= \beta \sum_{\mu_0,\mu_1 \in \mathbb{F}_p} \mu_1^{r+2i-1} \begin{pmatrix} \pi^2 & [\mu_0]+\pi[\mu_1] \\ 0 & 1 \end{pmatrix} f_{i-1} \\ z_i &= \sum_{\mu_0,\mu_1 \in \mathbb{F}_p} \mu_1^{p-r-2i} \begin{pmatrix} \pi^2 & [\mu_0]+\pi[\mu_1] \\ 0 & 1 \end{pmatrix} \beta f_i. \end{aligned}$$

Proposition 1.2. *Let* $0 < r \leq p - 2e - 1$*, and let the set* $\mathcal{D}$ *of weights be defined as above. Let* $\tau : \mathrm{ind}_{KZ}^G \sigma_0 / T(\mathrm{ind}_{KZ}^G \sigma_0) \twoheadrightarrow W$ *be a quotient. Suppose that* W *has no non-supersingular subrepresentations, that* $\mathrm{soc}_K(W) \simeq \otimes_{\sigma \in \mathcal{D}} \sigma$*, and that* $\tau(f_{e-1}) \in W$ *is non-zero. Then for each* $0 \leq i \leq e-1$ *the* K*-submodules of* W *generated by the elements* $\tau(f_i)$ *(resp.* $\tau(\beta f_i)$ *) are irreducible and isomorphic to* σ_i *(resp.* σ_i'*).*

Proof. This is Proposition 3.8 below. □

Admitting these two propositions, we can immediately establish the following irreducibility criterion, which is the main result of this paper. See Remark 3.9 for a variation.

Theorem 1.3. *Let* $0 < r \leq p - 2e - 1$*, and let the set* $\mathcal{D}$ *of weights be defined as above. Let* $\tau : \mathrm{ind}_{KZ}^G \sigma_0 \twoheadrightarrow W$ *be a quotient. Suppose that* W *has no non-supersingular subrepresentations, that* $\mathrm{soc}_K(W) \simeq \otimes_{\sigma \in \mathcal{D}} \sigma$*, and that* $\tau(f_{e-1}) \in W$ *is non-zero. Suppose also that* $\tau(z_i) \neq 0$ *for all* $1 \leq i \leq e-1$*. Then* W *is an irreducible* G*-module.*

Proof. Let $U \subseteq W$ be an irreducible G-submodule. Since f_0 generates $\mathrm{ind}_{KZ}^G \sigma_0$ as a G-module, to conclude $U = W$ it suffices to show that $\tau(f_0) \in U$. Note that if $\tau(\beta f_i) \in U$, then also $\tau(f_i) \in U$. By our assumption on the K-socle of W and the previous proposition, it then follows that any irreducible K-submodule of W must contain one of the elements $\tau(f_0), \dots, \tau(f_{e-1})$. Let $0 \leq l \leq e-1$ be the smallest number such that $\tau(f_l) \in U$, and suppose that $l > 0$.

By Frobenius duality there is a non-zero map $\psi_l : \mathrm{ind}_{KZ}^G \sigma_l' \to W$ such that $\psi_l(f_{\sigma_l'}) = \tau(\beta f_l)$. Since $\mathrm{Hom}_G(\mathrm{ind}_{KZ}^G \sigma_l', W)$ is a one-dimensional space, every non-zero element is an eigenvector for the action of the commutative algebra $\mathrm{End}_G(\mathrm{ind}_{KZ}^G \sigma_l')$. Therefore, ψ_l must factor through a quotient $\mathrm{ind}_{KZ}^G \sigma_l' / (T - \lambda)(\mathrm{ind}_{KZ}^G \sigma_l')$ for some $\lambda \in \overline{\mathbb{F}}_p$. We must have $\lambda = 0$, since otherwise the image of ψ_l in W would have a non-supersingular subrepresentation.

By assumption $\psi_l \left(\sum_{\mu_0,\mu_1 \in \mathbb{F}_p} \mu_1^{p-r-2l} \begin{pmatrix} \pi^2 & [\mu_0]+\pi[\mu_1] \\ 0 & 1 \end{pmatrix} f_{\sigma_l'} \right) = \tau(z_l)$ is a non-zero element of W. The second part of Lemma 1.1 then implies that $\tau(z_l)$ generates an irreducible K-submodule of W that is isomorphic to σ_{l-1}. But since each irreducible submodule in $\mathrm{soc}_K(W)$ appears with multiplicity one, it follows that $\tau(z_l) = c\tau(f_{l-1})$ for a suitable non-zero scalar $c \in \overline{\mathbb{F}}_p$, contradicting the minimality of l. It follows that $l = 0$, and hence $U = W$. $\square$

We briefly discuss previous work to place this theorem in context. A first result towards studying the supersingular representations of $\mathrm{GL}_2(F)$ was attained by Breuil, who showed in [Bre] that if $F = \mathbb{Q}_p$ then $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ is irreducible for all σ. He proved this by explicitly computing the $I(1)$-invariants of $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ and observing that every non-zero $I(1)$-invariant generates $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ as a G-module. Since $I(1)$ is a pro-p group, any irreducible submodule of $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ must have non-trivial $I(1)$-invariants, and the result follows. A more conceptual version of this argument was given by Ollivier in [Oll], and other proofs were found by Emerton ([Eme], Theorem 5.1) and Vignéras (unpublished). Moreover, $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ has the expected socle, and an explicit correspondence between irreducible Galois representations and supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ was stated in [Bre].

The smooth representation theory of $\mathrm{GL}_2(F)$ for $F \neq \mathbb{Q}_p$ is much more complicated, since $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ is of infinite length and there are many more supersingular representations of $\mathrm{GL}_2(F)$ than there are Galois representations to pair them with. When $F/\mathbb{Q}_p$ is unramified, Breuil and Paskunas [BP] have applied Paskunas' method of diagrams to prove the existence of many supersingular representations with socle $\oplus_{\sigma\in\mathcal{D}}\sigma$. These were again shown to be irreducible by an argument on $I(1)$-invariants, although the argument relies on the combinatorics of $\mathcal{D}$ and is considerably more complicated than in the case $F = \mathbb{Q}_p$. In fact, their method of construction essentially works for arbitrary extensions $F/\mathbb{Q}_p$. Alternatively, Hu [Hu] associated a canonical diagram to any supersingular representation (not necessarily irreducible) of $\mathrm{GL}_2(F)$ for arbitrary F. In general it has been difficult to show that the representation of $\mathrm{GL}_2(F)$ associated to a given diagram is irreducible, since the method of Breuil and Paskunas for proving irreducibility fails in this case. We note that the Breuil-Paskunas construction applied to totally ramified $F/\mathbb{Q}_p$ yields representations with no non-supersingular subrepresentations and with K-socle $\bigoplus_{\sigma\in\mathcal{D}}\sigma$. However, neither these representations nor Hu's canonical diagrams are understood explicitly enough at present to verify the non-vanishing of $\tau(f_{e-1})$ and $\tau(z_i)$ and establish irreducibility by means of Theorem 1.3 in any example.

The second section of the paper is rather technical. It uses the methods of Breuil's original paper [Bre] to prove Corollary 2.11, which will provide information about the $I(1)$-invariants of certain quotients of $\mathrm{ind}_{KZ}^G\sigma_0/T(\mathrm{ind}_{KZ}^G\sigma_0)$. Lemma 1.1 and Proposition 1.2 are proved in the third section. In fact, we obtain more precise information about V_{e-1}, which is used when constructing irreducible supersingular representations of $\mathrm{GL}_2(F)$. This work will appear in a separate article. We note that the constructions and results of this paper may be generalized to arbitrary extensions $F/\mathbb{Q}_p$, although the presence of an unramified subextension complicates the computations.

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1.1. **Notations and background results.** In this section we establish notation and recall some results that we will need. Let p be an odd prime. Recall that F is totally ramified extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}$ and $\pi \in \mathcal{O}$ is a uniformizer. Then $\mathcal{O}/(\pi) = \mathbb{F}_p$. Let $e = [F : \mathbb{Q}_p]$ be the ramification index. We assume that $e > 1$; note that in the case $F = \mathbb{Q}_p$ the questions we investigate have been resolved completely by Breuil. Let $G = \mathrm{GL}_2(F)$. Then $K = \mathrm{GL}_2(\mathcal{O}) \subset G$ is a maximal compact subgroup. Let $\overline{B} \subset \mathrm{GL}_2(\mathbb{F}_p)$ be the subgroup of upper triangular matrices. Fix the Iwahori subgroup $I = \omega^{-1}(\overline{B}) \subset K$, where $\omega : K \to \mathrm{GL}_2(\mathbb{F}_p)$ is the natural projection. Let $I(1)$ be the pro-p Sylow subgroup of I. Write Z for the center of G and $K(1)$ for the kernel of ω. We also define

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta = \alpha w = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}.$$

Recall that the distinct irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(\mathbb{F}_p)$ are $\sigma_{r,w} = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$, where $0 \leq w \leq p-2$ and $0 \leq r \leq p-1$. A model for $\sigma_{r,w}$ is given by the $(r+1)$-dimensional space $V_{\sigma_{r,w}}$ of homogeneous polynomials $P \in \overline{\mathbb{F}}_p[x,y]$ of degree r, where $\mathrm{GL}_2(\mathbb{F}_p)$ acts as follows. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_p)$, then $(\gamma P)(x,y) = (ad-bc)^w P(ax+cy, bx+dy)$.

Given $\lambda \in \mathbb{F}_p$, let $[\lambda] \in \mathcal{O}$ be its canonical lift. For $n \geq 1$ define the sets

$$I_n = \left\{ [\lambda_0] + \pi[\lambda_1] + \cdots + \pi^{n-1}[\lambda_{n-1}] : (\lambda_0, \ldots, \lambda_{n-1}) \in (\mathbb{F}_p)^n \right\} \subset \mathcal{O}.$$

We also set $I_0 = \{0\}$. Then for all $n \geq 0$ and $\lambda \in I_n$ we set

$$g^0_{n,\lambda} = \begin{pmatrix} \pi^n & \lambda \\ 0 & 1 \end{pmatrix}, \quad g^1_{n,\lambda} = \begin{pmatrix} 1 & 0 \\ \pi\lambda & \pi^{n+1} \end{pmatrix}.$$

In particular, $g^0_{0,0}$ is the identity matrix and $g^1_{0,0} = \alpha$. Also $g^1_{n,\lambda} = \beta g^0_{n,\lambda} w$ for all $n \geq 0$ and $\lambda \in I_n$. It follows from the Cartan decomposition that these $g^0_{n,\lambda}$ and $g^1_{n,\lambda}$ comprise a set of coset representatives for KZ in G:

$$G = \coprod_{\substack{i \in \{0,1\} \\ n \geq 0, \lambda \in I_n}} g^i_{n,\lambda} KZ.$$

For $n \geq 0$, we define $S^0_n = IZ\alpha^{-n}KZ = \coprod_{\lambda \in I_n} g^0_{n,\lambda} KZ$ and $S^1_n = IZ\beta\alpha^{-n}KZ = \coprod_{\lambda \in I_n} g^1_{n,\lambda} KZ$ as in [Bre]. We also set $S_n = S^0_n \coprod S^1_n$ and $B_n = B^0_n \coprod B^1_n$, where

$$B^0_n = \coprod_{m \leq n} S^0_m \quad \text{and} \quad B^1_n = \coprod_{m \leq n} S^1_m.$$

Given an irreducible $\overline{\mathbb{F}}_p$-representation σ of $\mathrm{GL}_2(\mathbb{F}_p)$, we can view it as a KZ-module where K acts via ω and the matrix $\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}$ acts trivially. Then a model for $\mathrm{ind}_{KZ}^G \sigma$ is the space of functions $f : G \to V_\sigma$ that are compactly supported modulo KZ and satisfy $f(kg) = \sigma(k)f(g)$ for all $k \in KZ$ and $g \in G$. The group G acts by $(hf)(g) = f(gh)$ for $h \in G$. Such a function is clearly determined by its values on the $(g^0_{n,\lambda})^{-1}$ and $(g^1_{n,\lambda})^{-1}$. Note that $\mathrm{ind}_{KZ}^G \sigma \simeq \overline{\mathbb{F}}_p[G] \otimes_{\overline{\mathbb{F}}_p[KZ]} V_\sigma$. If $g \in G$ and $v \in V_\sigma$, then the element $g \otimes v$ corresponds to the function defined by

$$(g \otimes v)(h) = \begin{cases} \sigma(hg)v & : h \in KZg^{-1} \\ 0 & : h \notin KZg^{-1}. \end{cases}$$

This is the element denoted $[g,v]$ in [Bre]. Observe that any function $f \in \mathrm{ind}_{KZ}^G \sigma$ may be written uniquely in the form

$$f = \sum_{n=0}^{\infty} \sum_{\lambda \in I_n} (g^0_{n,\lambda} \otimes v^0_{n,\lambda} + g^1_{n,\lambda} \otimes v^1_{n,\lambda})$$

for suitable $v^0_{n,\lambda}, v^1_{n,\lambda} \in V_\sigma$. We say that the support of f is the set of $g^i_{n,\lambda}$ such that $v^i_{n,\lambda} \neq 0$. We write $f \in S_n$ if the support of f is contained in S_n, and similarly for B_n, S^0_n, etc.

Observe that any element $z \in \mathcal{O}$ has a unique expansion $z = \sum_{i=0}^\infty z_i \pi^i$, where $z_i \in I_1$. Let $[z]_n$ denote the truncation $\sum_{i=0}^{n-1} z_i \pi^i \in I_n$. We will sometimes write $g^0_{n,z}$ to mean $g^0_{n,[z]_n}$.

Throughout this section and the following we assume that $\sigma = \sigma_{r,0}$, with $0 \leq r \leq p-1$. Then the formulae of section 2.5 and Lemme 3.1.1 of [Bre] imply the following explicit expressions for the action of the canonical endomorphism $T \in \mathrm{End}(\mathrm{ind}_{KZ}^G \sigma_r)$.

Lemma 1.4. *Let $v = \sum_{i=0}^r c_i x^{r-i} y^i \in V_{\sigma_r}$. If $n \geq 1$ and $\mu \in I_n$, then the action of T is given by:*

$$T(g^0_{n,\mu} \otimes v) = \sum_{\lambda \in I_1} g^0_{n+1,\mu+\pi^n\lambda} \otimes \left(\sum_{i=0}^r c_i (-\lambda)^i \right) x^r + g^0_{n-1,[\mu]_{n-1}} \otimes c_r (\mu_{n-1} x + y)^r,$$

$$T(g^1_{n,\mu} \otimes v) = \sum_{\lambda \in I_1} g^1_{n+1,\mu+\pi^n\lambda} \otimes \left(\sum_{i=0}^r c_{r-i} (-\lambda)^i \right) y^r + g^1_{n-1,[\mu]_{n-1}} \otimes c_0 (x + \mu_{n-1} y)^r.$$

In the remaining cases the action of T is given by:

$$T(\mathrm{Id}\otimes v) = \sum_{\lambda\in I_1} g^0_{1,\lambda}\otimes\left(\sum_{i=0}^r c_i(-\lambda)^i\right)x^r + \alpha\otimes c_r y^r,$$
$$T(\alpha\otimes v) = \sum_{\lambda\in I_1} g^1_{1,\lambda}\otimes\left(\sum_{i=0}^r c_{r-i}(-\lambda)^i\right)y^r + \mathrm{Id}\otimes c_0 x^r.$$

Corollary 1.5. *The endomorphism $T\in\mathrm{End}(\mathrm{ind}_{KZ}^G\sigma_r)$ is injective. In particular,*

$$\mathrm{ind}_{KZ}^G\sigma_r/T(\mathrm{ind}_{KZ}^G\sigma_r)\simeq T^{e-1}(\mathrm{ind}_{KZ}^G\sigma_r)/T^e(\mathrm{ind}_{KZ}^G\sigma_r).$$

Proof. Immediate from Lemma 1.4. □

Lemma 1.6. *Suppose that $v=\sum_{i=0}^r c_i x^{r-i}y^i\in V_\sigma$ and $n\geq 0$. Let $\mu=[\mu_0]+\pi[\mu_1]+\cdots+\pi^{n-1}[\mu_{n-1}]\in I_n$. If $k\geq 1$, then*

$$T^k(g^0_{n,\mu}\otimes v) = \sum_{(\nu_1,\ldots,\nu_k)\in(I_1)^k}\left(g^0_{n+k,\mu+\pi^n\nu_1+\cdots\pi^{n+k-1}\nu_k}\otimes\left(\sum_{i=0}^r c_i(-\nu_1)^i\right)x^r\right)+B_{n+k-1},$$
$$T^k(g^1_{n,\mu}\otimes v) = \sum_{(\nu_1,\ldots,\nu_k)\in(I_1)^k}\left(g^1_{n+k,\mu+\pi^n\nu_1+\cdots\pi^{n+k-1}\nu_k}\otimes\left(\sum_{i=0}^r c_{r-i}(-\nu_1)^i\right)y^r\right)+B_{n+k-1}.$$

In particular, if $1\leq k\leq n$ and $r>0$, then

$$\begin{aligned}T^k(g^0_{n,\mu}\otimes v) &= \sum_{(\nu_1,\ldots,\nu_k)\in(I_1)^k}\left(g^0_{n+k,\mu+\pi^n\nu_1+\cdots\pi^{n+k-1}\nu_k}\otimes\left(\sum_{i=0}^r c_i(-\nu_1)^i\right)x^r\right)+\\ &\quad\sum_{m=1}^{k-1}\sum_{\substack{(\nu_1,\ldots,\nu_{k-m})\\\in(I_1)^{k-m}}}\left(g^0_{n+k-2m,[\mu]_{n-m}+\sum_{j=1}^{k-m}\pi^{n-m+j}\nu_j}\otimes\left(c_r\sum_{i=0}^r\binom{r}{i}\mu_{n-m}^{r-i}(-\nu_1)^i\right)x^r\right)+\\ &\quad g^0_{n-k,[\mu]_{n-k}}\otimes c_r(\mu_{n-k}x+y)^r,\\ T^k(g^1_{n,\mu}\otimes v) &= \sum_{(\nu_1,\ldots,\nu_k)\in(I_1)^k}\left(g^1_{n+k,\mu+\pi^n\nu_1+\cdots\pi^{n+k-1}\nu_k}\otimes\left(\sum_{i=0}^r c_{r-i}(-\nu_1)^i\right)y^r\right)+\\ &\quad\sum_{m=1}^{k-1}\sum_{\substack{(\nu_1,\ldots,\nu_{k-m})\\\in(I_1)^{k-m}}}\left(g^1_{n+k-2m,[\mu]_{n-m}+\sum_{j=1}^{k-m}\pi^{n-m+j}\nu_j}\otimes\left(c_0\sum_{i=0}^r\binom{r}{i}\mu_{n-m}^{r-i}(-\nu_1)^i\right)y^r\right)+\\ &\quad g^1_{n-k,[\mu]_{n-k}}\otimes c_0(x+\mu_{n-k}y)^r.\end{aligned}$$

Proof. This is a straightforward calculation using the formulae of Lemma 1.4. □

2. Structured submodules and $I(1)$-invariants

Lemma 2.1. *Let $n\geq 1$. Then for any set-theoretic map $f:I_n\to\overline{\mathbb{F}}_p$ there exists a unique polynomial $P\in\overline{\mathbb{F}}_p[X_0,\ldots,X_{n-1}]$ in which each variable appears with degree at most $p-1$ and such that $f(\mu)=P(\mu_0,\ldots,\mu_{n-1})$ for all $\mu\in I_n$.*

Proof. When $n=1$ this is Lemme 3.1.6 of [Bre]. Suppose the claim is known for $n-1$. By the claim for $n=1$, for each $\mu\in I_{n-1}$ there exist unique $c_0^\mu,\ldots,c_{p-1}^\mu\in\overline{\mathbb{F}}_p$ such that $f(\mu+\pi^{n-1}[\lambda])=\sum_{j=0}^{p-1}c_j^\mu\lambda^j$. But by induction the map $\mu\mapsto c_j^\mu$ is itself expressible as a unique polynomial in $\mu_0,\ldots,\mu_{n-2}$ for each $0\leq j\leq p-1$. □

Lemma 2.2. *Let $\lambda_0,\lambda_1,\ldots,\lambda_e\in\mathbb{F}_p$. Then*

$$[\lambda_0]+\pi[\lambda_1]+\cdots+\pi^e[\lambda_e]+1\equiv$$
$$[\lambda_0+1]+\pi[\lambda_1]+\cdots+\pi^{e-1}[\lambda_{e-1}]+\pi^e[\lambda_e+\frac{\lambda_0^{p^e}+1-(\lambda_0+1)^{p^e}}{\pi^e}]\mod \pi^{e+1}.$$

Proof. Using the isomorphism $\mathcal{O} \simeq \varprojlim \mathcal{O}/(\pi)^n$, we see that $[\lambda]$ can be viewed as the following sequence on the right hand side: $(\lambda + (\pi), \lambda^p + (\pi^2), \lambda^{p^2} + (\pi^3), \dots)$. The claim then follows from a simple computation. □

Remark 2.3. An immediate consequence of the lemma is that if $n \leq e$, then

$$\sum_{i=0}^{n-1}[\lambda_i]\pi^i + \sum_{i=0}^{n-1}[\mu_i]\pi^i \equiv \sum_{i=0}^{n-1}[\lambda_i + \mu_i]\pi^i \mod \pi^n.$$

The computations in the sequel rely on this observation.

Remark 2.4. Observe that the binomial coefficient $\binom{p^e}{j}$ is divisible by p but not by p^2 precisely when $j = mp^{e-1}$ for $m = 1, 2, \dots, p-1$. Hence,

$$\frac{\lambda_0^{p^e} + 1 - (\lambda_0 + 1)^{p^e}}{\pi^e} = -\frac{1}{\pi^e}\sum_{m=1}^{p-1}\binom{p^e}{mp^{e-1}}\lambda_0^{mp^{e-1}} = -\frac{1}{\pi^e}\sum_{m=1}^{p-1}\binom{p^e}{mp^{e-1}}\lambda_0^m.$$

In particular, the expression above is a polynomial of degree $p-1$ in λ_0.

Lemma 2.5. *Fix elements $a, b, c, d \in \mathcal{O}$ and write $a = \sum_{i=0}^{\infty}[a_i]\pi^i$, and similarly for b, c, d. Suppose that $n \leq e$ and $\varepsilon \in I_n$. Let $\mu_\varepsilon = (1 + a\pi - c\varepsilon\pi)^{-1}(-b + \varepsilon + d\varepsilon\pi)$. Then,*

$$\mu_\varepsilon \equiv \sum_{u=0}^{n-1}[\varepsilon_u + P_u(\varepsilon_0, \dots, \varepsilon_{u-1})]\pi^u \mod \pi^n.$$

Here if $l \geq 1$ and $x \in \mathbb{N}$, we define $J(l, x)$ to be the set of ordered l-tuples $(j_1, \dots, j_l) \in \mathbb{N}^l$ such that $j_1 + \dots + j_l = x$. Then the polynomial $P_u(\varepsilon_0, \dots, \varepsilon_{u-1})$ is given by

$$P_u(\varepsilon_0, \dots, \varepsilon_{u-1}) = -b_u + \sum_{j=0}^{u-1}\varepsilon_j d_{u-j-1} +$$
$$\sum_{m=1}^{u-1}\left(-b_{u-m} + \varepsilon_{u-m} + \sum_{j=0}^{u-m-1}\varepsilon_j d_{u-m-j-1}\right)\left(\sum_{l=1}^{u}\sum_{J(l,m-l)}(-1)^l\prod_{k=1}^{l}(a_{j_k} - \sum_{j=0}^{j_k}\varepsilon_j c_{j_k-j})\right) +$$
$$(-b_0 + \varepsilon_0)\sum_{l=1}^{u}\sum_{J(l,u-l)}(-1)^l\prod_{k=1}^{l}(a_{j_k} - \sum_{j=0}^{j_k}\varepsilon_j c_{j_k-j}).$$

Proof. Since $n \leq e$ we see from Lemma 2.2 that π-adic decompositions behave well under addition and multiplication modulo π^n. For instance,

$$\varepsilon + b \equiv \textstyle\sum_{i=0}^{n-1}[\varepsilon_i + b_i]\pi^i \bmod \pi^n, \qquad \varepsilon a \equiv \sum_{i=0}^{n-1}\left[\sum_{j=0}^{i}\varepsilon_j a_{i-j}\right]\pi^i \bmod \pi^n.$$

The claim is then obtained by a straightforward calculation. □

For later reference we record here the first few polynomials P_u:

$$\begin{aligned}
P_0 &= -b_0\\
P_1(\varepsilon_0) &= c_0\varepsilon_0^2 + (d_0 - a_0 - b_0c_0)\varepsilon_0 + (-b_1 + a_0b_0)\\
P_2(\varepsilon_0, \varepsilon_1) &= -c_0^2\varepsilon_0^3 + (b_0c_0^2 - 2a_0c_0 + c_0d_0 + c_1)\varepsilon_0^2 + 2c_0\varepsilon_0\varepsilon_1 + (d_0 - a_0 - b_0c_0)\varepsilon_1 +\\
&\quad (d_1 - b_1c_0 - b_0c_1 - a_0d_0 + 2a_0b_0c_0 + a_0^2 - a_1)\varepsilon_0 + (-b_2 + a_0b_1 + a_1b_0 - a_0^2b_0).
\end{aligned}$$

A similar but easier computation produces the following result:

Lemma 2.6. *Suppose that $n \leq e$ and $\nu \in I_n$. Let $\lambda \in I_1$ be such that $\lambda\nu_0 \neq 1$ and set $\tilde{\nu} = [\nu(1 - \lambda\nu)^{-1}]_e$. Denote $u = 1 - \lambda\nu_0$. Then*

$$\tilde{\nu} = u^{-1}\nu_0 + \sum_{i=1}^{e-1}u^{-2}(\nu_i + R_i(\nu_0, \dots, \nu_{i-1}))\pi^i,$$

where

$$R_i(\nu_0,\ldots,\nu_{i-1}) = \sum_{l=1}^{i-1}\nu_{i-l}\left(\sum_{j=1}^{l}u^{-j}\lambda^j\sum_{J(j,l-j)}\prod_{k=1}^{j}\nu_{j_k+1}\right).$$

Proof. This is a straightforward calculation. At its end the answer is simplified using the identity $1+\lambda\nu_0 u^{-1} = u^{-1}$. □

Definition 2.7. Let $M \le e$ be a positive integer and let $\mathcal{Q} = (q_0,\ldots,q_{M-1})$ be a sequence of integers such that $0 \le q_i < p-1$ for each $0 \le i \le M-1$.

(1) A G-invariant submodule $W \subset \mathrm{ind}_{KZ}^G\sigma$ is called *$\mathcal{Q}$-structured* if every element $f \in W$ such that $f \notin B_0$ can be written in the form $f = f_n^0 + f_n^1 + f'$, where $f' \in B_{n-1}$, $f_n^0 \in S_n^0$, $f_n^1 \in S_n^1$, and f_n^0 and f_n^1 satisfy the following condition:

Let $N = \min\{n, M\}$. For each $0 \le i \le N-1$ and each $\mu \in I_{n-1-i}$ there exist polynomials $P_{\mu,i}^0(X), P_{\mu,i}^1(X) \in \overline{\mathbb{F}}_p[X]$ of degree at most q_i such that

$$\begin{aligned} f_n^0 &= \sum_{\mu\in I_{n-1}}\sum_{\lambda\in I_1} g^0_{n,\mu+\pi^{n-1}\lambda}\otimes P^0_{\mu,0}(\lambda)x^r + \\ &\sum_{i=1}^{N-1}\sum_{\mu\in I_{n-2-i}}\sum_{\lambda\in I_1}\sum_{\nu\in I_{i+1}} g^0_{\mu+\pi^{n-2-i}\lambda+\pi^{n-1-i}\nu}\otimes P^0_{\mu,i}(\lambda)\left(\prod_{j=1}^{i-1}\nu_j\right)\nu_i^{q_0+1}x^r,\\ f_n^1 &= \sum_{\mu\in I_{n-1}}\sum_{\lambda\in I_1} g^1_{n,\mu+\pi^{n-1}\lambda}\otimes P^1_{\mu,0}(\lambda)y^r + \\ &\sum_{i=1}^{N-1}\sum_{\mu\in I_{n-2-i}}\sum_{\lambda\in I_1}\sum_{\nu\in I_{i+1}} g^1_{\mu+\pi^{n-2-i}\lambda+\pi^{n-1-i}\nu}\otimes P^1_{\mu,i}(\lambda)\left(\prod_{j=1}^{i-1}\nu_j\right)\nu_i^{q_0+1}y^r. \end{aligned}$$

Moreover, we require that for every collection of polynomials $P_{\mu,i}^0(X), P_{\mu,i}^1(X)$, for $0 \le i \le N-1$ and every μ, there exists an element $f \in W$ of the above form.

(2) A G-invariant submodule $U \subset \mathrm{ind}_{KZ}^G\sigma$ is called *extended $\mathcal{Q}$-structured* if every element $f \in W$ such that $f \notin B_{e-1}$ can be written in the form $f = f_n^0 + f_n^1 + f'$, where $f' \in B_{n-1}$, and f_n^0 and f_n^1 satisfy the following condition:

Let $N = \min\{M, n+e-1\}$. For each $0 \le i \le N-1$ and each $\mu \in I_{n-i-e}$ there exist polynomials $P_{\mu,i}^0(X), P_{\mu,i}^1(X) \in \overline{\mathbb{F}}_p[X]$ of degree at most q_i such that

$$\begin{aligned} f_n^0 &= \sum_{\mu\in I_{n-e}}\sum_{\lambda\in I_1}\sum_{\zeta\in I_{e-1}} g^0_{n,\mu+\pi^{n-e}\lambda+\pi^{n-e+1}\zeta}\otimes P^0_{\mu,0}(\lambda)x^r + \\ &\sum_{i=1}^{N-1}\sum_{\substack{\mu\in I_{n-e-1-i}\\ \lambda\in I_1}}\sum_{\substack{\nu\in I_{i+1}\\ \zeta\in I_{e-1}}} g^0_{\mu+\pi^{n-e-1-i}\lambda+\pi^{n-e-i}\nu+\pi^{n-e+1}\zeta}\otimes P^0_{\mu,i}(\lambda)\left(\prod_{j=1}^{i-1}\nu_j\right)\nu_i^{q_0+1}x^r,\\ f_n^1 &= \sum_{\mu\in I_{n-1}}\sum_{\lambda\in I_1}\sum_{\zeta\in I_{e-1}} g^1_{n,\mu+\pi^{n-e}\lambda+\pi^{n-e+1}\zeta}\otimes P^1_{\mu,0}(\lambda)y^r + \\ &\sum_{i=1}^{N-1}\sum_{\substack{\mu\in I_{n-2-i}\\ \lambda\in I_1}}\sum_{\substack{\nu\in I_{i+1}\\ \zeta\in I_{e-1}}} g^1_{\mu+\pi^{n-e-1-i}\lambda+\pi^{n-e-i}\nu+\pi^{n-e+1}\zeta}\otimes P^1_{\mu,i}(\lambda)\left(\prod_{j=1}^{i-1}\nu_j\right)\nu_i^{q_0+1}y^r. \end{aligned}$$

Again we require that for every collection of polynomials $P_{\mu,i}^0(X), P_{\mu,i}^1(X)$, for all $0 \le i \le N-1$ and μ, there exists an element $f \in U$ of the above form.

Remark 2.8. From the formulae of Lemma 1.4 one sees that $T(\mathrm{ind}_{KZ}^G\sigma)$ is a $\mathcal{Q}$-structured submodule for $M = 1$ and $q_0 = r$. Similarly, Lemma 1.6 shows that if $W \subset \mathrm{ind}_{KZ}^G\sigma$ is a $\mathcal{Q}$-structured submodule, then $T^{e-1}(W)$ is extended $\mathcal{Q}$-structured.

Suppose that $\sigma = \det^w \otimes \mathrm{Sym}^r \mathbb{F}_p^2$ and $\mathcal{Q} = (q_0, \ldots, q_{M-1})$. We will now define some special elements of $\mathrm{ind}_{KZ}^G \sigma$. Put $\tilde{X}_0^0 = \mathrm{Id} \otimes x^r$ and $\tilde{X}_0^1 = \alpha \otimes y^r$, and for $1 \leq n \leq e-1$ we define

$$\begin{aligned}
\tilde{X}_n^0 &= \sum_{\mu \in I_{n+1}} g_{n+1,\mu}^0 \otimes \mu_1 \mu_2 \cdots \mu_{n-1} \mu_n^{r+1} x^r, \\
\tilde{X}_n^1 &= \sum_{\mu \in I_{n+1}} g_{n+1,\mu}^1 \otimes \mu_1 \mu_2 \cdots \mu_{n-1} \mu_n^{r+1} y^r.
\end{aligned}$$

Observe that $\tilde{X}_n^1 = \beta \tilde{X}_n^0$. If $1 \leq l \leq M-1$, then for arbitrary n we set

$$\begin{aligned}
X_{n,l}^{0,+} &= \sum_{\mu \in I_n} g_{n,\mu}^0 \otimes \mu_{n-l-2}^{q_l+1} \left(\prod_{i=1}^{l-1} \mu_{n-l-1+i} \right) \mu_{n-1}^{q_0+1} x^r, \\
X_{n,l}^{1,+} &= \sum_{\mu \in I_n} g_{n,\mu}^1 \otimes \mu_{n-l-2}^{q_l+1} \left(\prod_{i=1}^{l-1} \mu_{n-l-1+i} \right) \mu_{n-1}^{q_0+1} y^r.
\end{aligned}$$

We also define

$$\begin{aligned}
X_{n,0}^{0,-} &= \sum_{\mu \in I_n} g_{n,\mu}^0 \otimes \mu_{n-1}^{q_0+2} x^r, \\
X_{n,0}^{1,-} &= \sum_{\mu \in I_n} g_{n,\mu}^1 \otimes \mu_{n-1}^{q_0+2} y^r, \\
X_{n,l}^{0,-} &= \sum_{\mu \in I_n} g_{n,\mu}^0 \otimes \mu_{n-l-1}^2 \left(\prod_{j=1}^{l-1} \mu_{n-l-1+j} \right) \mu_{n-1}^{q_0+1} x^r, \\
X_{n,l}^{1,-} &= \sum_{\mu \in I_n} g_{n,\mu}^1 \otimes \mu_{n-l-1}^2 \left(\prod_{j=1}^{l-1} \mu_{n-l-1+j} \right) \mu_{n-1}^{q_0+1} y^r, \\
X_{n,M-1}^{0,-} &= \sum_{\mu \in I_n} g_{n,\mu}^0 \otimes \mu_{n-M} \left(\prod_{j=1}^{M-2} \mu_{n-M+j} \right) \mu_{n-1}^{q_0+1} x^r, \\
X_{n,M-1}^{1,-} &= \sum_{\mu \in I_n} g_{n,\mu}^1 \otimes \mu_{n-M} \left(\prod_{j=1}^{M-2} \mu_{n-M+j} \right) \mu_{n-1}^{q_0+1} y^r,
\end{aligned}$$

where in the middle two lines we have $1 \leq l \leq M-2$. Define $\tilde{X}_n^0 = \tilde{X}_n^1 = 0$ if $n \geq e$. Note also that $X_{n,j}^{1,s} = \beta X_{n,j}^{0,s}$ for all $0 \leq j \leq M-1$ and all $s \in \{+,-\}$. If $n \geq e$, then we define $Y_{n,l}^{j,s}$ to be the part of $T^{e-1}(X_{n-e+1,l}^{j,s})$ supported on S_n, for $j \in \{0,1\}$, all $s \in \{+,-\}$, and all $0 \leq l \leq M-1$ for which this makes sense. Similarly we obtain $\tilde{Y}_n^j$ from $\tilde{X}_n^j$. Explicit expressions for these elements may be obtained from Lemma 1.6. For instance, if $1 \leq l \leq M-1$, then

$$Y_{n,l}^{0,+} = \sum_{\mu \in I_n} g_{n,\mu}^0 \otimes \mu_{n-e-l-1}^{q_l+1} \left(\prod_{i=1}^{l-1} \mu_{n-e-l+i} \right) \mu_{n-e}^{q_0+1} x^r.$$

We state the following simple observation for later use; it implies that all the elements just defined are eigenvectors under the action of the set $D \subset K$ of diagonal matrices.

Lemma 2.9. *Let $a, d \in \mathbb{F}_p$ and let $\tilde{a}, \tilde{d} \in \mathcal{O}$ be any lifts of a, d. Let $\delta = \mathrm{diag}(\tilde{a}, \tilde{d}) \in D$. If $P(\mu)$ is a homogeneous polynomial of degree s in the variables $\mu_0, \ldots, \mu_{n-1}$ and $X^0 = \sum_{\mu \in I_n} g_{n,\mu}^0 \otimes P(\mu) x^r \in \mathrm{ind}_{KZ}^G \sigma$ (respectively, $X^1 = \sum_{\mu \in I_n} g_{n,\mu}^1 \otimes P(\mu) y^r$), then*

$$\begin{aligned}
\delta X^0 &= (a^{-1} d)^s a^r X^0, \\
\delta X^1 &= (a d^{-1})^s d^r X^1.
\end{aligned}$$

Proposition 2.10. *Suppose that $U \subset \mathrm{ind}_{KZ}^G \sigma$ is an extended $\mathcal{Q}$-structured G-submodule and that $q_0 \le p-3$. Let $n \ge e$ and let $M' = \min\{M, n-e\}$.*

Let $\mathcal{Y}$ be the $\overline{\mathbb{F}}_p$-vector subspace of $\mathrm{ind}_{KZ}^G \sigma$ spanned by

$$\{\tilde{Y}_n^0, \tilde{Y}_n^1\} \cup \{Y_{n,i}^{0,+}, Y_{n,i}^{1,+} : 1 \le i \le M'-1\} \cup \{Y_{n,i}^{0,-}, Y_{n,i}^{1,-} : 0 \le i \le M'-1\}.$$

Suppose that $f \in S_n$ is such that $\gamma f_n - f_n \in U + B_{n-1}$ for all $\gamma \in I(1)$. Then $f_n \in \mathcal{Y} + U + B_{n-1}$.

Proof. We largely follow the method of [Bre], Prop. 3.2.1, which considers the case of $e = 1$ and $U = T(\mathrm{ind}_{KZ}^G \sigma)$.

We write $f_n^0 = \sum_{\lambda \in I_n} g_{n,\lambda}^0 \otimes v_\lambda$. For $\lambda = [\lambda_0] + \pi[\lambda_1] + \cdots + \pi^{n-1}[\lambda_{n-1}]$, define $\tilde{\lambda} = [\lambda_0] + \cdots + \pi^{n-e-1}[\lambda_{n-e-1}] + \pi^{n-e}[\lambda_{n-e}+1] + \pi^{n-e+1}[\lambda_{n-e+1}] + \cdots + \pi^{n-1}[\lambda_{n-1}]$. A straightforward computation gives that

$$\begin{aligned}
\begin{pmatrix} 1 & \pi^n \\ 0 & 1 \end{pmatrix} (g_{n,\lambda}^0 \otimes v_\lambda) - (g_{n,\lambda}^0 \otimes v_\lambda) &= g_{n,\lambda}^0 \otimes \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} v_\lambda - v_\lambda \right), \\
\begin{pmatrix} 1 & \pi^{n-e} \\ 0 & 1 \end{pmatrix} (g_{n,\lambda}^0 \otimes v_\lambda) - (g_{n,\tilde{\lambda}}^0 \otimes v_{\tilde{\lambda}}) &= g_{n,\tilde{\lambda}}^0 \otimes \left(\begin{pmatrix} 1 & \frac{\lambda_{n-e}^{p^e}+1-(\lambda_{n-e}+1)^{p^e}}{\pi^e} \\ 0 & 1 \end{pmatrix} v_\lambda - v_{\tilde{\lambda}} \right).
\end{aligned}$$

The hypothesis on f_n then implies the following equalities for all $\lambda \in I_n$:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} v_\lambda - v_\lambda \in \overline{\mathbb{F}}_p x^r, \tag{2}$$

$$\begin{pmatrix} 1 & \frac{\lambda_{n-e}^{p^e}+1-(\lambda_{n-e}+1)^{p^e}}{\pi^e} \\ 0 & 1 \end{pmatrix} v_\lambda - v_{\tilde{\lambda}} \in \overline{\mathbb{F}}_p x^r. \tag{3}$$

The equality (2) is easily seen to imply $v_\lambda \in \overline{\mathbb{F}}_p x^r + \overline{\mathbb{F}}_p x^{r-1} y$, so we write $v_\lambda = c_\lambda x^r + d_\lambda x^{r-1} y$. Then (3) implies that

$$\left(c_\lambda - c_{\tilde{\lambda}} + d_\lambda \frac{\lambda_{n-e}^{p^e} + 1 - (\lambda_{n-e}+1)^{p^e}}{\pi^e} \right) x^r + (d_\lambda - d_{\tilde{\lambda}}) x^{r-1} y \in \overline{\mathbb{F}}_p x^r. \tag{4}$$

Given $\lambda \in I_n$, define $\langle \lambda \rangle = \lambda - [\lambda_{n-e}]\pi^{n-e} \in I_n$. In other words, $\langle \lambda \rangle$ is the same as λ, but with λ_{n-e} replaced by 0. Then the above formula implies that d_λ is independent of λ_{n-e}, so we write $d_\lambda = d_{\langle \lambda \rangle}$. Similarly, by Lemma 2.1 we can view $c_\lambda = c_{\langle \lambda \rangle}(\lambda_{n-e})$ as a polynomial in λ_{n-e} of degree at most $p-1$. From the definition of an extended $\mathcal{Q}$-structured module, we see that

$$c_{\langle \lambda \rangle}(\lambda_{n-e}) - c_{\langle \lambda \rangle}(\lambda_{n-e}+1) + d_{\langle \lambda \rangle} \cdot \frac{\lambda_{n-e}^{p^e} + 1 - (\lambda_{n-e}+1)^{p^e}}{\pi^e}$$

must be a polynomial of degree at most q_0+1 in λ_{n-e}. Since $c_{\langle \lambda \rangle}(\lambda_{n-e})$ has degree at most $p-1$ in λ_{n-e}, the difference $c_{\langle \lambda \rangle}(\lambda_{n-e}) - c_{\langle \lambda \rangle}(\lambda_{n-e}+1)$ has degree at most $p-2$. But $q_0+1 \le p-2$, so the remaining term $d_{\langle \lambda \rangle} \cdot \frac{\lambda_{n-e}^{p^e}+1-(\lambda_{n-e}+1)^{p^e}}{\pi^e}$ must also have degree at most $p-2$, and this forces $d_{\langle \lambda \rangle} = 0$ by the observation of Remark 2.4. Therefore $c_{\langle \lambda \rangle}(\lambda_{n-e}) - c_{\langle \lambda \rangle}(\lambda_{n-e}+1)$ has degree at most q_0+1 in the variable λ_{n-e}, and consequently $c_{\langle \lambda \rangle}(\lambda_{n-e})$ has degree at most q_0+2.

Using the deduction above we may rewrite

$$f_n^0 = \sum_{\mu \in I_{n-e}} \sum_{\lambda \in I_1} \sum_{\nu \in I_{e-1}} g_{n, \mu + \pi^{n-e}\lambda + \pi^{n-e+1}\nu}^0 \otimes c_{\mu,\nu}(\lambda) x^r, \tag{5}$$

where $c_{\mu,\nu}(X) \in \overline{\mathbb{F}}_p[X]$ is a polynomial of degree at most q_0+2. From the definition of an extended $\mathcal{Q}$-structured module it is easy to see (cf. [Bre], Lemme 3.1.5) that we may modify f by an element of $U + B_{n-1}^0$ if necessary and assume without loss of generality that for all $\mu \in I_{n-e}$ we have $c_{\mu,0}(X) = a_{\mu,0} X^{q_0+1} + b_{\mu,0} X^{q_0+2}$, where $a_{\mu,0}, b_{\mu,0} \in \overline{\mathbb{F}}_p$ are constants.

Now fix $\mu \in I_{n-e}$ and $\nu \in I_{e-1}$. Suppose that $\mu' \in I_{n-e}$, $\lambda' \in I_1$, and $\nu' \in I_{e-1}$ are such that

$$\begin{pmatrix} 1 & \mu + \pi^{n-e+1}\nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi^n & \mu' + \pi^{n-e}\lambda' + \pi^{n-e+1}\nu' \\ 0 & 1 \end{pmatrix} \in \begin{pmatrix} \pi^n & \mu + \pi^{n-e}\lambda + \pi^{n-e+1}\nu'' \\ 0 & 1 \end{pmatrix} KZ$$

for some $\lambda \in I_1$ and $\nu'' \in I_{e-1}$. It is easy to see that, equivalently, $(\mu+\mu') + \pi^{n-e}\lambda' + \pi^{n-e+1}(\nu + \nu') \equiv \mu + \pi^{n-e}\lambda + \pi^{n-e+1}\nu'' \mod \pi^n$. Considering this congruence modulo π, we find that $\mu'_0 = 0$, and it follows inductively that $\mu' = 0$. Similarly, $\lambda = \lambda'$, and $\nu' = \nu'' - \nu$ by Remark 2.3. We conclude that the terms of

$$h_\mu = \begin{pmatrix} 1 & \mu + \pi^{n-e+1}\nu \\ 0 & 1 \end{pmatrix} f_n^0 - f_n^0$$

with support in $\coprod_{\lambda,\nu''} KZ(g^0_{n,\mu+\pi^{n-e}\lambda+\pi^{n-e+1}\nu''})^{-1}$ are precisely:

$$\sum_{\lambda \in I_1} \sum_{\nu'' \in I_{e-1}} g^0_{n,\mu+\pi^{n-e}\lambda+\pi^{n-e+1}\nu''} \otimes (c_{\mu,\nu''-\nu}(\lambda) - c_{\mu,\nu}(\lambda))x^r. \tag{6}$$

By assumption, $h_\mu \in U + B_{n-1}$ and hence $c_{\mu,\nu}$ is independent of ν. Thus we may write $c_{\mu,\nu}(X) = c_\mu(X) = a_\mu X^{q_0+1} + b_\mu X^{q_0+2}$. From (6) we see that $(a_0 - a_\mu)\lambda^{q_0+1} + (b_0 - b_\mu)\lambda^{q_0+2}$ is a polynomial of degree at most $q_0 + 1$ in λ, and hence $b_\mu = b_0$ for all $\mu \in I_{n-e}$. Thus we may write

$$f_n^0 = \sum_{\mu \in I_{n-e}} \sum_{\substack{\lambda \in I_1 \\ \nu \in I_{e-1}}} g^0_{n,\mu+\pi^{n-e}\lambda+\pi^{n-e+1}\nu} \otimes (a(\mu_0, \ldots, \mu_{n-e-1})\lambda^{q_0+1} + b\lambda^{q_0+2})x^r. \tag{7}$$

Here a is a polynomial in the indicated variables and b is a constant. For all $0 \le j \le n-e$ denote

$$\gamma(j) = \begin{pmatrix} 1 & \pi^{n-e-j} \\ 0 & 1 \end{pmatrix}$$

and note that the action of $\gamma(j)$ preserves S_m^0 for each m. Using Lemma 2.2 and (7), we observe that if $1 \le j \le e-1$, then:

$$\gamma(j)f_n^0 - f_n^0 = \sum_{\mu \in I_{n-e}} \sum_{\substack{\lambda \in I_1 \\ \nu \in I_{e-1}}} g^0_{n,(\mu,\lambda,\nu)} \otimes (a(\mu_0, \ldots, \mu_{n-e-1}) - a(\mu_0, \ldots, \mu_{n-e-j} - 1, \ldots, \mu_{n-e-1}))\lambda^{q_0+1}x^r. \tag{8}$$

Here we have written $g^0_{n,(\mu,\lambda,\nu)}$ for $g^0_{n,\mu+\pi^{n-e}\lambda+\pi^{n-e+1}\nu}$. If $j \ge e$, then we get a similar formula for $\gamma(j)f_n^0 - f_n^0$, but with $a(\mu_0, \ldots, \mu_{n-e-j} - 1, \ldots, \mu_{n-e-1})$ replaced by an expression of the form $a(\mu_0, \ldots, \mu_{n-e-j-1}, \mu_{n-e-j} - 1, \mu_{n-e-j+1} + R_{n-e-j+1}, \ldots, \mu_{n-e-1} + R_{n-e-1})$, where each R_i is a polynomial in the variables $\mu_{n-e-j}, \ldots, \mu_{i-e}$.

By assumption, $\gamma(j)f_n^0 - f_n^0 \in U + B_{n-1}$. If $M' > 2$ then it is evident from the case $j = 1$ of the formula above that a has degree at most 2 in the variable μ_{n-e-1}. Therefore we may write $a = a^{(0)} + a^{(1)}\mu_{n-e-1} + a^{(2)}\mu_{n-e-1}^2$, where each $a^{(i)}$ is a polynomial in the variables $\mu_0, \ldots, \mu_{n-e-2}$.

We claim that the polynomial $a^{(2)}$ is constant. Indeed, suppose it is not and consider the minimal $j \ge 2$ such that μ_{n-e-j} appears in $a^{(2)}$. Then we see from (8) and the remark following it that $\gamma(j)f_n^0 - f_n^0$ has a term of the form $\sum_{(\mu,\lambda,\nu)} g^0_{n,(\mu,\lambda,\nu)} \otimes R(\mu_0, \ldots, \mu_{n-e-2})\mu_{n-e-1}^2\lambda^{q_0+1}x^r$, contradicting $\gamma(j)f_n^0 - f_n^0 \in U + B_{n-1}$.

It is immediate from the case $j = 1$ of (8) that $a^{(0)}$ and $a^{(1)}$ have degrees at most $q_1 + 1$ and 2, respectively, in the variable μ_{n-e-2}. Modifying $a^{(0)}$ by an element of $U + B_{n-1}$, we may assume that it has the form $a^{(0)} = \hat{a}^{(0)}(\mu_0, \ldots, \mu_{n-e-3})\mu_{n-e-2}^{q_1+1}$. But then we can show that $\hat{a}^{(0)}$ is a scalar by the same argument that was used for $a^{(2)}$.

Therefore, after modifying f_n^0 by an element of $\overline{\mathbb{F}}_p \cdot Y^{0,-}_{n,0} + \overline{\mathbb{F}}_p \cdot Y^{0,-}_{n,1} + \overline{\mathbb{F}}_p \cdot Y^{0,+}_{n,1} + U + B_{n-1}$, we may assume that

$$f_n^0 = \sum_{\mu \in I_{n-e}} \sum_{\substack{\lambda \in I_1 \\ \nu \in I_{e-1}}} g^0_{n,(\mu,\lambda,\nu)} \otimes a^{(1)}(\mu_0, \ldots, \mu_{n-e-2})\mu_{n-e-1}\lambda^{q_0+1}x^r,$$

where $a^{(1)}$ has degree at most 2 in the variable μ_{n-e-1}. We may now go back to the expression (7) and repeat the entire argument with $a^{(1)}$ in place of a.

Iterating the argument, we obtain inductively that, after adding to f_n^0 an element of $\overline{\mathbb{F}}_p \cdot Y_{n,0}^{0,-} + \sum_{i=1}^{M-2}(\overline{\mathbb{F}}_p \cdot Y_{n,i}^{0,-} + \overline{\mathbb{F}}_p \cdot Y_{n,i}^{0,+}) + U + B_{n-1}$, we get

$$f_n^0 = \sum_{\mu \in I_{n-e}} \sum_{\substack{\lambda \in I_1 \\ \nu \in I_{e-1}}} g_{n,(\mu,\lambda,\nu)}^0 \otimes a(\mu_0, \dots, \mu_{n-e-M+1}) \left(\prod_{j=1}^{M-2} \mu_{n-e-j} \right) \lambda^{q_0+1} x^r,$$

and this time a has degree at most 1 in the variable $\mu_{n-e-M+1}$. Thus we may write $a = a^{(0)} + a^{(1)}\mu_{n-e-M+1}$, where the $a^{(i)}$ are polynomials in the variables $\mu_0, \dots, \mu_{n-e-M}$. We can show as before that, modulo $U + B_{n-1}$, we may take $a^{(0)} = c\mu_{n-e-M}^{q_{M-1}+1}$ for some scalar $c \in \overline{\mathbb{F}}_p$. On the other hand, $a^{(1)}$ must have degree at most 1 in μ_{n-e-M}. Using the same method as before, we show that $a^{(1)} = d_1\mu_{n-e-M} + d_0$ for scalars $d_0, d_1 \in \overline{\mathbb{F}}_p$. Moreover, since $q_{M-1} \geq 0$ we may modify f_n^0 yet again by an element of $U + B_{n-1}$ and take $d_0 = 0$. This proves that $f^0 \in \overline{\mathbb{F}}_p\tilde{Y}_n^0 + \sum_i \overline{\mathbb{F}}_p Y_{n,i}^{0,+} + \sum_i \overline{\mathbb{F}}_p Y_{n,i}^{0,-} + U + B_{n-1}$.

Observe that $\beta^{-1}f$ also satisfies the hypotheses of the lemma, and hence there exist scalars $c, c_i^{1,+}, c_i^{1,-} \in \overline{\mathbb{F}}_p$ such that $\beta^{-1}f^1 = (\beta^{-1}f)^0 \equiv c\tilde{Y}_n^0 + c_0^{1,-}Y_{n,0}^{0,-} + \sum_{i=1}^{M'}(c_i^{1,+}Y_{n,i}^{0,+} + c_i^{1,-}Y_{n,i}^{0,-})$ modulo $U + B_{n-1}$. But this means that $f^1 \equiv c\tilde{Y}_n^1 + c_0^{1,-}Y_{n,0}^{1,-} + \sum_{i=1}^{M'}(c_i^{1,+}Y_{n,i}^{1,+} + c_i^{1,-}Y_{n,i}^{1,-})$ modulo $U + B_{n-1}$. □

Corollary 2.11. *Suppose that $W \subset \mathrm{ind}_{KZ}^G\sigma$ is a $\mathcal{Q}$-structured G-submodule and that $q_0 \leq p-3$. Let $n \geq 1$ and let $M' = \min\{M, n-1\}$. Let $\mathcal{X}$ be the $\overline{\mathbb{F}}_p$-vector subspace of $\mathrm{ind}_{KZ}^G\sigma$ spanned by*

$$\{\tilde{X}_n^0, \tilde{X}_n^1\} \cup \{X_{n,i}^{0,+}, X_{n,i}^{1,+} : 1 \leq i \leq M'-1\} \cup \{X_{n,i}^{0,-}, X_{n,i}^{1,-} : 0 \leq i \leq M'-1\}.$$

Suppose that $f \in S_n$ is such that $\gamma f_n - f_n \in W + B_{n-1}$ for all $\gamma \in I(1)$. Then $f_n \in \mathcal{X} + U + B_{n-1}$.

Proof. In view of Remark 2.8 and the injectivity of T (Corollary 1.5), the claim is immediate from Proposition 2.10. □

3. Construction of a quotient

Let F^{nr} be the maximal unramified extension of F, and let $L = F^{nr}(\pi^{1/(p^2-1)})$. Choose a field embedding $\mathbb{F}_{p^2} \hookrightarrow \overline{\mathbb{F}}_p$. It induces a character $\omega_2 : I_F \twoheadrightarrow \mathrm{Gal}(L/F^{nr}) \simeq \mathbb{F}_{p^2}^* \to \overline{\mathbb{F}}_p^*$ of the inertia subgroup $I_F \subset \mathrm{Gal}(\overline{F}/F)$. Then $\omega_2^{e(p+1)}$ is the restriction to I_F of the mod p cyclotomic character.

Suppose that $0 < r \leq p - 2e - 1$. Consider a continuous irreducible tamely ramified Galois representation $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ whose restriction to I_F has the form

$$\rho_{|I_F} \sim \begin{pmatrix} \omega_2^{r+e} & 0 \\ 0 & \omega_2^{p(r+e)} \end{pmatrix}.$$

Recall that a *Serre weight* (in this context) is an irreducible $\overline{\mathbb{F}}_p$-representation of the finite group $\mathrm{GL}_2(\mathbb{F}_p)$. In our case the author [Sch1], [Sch2] has conjectured that the set of modular weights of ρ is the set $\mathcal{D}$ defined in (1). See [Sch1], Def. 1.2 for the definition of the modular weights of a Galois representation. In almost all of the cases under consideration here ($F/\mathbb{Q}_p$ totally ramified, with restrictions on r) the conjecture has been proved by Gee and Savitt [GS].

We will inductively construct a sequence of quotients of $V_0 = \mathrm{ind}_{KZ}^G\sigma_0/T(\mathrm{ind}_{KZ}^G\sigma_0)$ as follows. Let $1 \leq i \leq e-1$ and suppose that V_{i-1} has been constructed. We claim that the image of $\tilde{X}_i^0$ in V_{i-1} is invariant under the action of $I(1)$ and, furthermore, that the KZ-submodule of V_{i-1} generated by $\tilde{X}_i^0$ is isomorphic to σ_i'. This gives us a map $\psi \in \mathrm{Hom}_{KZ}(\sigma_i', (V_{i-1})_{|KZ})$ determined by $\psi(v) = \tilde{X}_i^0$, where $v \in V_{\sigma_i'}^{I(1)}$ is a highest weight vector. By Frobenius reciprocity we obtain a map $\Psi_i \in \mathrm{Hom}_G(\mathrm{ind}_{KZ}^G\sigma_i', V_{i-1})$, and finally we define

$$V_i = V_{i-1}/\Psi_i(T(\mathrm{ind}_{KZ}^G\sigma_i')).$$

Note that v is determined up to scalar, and hence V_i is independent of the choice of v. In this section we show that the idea outlined here actually works. If V_i has been constructed, let N_i denote the kernel of the natural projection $\mathrm{ind}_{KZ}^G\sigma_0 \to V_i$.

Lemma 3.1. *Suppose that $1 \leq n \leq e-1$ and the quotient V_{n-1} has been constructed. Then $\tilde{X}_n^0 \in (V_{n-1})^{I(1)}$ and $\tilde{X}_n^1 \in (V_{n-1})^{I(1)}$.*

Proof. Since $\tilde{X}_n^1 = \beta \tilde{X}_n^0$ and β normalizes $I(1)$, it suffices to show that $\tilde{X}_n^0$ is an $I(1)$-invariant. Let $\gamma \in I(1)$, and write

$$\gamma = \begin{pmatrix} 1 + a\pi & b \\ c\pi & 1 + d\pi \end{pmatrix},$$

where $a, b, c, d \in \mathcal{O}$. Expand $a = \sum_{i=0}^{\infty} [a_i]\pi^i$, where $[a_i] \in I_1$, and do similarly for b, c, d. For any $\mu \in I_{n+1}$, define

$$\varepsilon_\mu = (1 + d\pi + \mu c\pi)^{-1}(b + \mu + \mu a\pi)$$

and expand $\varepsilon_\mu = \sum_{i=0}^{\infty} [\varepsilon_i]\pi^i$. Observe that

$$\gamma g^0_{n+1,\mu} = \begin{pmatrix} \pi^{n+1} & \varepsilon_\mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + \pi(a - c\varepsilon_\mu) & 0 \\ 0 & 1 + \pi(d + \mu c) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\pi^{n+2}c\cdot}{(1+\pi(d+\mu c))^{-1}} & 1 \end{pmatrix}.$$

On the other hand,

$$\begin{pmatrix} \pi^{n+1} & \varepsilon_\mu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi^{n+1} & [\varepsilon_\mu]_{n+1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},$$

where $z \in \mathcal{O}$. We conclude that $\gamma g^0_{n+1,\mu} = g^0_{n+1,\varepsilon_\mu} u$ for some $u \in I(1)$.

Recall the definition of μ_ε from Lemma 2.5 and observe that $\mu_{(\varepsilon_\mu)} = \mu$. Therefore, by the same lemma,

$$\begin{aligned} \gamma \tilde{X}_n^0 - \tilde{X}_n^0 \quad &= \sum_{\mu \in I_{n+1}} (g^0_{n+1,\varepsilon_\mu} \otimes \mu_1 \cdots \mu_{n-1} \mu_n^{r+1} x^r - g^0_{n+1,\mu} \otimes \mu_1 \cdots \mu_{n-1} \mu_n^{r+1} x^r) = \\ &\sum_{\varepsilon \in I_{n+1}} g^0_{n+1,\varepsilon} \otimes \left((\varepsilon_1 + P_1) \cdots (\varepsilon_{n-1} + P_{n-1})(\varepsilon_n + P_n)^{r+1} - \varepsilon_1 \cdots \varepsilon_{n-1} \varepsilon_n^{r+1} \right) x^r. \end{aligned}$$

Observe that we may write $\gamma \tilde{X}_n^0 - \tilde{X}_n^0 = \sum_{i=1}^n C_n$, where

$$C_n = \sum_{\varepsilon \in I_{n+1}} g^0_{n+1,\varepsilon} \otimes (\varepsilon_1 + P_1) \cdots (\varepsilon_{n-1} + P_{n-1})((\varepsilon_n + P_n)^{r+1} - \varepsilon_n^{r+1}) x^r$$

and if $1 \leq i \leq n-1$ then

$$C_i = \sum_{\varepsilon \in I_{n+1}} g^0_{n+1,\varepsilon} \otimes \left(\prod_{j=1}^{i-1} (\varepsilon_j + P_j) \right) P_i(\varepsilon_0, \ldots, \varepsilon_{i-1}) \left(\prod_{j=i+1}^{n-1} \varepsilon_j \right) \varepsilon_n^{r+1} x^r.$$

For each $1 \leq i \leq n$ we claim that $C_i \in N_{n-i} \subset N_{n-1}$. This would imply that the image in V_{n-1} of $\gamma \tilde{X}_n^0 - \tilde{X}_n^0$ vanishes for all $\gamma \in I(1)$, hence that $\tilde{X}_n^0$ is indeed an $I(1)$-invariant in V_{n-1}.

Let $c_j(\varepsilon_0, \ldots, \varepsilon_{n-1})$, for $0 \leq j \leq r$, be the polynomials such that $\sum_{j=0} c_j(-\varepsilon_n)^j = (\varepsilon_1 + P_1) \cdots (\varepsilon_{n-1} + P_{n-1})((\varepsilon_n + P_n)^{r+1} - \varepsilon_n^{r+1})$. In particular, $c_r(\varepsilon_0, \ldots, \varepsilon_{n-1}) = (-1)^r (r+1)(\varepsilon_1 + P_1) \cdots (\varepsilon_{n-1} + P_{n-1}) P_n$. For each $\tilde{\varepsilon} = \sum_{i=0}^{n-1} \varepsilon_i \pi^i \in I_n$, define $v_{\tilde{\varepsilon}} = \sum_{j=0}^r c_j(\varepsilon_0, \ldots, \varepsilon_{n-1}) x^{r-j} y^j \in V_{\sigma_0}$. Then it follows from Lemma 1.4 that

$$C_n = T \left(\sum_{\tilde{\varepsilon} \in I_n} g^0_{n,\tilde{\varepsilon}} \otimes v_{\tilde{\varepsilon}} \right) - \sum_{\tilde{\varepsilon} \in I_{n-1}} \otimes \left(\sum_{\varepsilon_{n-1} \in I_1} c_r(\varepsilon_0, \ldots, \varepsilon_{n-1})(\varepsilon_{n-1} x + y)^r \right). \tag{9}$$

It is easy to see from Lemma 2.5 that $P_1(\varepsilon_0)$ is a quadratic polynomial in ε_0, whereas if $n > 1$ then ε_{n-1} appears with degree at most one in any term of P_n. Therefore, ε_{n-1} appears with degree at most $r+3$ in any term of $c_r(\varepsilon_{n-1} x + y)^r$. Since $r + 3 < p - 1$ by assumption, we see that the second term on the right-hand side of (9) vanishes and thus $C_n \in T(\mathrm{ind}_{KZ}^G \sigma_0) = N_0$.

One proves in a similar way that $C_i \in \Psi_{n-i}(T(\mathrm{ind}_{KZ}^G \sigma'_{n-i}))$ if $1 \leq i \leq n-1$. For more detail the reader is directed to the analogous argument for A_i in the proof of Lemma 3.2. $\square$

Lemma 3.2. *For any $\lambda \in I_1$, the following identities hold in V_{n-1}:*

$$\begin{aligned} h_n^\lambda &= \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \tilde{X}_n^0 &= \sum_{\nu \in I_{n+1}} g^0_{n+1,\nu} \otimes (1-\lambda\nu_0)^{p-r-2n-1} \nu_1 \cdots \nu_{n-1} \nu_n^{r+1} x^r + \\ &&\sum_{\tau \in I_n} g^1_{n,\tau} \otimes (-1)^{r+n} \lambda^{p-r-2n-1} \tau_0 \cdots \tau_{n-2} \tau_{n-1}^{r+1} y^r, \\ h_n^\infty &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{X}_n^0 &= \sum_{\nu \in I_{n+1}} g^0_{n+1,\nu} \otimes (-1)^n \nu_0^{p-r-2n-1} \nu_1 \cdots \nu_{n-1} \nu_n^{r+1} x^r + \\ &&\sum_{\tau \in I_n} g^1_{n,\tau} \otimes \tau_0 \cdots \tau_{n-2} \tau_{n-1}^{r+1} y^r. \end{aligned}$$

Proof. For $\lambda = 0$ the claim is obvious, so assume $\lambda \neq 0$. First we observe that

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} g^0_{n+1,\mu} = g^0_{n+1,\mu(\lambda\mu+1)^{-1}} \begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\lambda\mu+1)^{-1} & 0 \\ 0 & \lambda\mu+1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi^{n+1} z_2 & 1 \end{pmatrix}$$

if $\lambda\mu+1 \in \mathcal{O}^*$, where $z_1, z_2 \in \mathcal{O}$. On the other hand, if $\lambda\mu+1 \notin \mathcal{O}^*$ (in other words, if $\mu_0 = -\lambda^{-1}$), then

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} g^0_{n+1,\mu} = g^1_{n,\pi^{-1}\mu^{-1}(1+\lambda\mu)} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & -\mu^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi^{n+1}\mu^{-1} & 1 \end{pmatrix}.$$

Given our restrictions on r, it follows immediately that

$$h_n^\lambda = \sum_{\mu \in I_{n+1}} g^0_{n+1,\mu(\lambda\mu+1)^{-1}} \otimes \mu_1 \cdots \mu_{n-1} \mu_n^{r+1} (\lambda\mu_0+1)^{p-1-r} x^r + \sum_{\substack{\mu \in I_{n+1} \\ \lambda\mu_0+1=0}} g^1_{n,\pi^{-1}\mu^{-1}(\lambda\mu+1)} \otimes (-1)^r \mu_1 \cdots \mu_{n-1} \mu_n^{r+1} \mu_0^{p-1-r} y^r. \quad (10)$$

For $\nu \in I_{n+1}$, set $\tilde{\nu} = \nu(1-\lambda\nu)^{-1}$ if this is defined. For $\tau \in I_n$ we set $\tilde{\tau} = -(\lambda - \pi\tau)^{-1}$, which exists for all τ since we have assumed $\lambda \neq 0$. Then the expression above may be rewritten as

$$h_n^\lambda = \sum_{\nu \in I_{n+1}} g^0_{n+1,\nu} \otimes (1-\lambda\nu_0)^r \tilde{\nu}_1 \cdots \tilde{\nu}_{n-1} \tilde{\nu}_n^{r+1} x^r + \sum_{\tau \in I_n} g^1_{n,\tau} \otimes \lambda^r \tilde{\tau}_1 \cdots \tilde{\tau}_{n-1} \tilde{\tau}_n^{r+1} y^r.$$

Let $\hat{h}_n^\lambda$ denote the claimed expression for h_n^λ in the statement of the lemma. We need to show that $h_n^\lambda - \hat{h}_n^\lambda$ lies in N_{n-1}. By Lemma 2.6 we see that the first summand of h_n^λ (consisting of the terms supported on S^0_{n+1}) can be expressed as

$$\sum_{\nu \in I_{n+1}} g^0_{n+1,\nu} \otimes (1-\lambda\nu_0)^{p-1-2n-r} (\nu_1 + R_1) \cdots (\nu_{n-1} + R_{n-1})(\nu_n + R_n)^{r+1} x^r.$$

As in the proof of the preceding lemma, the difference between this expression and the corresponding summand of $\hat{h}_n^\lambda$ can be written as a sum $\sum_{i=1}^n A_i$, where

$$A_n = \sum_{\nu \in I_{n+1}} g^0_{n+1,\nu} \otimes (1-\lambda\nu_0)^{p-r-2n-1} (\nu_1 + R_1) \cdots (\nu_{n-1} + R_{n-1})((\nu_n + R_n)^{r+1} - \nu_n^{r+1}) x^r$$

and for $1 \le i \le n-1$ we have

$$A_i = \sum_{\nu \in I_{n+1}} g^0_{n+1,\nu} \otimes (1-\lambda\nu_0)^{p-r-2n-1} \left(\prod_{j=1}^{i-1} (\nu_j + R_j) \right) R_i(\nu_0, \ldots, \nu_{i-1}) \left(\prod_{j=i+1}^{n-1} \nu_j \right) \nu_n^{r+1} x^r.$$

We claim that $A_i \in N_{n-i} \subset N_{n-1}$ for all $1 \le i \le n$. Indeed, consider first the case $i = n$. Define polynomials $c_j(\nu_0, \ldots, \nu_{n-1})$ such that $\sum_{j=0}^r c_j(-\nu_n)^j = (1-\lambda\nu_0)^{p-r-2n-1}(\nu_1 + R_1) \cdots (\nu_{n-1} +$

$R_{n-1})((\nu_n + R_n)^{r+1} - \nu_n^{r+1})$. For each $\nu \in I_n$ define $v_\nu = \sum_{j=0}^{r} c_j x^{r-j} y^j \in V_{\sigma_0}$. Then we see from the formulae of Lemma 1.4 that

$$A_n = T\left(\sum_{\nu\in I_n} g^0_{n,\nu} \otimes v_\nu\right) - \sum_{\nu\in I_n} g^0_{n-1,[\nu]_{n-1}} \otimes B(\nu_0,\ldots,\nu_{n-1})(\nu_{n-1}x+y)^r =$$
$$T\left(\sum_{\nu\in I_n} g^0_{n,\nu} \otimes v_\nu\right) - \sum_{\nu\in I_{n-1}} g^0_{n-1,\nu} \otimes \left(\sum_{\nu_{n-1}\in I_1} B(\nu_0,\ldots,\nu_{n-1})(\nu_{n-1}x+y)^r\right),$$

where

$$B(\nu_0,\ldots,\nu_{n-1}) = (-1)^r(r+1)(1-\lambda\nu_0)^{p-r-2n-1}(\nu_1+R_1)\cdots(\nu_{n-1}+R_{n-1})R_n.$$

Observe that if $n > 1$, then ν_{n-1} appears with degree at most two in any term of $R_n(\nu_0,\ldots,\nu_{n-1})$. Thus ν_{n-1} appears with degree at most $r+3$ in any term of $B(\nu_0,\ldots,\nu_{n-1})\cdot(\nu_{n-1}x+y)^r$. But $r+3 < p-1$ by assumption, so $\sum_{\nu_{n-1}\in I_1} B(\nu_0,\ldots,\nu_{n-1})(\nu_{n-1}x+y)^r = 0$ and $A_n \in T(\mathrm{ind}_{KZ}^G \sigma_0) = N_0$. The case $i = n = 1$ is handled separately but analogously.

Now suppose that $1 \le i < n$. Observe that ν_i does not appear in A_i, and that the projection of A_i to V_{n-i-1} is a scalar multiple of the image under Ψ_{n-i} of the element

$$\sum_{\nu\in I_i} g^0_{i,\nu} \otimes (1-\lambda_0)^{p-r-2n-1}\left(\prod_{j=1}^{i-1}(\nu_j+R_j)\right) R_i(\nu_0,\ldots,\nu_{i-1})\hat{x}^{p-r-1-2(n-i)} \in \mathrm{ind}_{KZ}^G \sigma'_{n-i}.$$

Here we denote the usual basis of $V_{\sigma'_{n-i}}$ by $\{\hat{x}^{p-r-1-2(n-i)-j}\hat{y}^j : 0 \le j \le p-r-1-2(n-i)\}$. Using the assumption that $p-r-1-2(n-i) > 3$, we find that this element actually lies in $T(\mathrm{ind}_{KZ}^G \sigma'_{n-i})$ and hence that $A_i \in N_{n-i}$ as claimed.

The remaining terms of h_n^λ and $\hat{h}_n^\lambda$, those supported on S_n^1, are shown to be equal modulo N_{n-1} in a similar way, proving the claim about h_n^λ. The case of h_n^∞ is also treated by easy but somewhat tedious computations. □

Proposition 3.3. *The KZ-submodule $U \subset V_{n-1}$ generated by $\tilde{X}_n^0$ is irreducible and isomorphic to σ'_n.*

Proof. Since $KZ = \coprod_{\lambda\in I_1}\begin{pmatrix}1 & 0\\ \lambda & 1\end{pmatrix} I(1) \coprod \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} I(1)$ and since $\tilde{X}_n^0 \in V_{n-1}$ is an $I(1)$-invariant by Lemma 3.1, we see that U is spanned by the $p+1$ elements h_n^λ and h_n^∞. Inspecting the expressions of Lemma 3.2, we see easily by Corollary 2.11 that $U^{I(1)} = \overline{\mathbb{F}}_p \cdot \tilde{X}_n^0$. Since $I(1)$ is a pro-p-group, any irreducible KZ-submodule $U' \subset U$ must contain an $I(1)$-invariant vector, hence $\tilde{X}_n^0 \in U'$, so $U' = U$ and U is irreducible. Finally, observe that for any $a, b \in \mathbb{F}_p^*$ we have

$$\begin{pmatrix}[a] & 0\\ 0 & [b]\end{pmatrix}\tilde{X}_n^0 = \sum_{\mu\in I_{n+1}} g^0_{n+1,[ab^{-1}]\mu} \otimes \mu_1\cdots\mu_{n-1}\mu_n^{r+1}a^r x^r = (ab)^{r+n}a^{p-r-2n-1}.$$

The claim follows from the classification of irreducible KZ-representations. See, for instance, Proposition 4 of [BL]. □

Remark 3.4. It can be shown that the KZ-submodule of V_{n-1} generated by $\tilde{X}_n^1$ is not irreducible, but rather a principal series of dimension $p+1$.

The preceding proposition shows that the inductive program described at the beginning of this section may indeed be carried out to produce a sequence of quotients $\mathrm{ind}_{KZ}^G\sigma_0 \to V_0 \to V_1 \to \cdots \to V_{e-1}$. Let $\kappa_i : \mathrm{ind}_{KZ}^G \sigma_0 \to V_i$ be the natural surjections. For each $0 \le i \le e-1$, let $p_i : V_i \to V_{e-1}$ be the natural surjection map. For $1 \le i \le e-1$ and $j \in \{0,1\}$, let $X_i^j = \kappa_{e-1}(\tilde{X}_i^j)$. For any weight $\sigma = \sigma_{s,w}$, let $\tilde{X}_1^0(\sigma)$ denote the element

$$\tilde{X}_1^0(\sigma) = \sum_{\mu\in I_2} g^0_{2,\mu} \otimes \mu_1^{s+1}\hat{x}^s \in \mathrm{ind}_{KZ}^G \sigma.$$

We denote $\sigma_e = \det^{-e} \otimes \mathrm{Sym}^{r+2e}\overline{\mathbb{F}}_p^2$ and $\sigma'_e = \det^{r+e} \otimes \mathrm{Sym}^{p-r-1-2e}\overline{\mathbb{F}}_p^2$, compatibly with (1).

Proposition 3.5. *Let σ be a weight such that there exists a non-trivial G-equivariant map $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma) \to V_{e-1}$. Then $\sigma \in \mathcal{D} \cup \{\sigma_e, \sigma'_e\}$.*

Proof. Observe that D acts on the $I(1)$-invariants of σ_i via the character $\chi_i : \mathrm{diag}(\tilde{a}, \tilde{d}) \mapsto (ad^{-1})^i a^r$ and on the $I(1)$-invariants of σ'_i via the character $\chi'_i : \mathrm{diag}(\tilde{a}, \tilde{d}) \mapsto (a^{-1}d)^i d^r$. So long as $0 < r < p - 1 - 2e$, each of these characters arises from a unique Serre weight.

Observe also that N_i is a $\mathcal{Q}$-structured G-submodule of $\mathrm{ind}_{KZ}^G\sigma_0$ in the sense of Definition 2.7 for the $(i+1)$-tuple $\mathcal{Q} = (q_0, \dots, q_i)$ given by $q_0 = r$ and $q_j = p - r - 2j - 1$ for $j > 0$. Indeed, for $i = 0$ we already noted this above in Remark 2.8. If $i > 0$ this follows from the explicit expression of the map Ψ_i (see, for instance, equation (7) on page 266 of [BL]) and the observation that, for any $n \geq 0$ and any $\nu \in I_n$,

$$\begin{aligned} \Psi_i(g^0_{n,\nu} \otimes \hat{x}^{p-r-2i-1}) &= g^0_{n,\nu} \cdot \tilde{X}^0_i &= \sum_{\mu \in I_{i+1}} g^0_{n+i+1,\nu+\pi^n\mu} \otimes \mu_1 \cdots \mu_{i-1}\mu_i^{r+1} x^r, \\ \Psi_i(g^1_{n,\nu} \otimes \hat{y}^{p-r-2i-1}) &= g^1_{n,\nu} \cdot w\tilde{X}^0_i &= \sum_{\mu \in I_{i+1}} g^1_{n+i+1,\nu+\pi^n\mu} \otimes \mu_1 \cdots \mu_{i-1}\mu_i^{r+1} y^r. \end{aligned}$$

Let $0 \neq v \in V_\sigma^{I(1)}$ be a highest weight vector, and suppose that D acts on v via the character χ. By assumption there exists a non-zero map $\Phi \in \mathrm{Hom}_G(\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma), V_{e-1})$. By Frobenius reciprocity Φ corresponds to a non-zero map $\varphi \in \mathrm{Hom}_{KZ}(\sigma, (V_{e-1})_{|KZ})$. Then $\varphi(v) \in V_{e-1}^{I(1)}$. For the e-tuple $\mathcal{Q} = (q_0, \dots, q_{e-1})$ just defined, we see by Lemma 2.9 that the subgroup $D \subset K$ of diagonal matrices acts as follows:

$$\begin{array}{lll} \mathrm{diag}(\tilde{a}, \tilde{d})X^{0,+}_{n,l} = (ad^{-1})^{l-1}a^r X^{0,+}_{n,l} & \mathrm{diag}(\tilde{a}, \tilde{d})X^{1,+}_{n,l} = (a^{-1}d)^{l-1}d^r X^{0,+}_{n,l}, & 1 \leq l \leq e-1 \\ \mathrm{diag}(\tilde{a}, \tilde{d})X^{0,-}_{n,0} = (a^{-1}d)^2 d^r X^{0,-}_{n,0} & \mathrm{diag}(\tilde{a}, \tilde{d})X^{1,-}_{n,0} = (ad^{-1})^2 a^r X^{1,-}_{n,0} & \\ \mathrm{diag}(\tilde{a}, \tilde{d})X^{0,-}_{n,l} = (a^{-1}d)^{l+2} d^r X^{0,-}_{n,l} & \mathrm{diag}(\tilde{a}, \tilde{d})X^{1,-}_{n,l} = (ad^{-1})^{l+2} a^r X^{1,-}_{n,l}, & 1 \leq l \leq e-2 \\ \mathrm{diag}(\tilde{a}, \tilde{d})X^{0,-}_{n,e-1} = (a^{-1}d)^e d^r X^{0,-}_{n,e-1} & \mathrm{diag}(\tilde{a}, \tilde{d})X^{1,-}_{n,e-1} = (ad^{-1})^e a^r X^{1,-}_{n,e-1} & \\ \mathrm{diag}(\tilde{a}, \tilde{d})\tilde{X}^0_n = (a^{-1}d)^n d^r \tilde{X}^0_n & \mathrm{diag}(\tilde{a}, \tilde{d})\tilde{X}^1_n = (ad^{-1})^n a^r \tilde{X}^1_n & 1 \leq n \leq e-1 \end{array}$$

By Corollary 2.11, $\varphi(v)$ must be a linear combination of the elements $X^{j,+}_{n,l}$ for $j \in \{0, 1\}$ and $1 \leq l \leq e-1$ and $X^{j,-}_{n,l}$ for $j \in \{0, 1\}$ and $0 \leq l \leq e-1$, as well as $\tilde{X}^j_n$ if n is in the suitable range. Hence χ is one of the characters appearing in the list above, all of which are equal to χ_i or χ'_i for some $0 \leq i \leq e$. Therefore $\sigma \in \mathcal{D} \cup \{\sigma_e, \sigma'_e\}$. □

Lemma 3.6. *The element $\alpha \otimes y^r \in \mathrm{ind}_{KZ}^G\sigma_0/T(\mathrm{ind}_{KZ}^G\sigma_0)$ generates a KZ-submodule isomorphic to σ'_0.*

Proof. One shows by explicit computation, starting from Corollary 2.11, that if the image of $f \in B_1 \subset \mathrm{ind}_{KZ}^G\sigma_0$ is an $I(1)$-invariant in $\mathrm{ind}_{KZ}^G\sigma_0/T(\mathrm{ind}_{KZ}^G\sigma_0)$, then $f \in \overline{\mathbb{F}}_p\cdot(\mathrm{Id}\otimes x^r) \oplus \overline{\mathbb{F}}_p\cdot(\alpha\otimes y^r)$. The claim now follows by the argument of [Bre], Prop. 4.1.2. □

Proposition 3.7. *Suppose we are given a map of G-modules $\tau : V_0 = \mathrm{ind}_{KZ}^G\sigma_0/T(\mathrm{ind}_{KZ}^G\sigma_0) \to W$, where W satisfies $\mathrm{soc}_K(W) = \bigoplus_{\sigma\in\mathcal{D}}\sigma$ and has no non-supersingular subrepresentations. Then τ factors through V_{e-1}.*

Proof. We argue by induction. Let $1 \leq n \leq e-1$ and suppose it is known that τ factors through V_{n-1}. We claim it factors through V_n. Let X^0_n be the image in V_{n-1} of $\tilde{X}^0_n$. Note that the image of the map $\Psi_n : \mathrm{ind}_{KZ}^G\sigma'_n \to V_{n-1}$ is contained in (and in fact equal to) the submodule of V_{n-1} generated by the element X^0_n. Hence if $\tau(\tilde{X}^0_n) = 0$, then obviously τ factors through $V_{n-1}/\Psi_n(\mathrm{ind}_{KZ}^G\sigma'_n)$ and hence through V_n.

Now suppose $\tau(\tilde{X}^0_n) \neq 0$. By our assumption on the socle of W, the space $\mathrm{Hom}_G(\mathrm{ind}_{KZ}^G\sigma'_n, W) \simeq \mathrm{Hom}_{KZ}(\sigma'_n, W_{|KZ})$ is one-dimensional. Thus every non-zero element of this space, and in particular $\tau \circ \Psi_n$, is an eigenvector for the action of the commutative algebra $\mathrm{End}_G(\mathrm{ind}_{KZ}^G\sigma'_n)$. It follows that $\tau \circ \Psi_n$ factors through $\mathrm{ind}_{KZ}^G\sigma'_n/(T-\xi)(\mathrm{ind}_{KZ}^G\sigma'_n)$ for some scalar $\xi \in \overline{\mathbb{F}}_p$. But $\xi = 0$, since otherwise the classification of [BL] would imply that the image of $\tau \circ \Psi_n$ in W contains a non-supersingular representation. □

Recall the elements f_i defined in the introduction for $0 \leq i \leq e-1$.

Proposition 3.8. *Suppose that* $\tau : \mathrm{ind}_{KZ}^G \sigma_0 / T(\mathrm{ind}_{KZ}^G \sigma_0) \twoheadrightarrow W$ *is a quotient, that* W *has no non-supersingular subrepresentations, and that* $\mathrm{soc}_K(W) = \bigoplus_{\sigma \in \mathcal{D}} \sigma$. *If* $\tau(f_{e-1}) \neq 0$, *then for each* $0 \leq i \leq e-1$ *the* K*-submodule of* W *generated by* $\tau(f_i)$ *(resp.* $\tau(\beta f_i)$*) is irreducible and isomorphic to* σ_i *(resp.* σ_i'*). Moreover, if* $1 \leq i \leq e-1$ *and* $\tau(\tilde{X}_i^0) \neq 0$, *then there exists a non-zero scalar* $c_i \in \overline{\mathbb{F}}_p$ *such that* $\tau(f_i) = c_i \tau(\tilde{X}_i^1)$ *and* $\tau(\beta f_i) = c_i \tau(\tilde{X}_i^0)$.

Proof. Recalling that β acts as an involution on $\mathrm{ind}_{KZ}^G \sigma_0$, observe from the definitions that $f_0 = \tilde{X}_0^0$ and $f_1 = \tilde{X}_1^1$. Hence $\beta f_0 = \tilde{X}_0^1$ and $\beta f_1 = \tilde{X}_1^0$. Since these elements already generate irreducible K-submodules of V_{e-1} isomorphic to the specified Serre weights by Proposition 3.3 and Lemma 3.6, and since τ factors through V_{e-1} by Proposition 3.7, the claim is established for $i \in \{0,1\}$.

Now suppose that the claim is known for $i-1$. By Frobenius reciprocity the non-zero element $\tau(f_{i-1}) \in W$ defines a map $\Psi_{i-1} : \mathrm{ind}_{KZ}^G \sigma_{i-1} \to W$, which factors through $\mathrm{ind}_{KZ}^G \sigma_{i-1}/T(\mathrm{ind}_{KZ}^G \sigma_{i-1})$ as in the proof of the previous proposition. By the second part of Lemma 1.1, which follows from results that were proved earlier in this section, the element $h = \sum_{\mu \in I_2} g_{2,\mu}^0 \otimes \mu_1^{r+2i-1} \hat{x}^{r+2i-2} \in \mathrm{ind}_{KZ}^G \sigma_{i-1}/T(\mathrm{ind}_{KZ}^G \sigma_{i-1})$ generates an irreducible K-submodule isomorphic to σ_i'. Hence $\tau(\beta f_i) = \Psi_{i-1}(h) \in W$, which is non-zero by assumption, generates an irreducible K-submodule isomorphic to σ_i'. Moreover, $\tau(\beta f_i)$ is an $I(1)$-invariant since h is. Similarly, $\tau(\beta f_i) \in W$ determines a map $\Psi_i : \mathrm{ind}_{KZ}^G \sigma_i' / T(\mathrm{ind}_{KZ}^G \sigma_i') \to W$ with $\Psi_i(f_{\sigma_i'}) = \tau(\beta f_i)$. Then $\Psi_i(\beta f_{\sigma_i'}) = \tau(f_i)$ generates an irreducible K-submodule isomorphic to σ_i by Lemma 3.6. Since $\tilde{X}_i^0$ is an $I(1)$-invariant generating an irreducible K-submodule of V_{e-1} isomorphic to σ_i' if $1 \leq i \leq e-1$, and since the Serre weights in $\mathrm{soc}_K(W)$ appear with multiplicity one, $\tau(\tilde{X}_i^0)$ is necessarily a scalar multiple of $\tau(\beta f_i)$. $\square$

Remark 3.9. For each $1 \leq i \leq e-1$, define the elements

$$Z_i = \sum_{\lambda \in I_2} \lambda_1^{p-r-2i} g_{2,\lambda}^0 \tilde{X}_i^0 = \sum_{\lambda \in I_2} \sum_{\mu \in I_{i+1}} g_{i+3,\lambda+\pi^2\mu}^0 \otimes \lambda_1^{p-r-2i} \mu_1 \cdots \mu_{i-1} \mu_i^{r+1} x^r \in \mathrm{ind}_{KZ}^G \sigma_0.$$

Let $\tau : \mathrm{ind}_{KZ}^G \sigma_0 / T(\mathrm{ind}_{KZ}^G \sigma_0) \twoheadrightarrow W$ be such that W has no non-supersingular subrepresentations and that $\mathrm{soc}_K(W) = \bigoplus_{\sigma \in \mathcal{D}} \sigma$. Assume that $\tau(Z_i) \neq 0$ for all $1 \leq i \leq e-1$. Then $\tau(\tilde{X}_i^0) \neq 0$ for all $0 \leq i \leq e-1$, and one can show that W is irreducible by the same proof as that of Theorem 1.3, but with $\tilde{X}_i^1$ and Z_i playing the roles of f_i and z_i respectively.

To conclude, we observe that the construction of the quotient V_{e-1} is very natural. Indeed, we began with a weight σ and took $V_0 = \mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$. In each intermediate quotient V_{i-1}, we used Corollary 2.11 to compute enough information about the $I(1)$-invariants in V_{i-1} to find a unique (up to scalar multiplication) pair of minimal $I(1)$-invariants $\tilde{X}_i^0, \tilde{X}_i^1$ on which D acts via characters that have not appeared in $V_{i-2}^{I(1)}$. Here we denote $V_{-1} = \mathrm{ind}_{KZ}^G \sigma$, and by "minimal" we mean that among all $I(1)$-invariants with this property, $\tilde{X}_i^0$ and $\tilde{X}_i^1$ are supported closest to the origin of the Bruhat-Tits tree of G. One of the elements of this pair generates an irreducible KZ-module τ, which by Frobenius reciprocity gives a map $\Psi : \mathrm{ind}_{KZ}^G \tau \to V_{i-1}$. We defined $V_i = V_{i-1}/\Psi(T(\mathrm{ind}_{KZ}^G \tau))$ and repeated the process. Note that the quotient V_{e-1} is very large; it is non-admissible if $e > 1$. However, the main idea behind this paper is that Breuil's [Bre] original computational proof of irreducibility still applies for a totally ramified extension of $\mathbb{Q}_p$ if one works over V_{e-1} rather than over $V_0 = \mathrm{ind}_{KZ}^G \sigma_0 / T(\mathrm{ind}_{KZ}^G \sigma_0)$ itself.

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