WEIGHTS IN SERRE'S CONJECTURE FOR GL_n VIA THE BERNSTEIN-GELFAND-GELFAND COMPLEX

MICHAEL M. SCHEIN

Abstract. Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be an n-dimensional mod p Galois representation. If ρ is modular for a weight in a certain class, called p-minute, then we restrict the Fontaine-Laffaille numbers of ρ; in other words, we specify the possibilities for the restriction of ρ to inertia at p. Our result agrees with the Serre-type conjectures for GL_n formulated by Ash, Doud, Pollack, Sinnott, and Herzig; to our knowledge, this is the first unconditional evidence for these conjectures for arbitrary n.

1. Introduction

Let p be a prime and let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be a mod p Galois representation. There exists a notion of ρ being modular of a certain weight; see section 1.2. If $n = 2$, then Serre's well-known conjecture specified when ρ is modular, and, if so, of which weights. Ash, Doud, Pollack, and Sinnott [AS], [ADP] have extended this conjecture by specifying some modular weights of Galois representations of arbitrary dimension. Their work was conceptually reformulated and extended by Herzig [Her] if ρ is tamely ramified at p; in this case, he gives a set of weights that strictly contains theirs and conjectures that it is the complete set of modular weights of ρ. We note that these conjectures were formulated using a different generalization of the classical definition of modularity from the one we use, and it has not been proved that the two definitions are equivalent; see section 1.4. In fact, the compatibility of Theorem 1.2 with the conjectures may be seen as evidence for the equivalence of the notions of modularity.

A *Serre weight* is an isomorphism class of irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_n(\mathbb{F}_p)$. We recall a description of these. An n-tuple $\lambda = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ is called p-restricted if it satisfies the following conditions:

- $a_1 \geq a_2 \geq \cdots \geq a_n$
- $0 \leq a_i - a_{i+1} \leq p - 1$ for all $i = 1, \dots, n-1$

Let $B \subset \mathrm{GL}_n$ and $B^- \subset \mathrm{GL}_n$ be the Borel subgroups of upper and lower triangular matrices, respectively, and let $T \subset B$ be the diagonal torus. We view them as algebraic groups over $\overline{\mathbb{F}}_p$. Given a character $\lambda : T \to \mathbb{G}_m$, consider its restriction $\lambda : T(\overline{\mathbb{F}}_p) \to \overline{\mathbb{F}}_p^*$ to

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$\overline{\mathbb{F}}_p$-points, which necessarily has the following form for some n-tuple $(a_1, \ldots, a_n) \in \mathbb{Z}^n$:

$$\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \mapsto t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}.$$

We will abusively write λ for the n-tuple $(a_1, \ldots, a_n)$. We may consider λ as a character of B^- by means of the natural surjection $B^- \to T$. Let $\overline{\mathbb{F}}_p(\lambda)$ denote a one-dimensional $\overline{\mathbb{F}}_p$-vector space on which B^- acts via λ. Let $W(\lambda) = \mathrm{ind}_{B^-}^{\mathrm{GL}_n} \overline{\mathbb{F}}_p(\lambda)$ be the corresponding algebraic Weyl module. Here we are taking the "algebraic induction" as in [Jan], I.3.3; it consists of B^--invariant functions $\mathrm{GL}_n \to \overline{\mathbb{F}}_p(\lambda)$ that are morphisms of algebraic varieties. Then $W(\lambda)$ has a unique irreducible GL_n-submodule of highest weight λ, which we denote $F(\lambda)$ or $F(a_1, \ldots, a_n)$. Restricting to $\mathbb{F}_p$-points, the construction $\lambda \mapsto F(\lambda)$ gives a bijective correspondence between Serre weights and p-restricted n-tuples such that $0 \leq a_n < p - 1$.

Definition 1.1. Let $\lambda = (a_1, \ldots, a_n)$ be a p-restricted n-tuple.

(1) The weight $F(\lambda)$ is called *p-small* if $a_1 - a_n < p - n$.
(2) $F(\lambda)$ is called *p-minute* if it is p-small and $a_1 + a_2 + \cdots + a_{n-1} - (n-1)a_n < p - 1$.

The aim of this paper is to restrict the p-minute weights of which a Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ can be modular. We will recall below that, for any integer x, Fontaine and Laffaille constructed a fully faithful functor $\mathbf{T}_x^*$ from a certain category of filtered modules, where the jumps in the filtration lie between x and $x+p-2$, to that of local Galois representations. See [Bre] for an exposition of this theory. Let $G_p \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be a decomposition subgroup at p, and let $I_p' \subset I_p$ be the wild inertia and inertia subgroups inside G_p, respectively. If $\rho|_{G_p}$ is in the essential image of such a functor $\mathbf{T}_x^*$, then its *Fontaine-Laffaille numbers* are the jumps in the filtration of a module giving rise to it. Our result is the following:

Theorem 1.2. *Suppose that $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is modular of a p-minute weight $F(a_1, \ldots, a_n)$. Then $\rho|_{G_p}$ lies in the essential image of $\mathbf{T}_{a_n}^*$, and its Fontaine-Laffaille numbers are contained in the set $\{a_i + (n-i) : i = 1, 2, \ldots, n\}$.*

Remark 1.3. By Proposition 4.1, we can immediately restrict to the case when $a_n = 0$.

Observe that if λ is p-small, then the numbers $a_i + (n-i)$ for $1 \leq i \leq n$ are distinct and lie in the range $[a_n, a_n + p - 2]$.

From the definition of modularity in section 1.2 below, it will be clear that the proof of Theorem 1.2 comes down to a study of the étale cohomology of a certain Shimura variety X. We will use Faltings' comparison theorem to relate this to the crystalline cohomology of a crystal on the special fiber of X, and the crucial tool in the subsequent argument will be the integral Bernstein-Gelfand-Gelfand resolution of Polo and Tilouine [PT]. This method has been applied by Mokrane and Tilouine [MT] to analyze the cohomology of Siegel modular varieties, and by Dimitrov [Dim] to study that of Hilbert modular varieties. Our adaptation of it to Shimura varieties owes much to their work, as will be apparent to the reader.

We have also applied these methods to the study of mod p representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$, for a totally real field F. We will report on this in a future article.

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1.1. **Fontaine-Laffaille numbers.** Theorem 1.2 can be restated more explicitly as follows. The semisimplification $\rho|_{I_p}^{ss}$ of the restriction to I_p of any continuous representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\overline{\mathbb{F}}_p)$ factors through the tame inertia $I_p/I_p' \simeq \varprojlim \mathbb{F}_{p^n}^*$. Since I_p/I_p' is abelian, we see that $\rho|_{I_p}^{ss}$ is a sum of characters. Moreover, G_p/I_p acts by conjugation on the tame inertia, with the topological generator Frob_p acting by "raising to the p-th power." It follows that if a character appears in $\rho|_{I_p}^{ss}$, then so do all of its Galois conjugates. Therefore, there is a partition $n_1 + n_2 + \cdots n_r = n$ such that

$$\rho|_{I_p}^{ss} \sim \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_r \end{pmatrix},$$

where $B_l \in \mathrm{GL}_{n_l}(\overline{\mathbb{F}}_p)$ for each $1 \leq l \leq r$, and there exists a character $\phi_l : I_p/I_p' \to \overline{\mathbb{F}}_p^*$ of level n_l (i.e. ϕ_l factors through the quotient $\mathbb{F}_{p^{n_l}}^*$ and not through any smaller quotient) such that

$$B_l \sim \begin{pmatrix} \phi_l & & & \\ & \phi_l^p & & \\ & & \ddots & \\ & & & \phi_l^{p^{n_l-1}} \end{pmatrix}.$$

Now for each $1 \leq l \leq r$, choose a fundamental character $\psi_l : I_p/I_p' \to \overline{\mathbb{F}}_p^*$ of level n_l; this means that ψ_l is induced from an embedding of fields $\mathbb{F}_{p^{n_l}} \hookrightarrow \overline{\mathbb{F}}_p$. Then $\phi_l = \psi_l^{b_l}$ for some $0 \leq b_l \leq p^{n_l} - 2$. We can write b_l in base p as

$$b_l = \sum_{j=0}^{n_l-1} b_{lj} p^j,$$

where $0 \leq b_{lj} \leq p-1$. Note that the set $\{b_{lj} : 0 \leq j \leq n_l - 1\}$ is independent of the choice of ψ_l.

If $\rho|_{G_p}$ is in the image of the Fontaine-Laffaille functor $\mathbf{T}_0^*$, then the collection of n integers $\{b_{lj} : 1 \leq l \leq r, 0 \leq j \leq n_l - 1\}$ is its set of Fontaine-Laffaille numbers, so that, in view of Proposition 4.1, we see that Theorem 1.2 is equivalent to the following:

Theorem 1.2′. *Suppose that $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is modular of a p-minute weight $F(a_1, \ldots, a_n)$, with $a_n = 0$. Then there exists a partition $n = n_1 + \cdots + n_r$ such that $\rho|_{I_p}$ can be written as above, where $b_{lj} \in \{a_i + (n-i) : 1 \leq i \leq n\}$ for all $1 \leq l \leq r$ and $0 \leq j \leq n_l - 1$.*

To make sense of Theorem 1.2 we must define what it means for ρ to be modular of a weight $\lambda = F(a_1, \ldots, a_n)$.

1.2. **Modularity.** Let $E/\mathbb{Q}$ be an imaginary quadratic extension in which p splits, and write v and $\overline{v}$ for the places of E dividing p. Choose a division algebra D of degree n^2 over E as in [HT1] I.7; in particular, D is split at v. As there, we use D to define a reductive group $\Gamma/\mathbb{Q}$ (in [HT1] it is denoted G) such that $\Gamma(\mathbb{Q}_p) \simeq \mathbb{Q}_p^* \times \mathrm{GL}_n(E_v)$. If $U \subset \Gamma(\mathbb{A}^\infty)$ is an open compact subgroup, where $\mathbb{A}^\infty$ denotes the finite adèles over $\mathbb{Q}$, then we obtain a compact Shimura variety X_U/E of dimension $n-1$ as in [HT1] III.1. Let $\mathcal{O}_v$ be the ring of integers of E_v; its residue field is $k = \mathbb{F}_p$. Set $\mathrm{GL}_n^1(\mathcal{O}_v) = \ker(\mathrm{GL}_n(\mathcal{O}_v) \to \mathrm{GL}_n(k))$, and let $U^p \subset \Gamma(\mathbb{A}^{\infty,p})$. Now define

$$\begin{aligned} U &= \mathbb{Q}_p^* \times \mathrm{GL}_n(\mathcal{O}_v) \times U^p \\ U_1 &= \mathbb{Q}_p^* \times \mathrm{GL}_n^1(\mathcal{O}_v) \times U^p. \end{aligned}$$

If U^p is sufficiently small in the sense of [HT1] III.1, then Kottwitz [Kot] proved that the natural map $X_{U_1} \to X_U$ is a Galois cover with Galois group $U/U_1 \simeq \mathrm{GL}_n(k)$. Moreover, these schemes have integral models over $\mathcal{O}_v$, which are described in [HT1] III.4. We denote the integral models of X_U and X_{U_1} by X and X^1, respectively.

Let λ be an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_n(k)$, and let V_λ be its underlying $\overline{\mathbb{F}}_p$-vector space. Then we can define a mod p étale sheaf $\mathcal{F}_\lambda$ on X_U in the usual manner. Namely, if $Y \to X_U$ is an étale cover, then the sections above it are locally constant functions $f : X_{U_1} \times_{X_U} Y \to V_\lambda$ such that for all $\gamma \in \mathrm{GL}_n(k)$ and any connected component $C \in \pi_0(X_{U_1} \times_{X_U} Y)$ we have $f(C\gamma) = \gamma^{-1} f(C)$. Similarly we can define a sheaf $\mathcal{F}_\lambda$ on the integral model X. The étale cohomology $H^*(X_U \otimes \overline{E}, \mathcal{F}_\lambda)$ carries an action of $\mathrm{Gal}(\overline{E}/E)$.

Definition 1.4. Given $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, let λ be a weight. We say that ρ *is modular of weight* λ if there exist a quadratic imaginary extension $E/\mathbb{Q}$, a division algebra D/E, and a reductive group $\Gamma/\mathbb{Q}$ as above, such that $\rho|_{\mathrm{Gal}(\overline{E}/E)}$ appears in $H^*(X_U \otimes \overline{E}, \mathcal{F}_\lambda)$.

Remark 1.5. The reader should not be disturbed that this definition only involves the restriction of ρ to the index two subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}/E) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The conjectures of Ash et al. and Herzig, as well as Serre-type conjectures in other settings and the Langlands philosophy in general, suggest that the collection of modular weights of ρ should only depend on local information at p, in other words on the restriction of ρ to a decomposition subgroup G_p

at p. However, $G_p \simeq \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) = \mathrm{Gal}(\overline{E}_v/E_v)$, so we do not lose any local information by restricting to $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$.

In the rest of this paper we will compute the Fontaine-Laffaille numbers of $H^*(X_U \otimes \overline{E}, \mathcal{F}_\lambda)$ for any p-minute weight λ. We note that if ξ is an irreducible *p-adic* representation of $\mathrm{GL}_n(k)$, then a p-adic étale sheaf $\mathcal{F}_\xi$ on X_U can be defined similarly, and the Hodge-Tate numbers of $H^*(X_U \otimes \overline{E}, \mathcal{F}_\xi)$ are computed on pp. 99-104 of [HT1]. The argument there follows the same lines as ours, but is somewhat easier. It does not imply our results, however; if $\bar{\lambda}$ is a $\mathrm{GL}_n(k)$-action on a $\mathbb{Z}_p$-lattice lifting λ, the cohomology $H^*(X_U \otimes \overline{E}, \mathcal{F}_{\bar{\lambda}})$ may have torsion.

1.3. **Notation.** We now establish notation that will accompany us for the rest of the paper. Let G be the algebraic group GL_n, and let B and T be the upper triangular Borel subgroup and the diagonal torus, respectively. Let $P \subset G$ be the standard parabolic subgroup with blocks of size $n-1$ and 1. In other words,

$$P = \left\{ \begin{pmatrix} A_{n-1} & * \\ 0 & A_1 \end{pmatrix} : \begin{matrix} A_{n-1} \in \mathrm{GL}_{n-1} \\ A_1 \in \mathrm{GL}_1 \end{matrix} \right\} \subset \mathrm{GL}_n.$$

Let N_P be the unipotent radical of P; this is the subgroup of the same form as P, where A_{n-1} and A_1 are replaced with identity matrices of the appropriate size. Let P^- be the opposite parabolic, and let $L = P \cap P^- = \mathrm{GL}_{n-1} \times \mathrm{GL}_1$ be the corresponding Levi subgroup. Denote by N and N_P^- the unipotent radicals of B and P^-, respectively.

As usual, the Lie algebra of each group is denoted by the appropriate lower-case Fraktur letter. For instance, $\mathfrak{g}$ and $\mathfrak{n}_P^-$ are the Lie algebras of G and N_P^-, respectively.

1.4. **Comparison of our result with Herzig's conjecture.** Our motivation for proving Theorem 1.2 was to make progress towards the conjectures of Ash-Doud-Pollack-Sinnott and of Herzig. We remark that while in general Herzig gave a larger list of modular weights than Ash et al., in the case of p-small weights the conjectures coincide.

We briefly compare Theorem 1.2 with the conjecture, in Herzig's formulation. Let us say that a p-small weight $F(\lambda)$, where $\lambda = (a_1, a_2, \ldots, a_n)$, *has property* $(*)$ if the Galois representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ which are tamely ramified at p and are predicted by Herzig to be modular of this weight are precisely those with Fontaine-Laffaille numbers $\{a_i + n - i : 1 \le i \le n\}$. We will see below that p-small weights generically have property $(*)$, in the sense that the proportion of p-small weights with property $(*)$ tends to 1 as p increases.

Note that Ash et al. and Herzig use a group-cohomological definition of the notion of ρ being modular of a weight λ; see [AS], Conjecture 2.1 and [Her], Definition 6.3. For an integer N, let $\Gamma_0(N) \subset \mathrm{SL}_n(\mathbb{Z})$ be the subgroup of matrices whose top row has the form $(*, 0, \ldots, 0)$ modulo N. If p divides N, then a weight $F(\lambda)$, which is a $\mathrm{GL}_n(\mathbb{F}_p)$-module by definition, can be viewed as a $\Gamma_0(N)$-module via reduction modulo p of matrices. Then ρ is said to be modular of weight $F(\lambda)$ if it is attached, in the sense of [AS], Definition 1.1, to a class $\beta \in H^*(\Gamma_0(N'), V)$ for a suitable N', where the $\Gamma_0(N')$-module V is isomorphic to $F(\lambda)$ up to a twist. The cohomology class β is furthermore required to be an eigenvector for the Hecke action defined in [AS]. If $n = 2$, then such classes correspond to Hecke modular eigenforms, and ρ is attached to an eigenform if it arises from it by the Eichler-Shimura construction. In this case it is well-known that ρ appears in the cohomology of an appropriate modular curve; this formulation of modularity motivates Definition 1.4.

We believe that Definition 1.4 is equivalent to the notion of modularity of [AS] and that the proof for modular curves can be modified to our case, but we have not established this. Finally, note that Theorem 1.2 is actually a statement about automorphic representations for the reductive group $\Gamma/\mathbb{Q}$ of section 1.2, and that $\Gamma(\mathbb{Q}_p) \not\simeq \mathrm{GL}_n(\mathbb{Q}_p)$. Yet for the rest of this section, we suppose that the two notions of modularity are equivalent; we will then see that Theorem 1.2 is compatible with the conjectures of Ash et al. and Herzig.

We will prove in section 4 that the Fontaine-Laffaille numbers of $H = H^*(X_U \otimes \overline{E}, \mathcal{F}_\lambda)$, and hence of any irreducible subquotient, lie in the set $\{a_i + (n - i) : 1 \leq i \leq n\}$; this is Theorem 1.2. If one could show, for a p-minute weight $F(\lambda)$ with property $(*)$, that all the elements of this set appear as Fontaine-Laffaille numbers of a given subquotient ρ of H, then we would show that $F(\lambda)$ is a modular weight of a Galois representation ρ only if Herzig predicts it to be a modular weight.

Observe that the hypotheses of Theorem 1.2 contain no restriction on the ramification of ρ at p. Herzig's conjecture depends only on $\rho|_{I_p}^{ss}$, and it is expected that if ρ is not tamely ramified at p, then its modular weights will be a subset of those that Herzig predicts. Theorem 1.2 conforms to this expectation.

Proposition 1.6 (Herzig). *Let $F = F(a_1, a_2, \ldots, a_n)$ be a p-small weight.*

1. *If $n = 3$, then F has property $(*)$.*
2. *For all $n \geq 3$ there exists a constant δ_n, independent of p, such that if $\delta_n < a_1 - a_n + n - 1 < p - \delta_n$, then F has property $(*)$.*

Proof. Recall that Herzig in [Her], 6.4 defines a p-adic representation $V(\rho)$ of $\mathrm{GL}_n(\mathbb{F}_p)$ for each Galois representation ρ that is tamely ramified at p. Let $JH(\overline{V(\rho)})$ be the set of Jordan-Hölder constituents of its reduction modulo p. The elements of this set are irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$, i.e. weights. Herzig defined a map $\mathcal{R}$ from the set of weights to itself and conjectured that the regular modular weights of ρ are $\mathcal{R}(JH(\overline{V(\rho)}))$. See [Her] for the definition of regular weights; all p-small weights are regular.

If $F(a_1, \ldots, a_n)$ is p-small, then its preimage under $\mathcal{R}$ consists of the single weight

$$F(a_n+(n-1)(p-1), a_{n-1}+1+(n-2)(p-1), \ldots, a_{n+1-i}+i-1+(n-i)(p-1), \ldots, a_1+n-1).$$

Since this weight lies in the highest alcove of the p-restricted region, the only Weyl module in which it appears as a Jordan-Hölder constituent is

$$W(a_n+(n-1)(p-1), a_{n-1}+1+(n-2)(p-1), \ldots, a_{n+1-i}+i-1+(n-i)(p-1), \ldots, a_1+n-1).$$

If $n = 3$, then it is easy to see from Lemma 7.6 of [Her] that $\overline{V(\rho)}$ contains this Weyl module if and only if ρ has the Fontaine-Laffaille numbers claimed. This proves claim (1). Claim (2) is precisely [Her], Proposition 6.25(b). □

2. Crystals

2.1. **Definitions.** We emphasize that the second and third sections of this paper closely follow the method established by Mokrane and Tilouine in [MT].

Let k be a perfect field of characteristic p, $W = W(k)$ its ring of Witt vectors, and let R be a W-algebra. Its p-adic completion $\hat{R}$ has an endomorphism $\Phi : \hat{R} \to \hat{R}$ that lifts the Frobenius endomorphism of R/pR. Let $a \geq 0$ be an integer. We recall the definition of the category $\mathcal{MF}^{\nabla,R}_{[0,a]}$ from section 2 of [Fal]. Its objects are quadruples (M, M^i, ϕ^i, ∇), where:

(1) M is a finitely generated p-torsion R-module equipped with a descending filtration by finitely generated R-modules
$$M = M^0 \supset M^1 \supset \cdots \supset M^{a+1} = 0.$$
(2) The ϕ^i are R-linear maps $\phi^i : M^i \otimes_{R,\Phi} R \to M$ such that $\phi^{i-1}|_{M^i} = p\phi^i$.
(3) The integrable connection $\nabla : M \to M \otimes_R \Omega_{R/W}$ satisfies
   (a) $\nabla(M^i) \subset M^{i-1} \otimes_R \Omega_{R/W}$ (Griffiths transversality).
   (b) $\nabla \circ \phi^i = (\phi^{i-1} \otimes_R d\Phi_*/p) \circ \nabla$ as maps from $M^i \otimes_{R,\Phi} R$ to $M \otimes_R \Omega_{R/W}$, i.e. the ϕ^i are parallel with respect to ∇.

Morphisms in $\mathcal{MF}^{\nabla,R}_{[0,a]}$ are R-module homomorphisms compatible with the additional structure. In practice we will have $k = \mathbb{F}_p$ and $W = \mathbb{Z}_p$, and R will be a ring of sections $\mathcal{O}_{X_0}(U)$ of the structure sheaf of the special fiber $X_0 = X \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. We will then suppress R from the notation and view a crystal $M \in \mathcal{MF}^{\nabla}_{[0,a]}$ as a quasi-coherent sheaf on X_0.

2.2. **The crystal associated to λ.** Let $\lambda = (a_1, a_2, \ldots, a_n)$ be a p-minute weight with $a_n = 0$. The aim of this section is to construct an $\mathcal{O}_{X_0}$-module $\mathcal{V}_\lambda$ associated to λ whose dual $\mathcal{V}_\lambda^\vee$ will be an object of $\mathcal{MF}^{\nabla}_{[0,a_1]}$. For the most part we follow section 5 of [MT], but our life is easier because X is compact. First, let (r, V) be a $\mathbb{Z}_p$-representation of G. For example, it could be the irreducible $\mathbb{Z}_p$-representation whose highest weight is the dominant weight $\lambda = (a_1, \ldots, a_n)$; we will abusively refer to its representation space as V_λ. As in [HT1] III.4, X represents a certain moduli problem of abelian varieties. Let $\pi : A \to X$ be the universal abelian variety, let $\Omega^\bullet_{A/X}$ be its complex of relative differentials, and consider the sheaf $\mathcal{A} = (R^1\pi_*\Omega^\bullet_{A/X})^\vee$. It is canonically isomorphic to Lie A and is equipped with a Gauss-Manin connection $\nabla : \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{O}_X} \Omega_X$. We have a decomposition Lie $A =$ Lie $^+A \oplus$ Lie ^-A, and the moduli problem defining X forces Lie ^+A to have rank n over $\mathcal{O}_X$. Let $\mathcal{A}^+$ be the corresponding direct summand of $\mathcal{A}$. Note that ∇ restricts to a connection $\nabla : \mathcal{A}^+ \to \mathcal{A}^+ \otimes_{\mathcal{O}_X} \Omega_X$.

Set $\mathcal{T} = \underline{Isom}(\mathcal{O}_X^{\oplus n}, \mathcal{A}^+)$. Since G acts on the left on $\mathcal{O}_X^{\oplus n}$ in the obvious way, it acts on the right on $\mathcal{T}$. Let $\mathcal{V} = \mathcal{T} \overset{G}{\times} V$ be the amalgamated product; this is $\mathcal{T} \times V$ modulo the equivalence relation $(t, gv) = (tg, v)$.

We define an integrable connection on $\mathcal{V}$ as follows. For any scheme $Y \to X$, let $\psi \in \mathcal{T}(Y)$. Then we have a diagram

$$\begin{array}{ccc} \mathcal{O}_Y^{\oplus n} & \xrightarrow{\nabla_\psi} & \mathcal{O}_Y^{\oplus n} \otimes \Omega_X^1 \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi \otimes 1} \\ \mathcal{A}_Y^+ & \xrightarrow{\nabla} & \mathcal{A}_Y^+ \otimes \Omega_X^1 \end{array}$$

defining a map $\nabla_\psi \in \mathrm{End}_{\mathcal{O}_Y}(\mathcal{O}_Y^{\oplus n}) \otimes_{\mathcal{O}_X} \Omega_X$. Moreover, ∇_ψ is contained in $\mathfrak{g} \otimes \Omega_X$. The differential of r gives a map $dr : \mathfrak{g} \to \mathrm{End}(V)$, whence we can define a connection on $V \otimes \mathcal{O}_Y$:

$$\nabla_{V,\psi} = (dr \otimes 1_{\mathcal{O}_Y} \otimes_{\mathcal{O}_X} 1_{\Omega_X}) \circ \nabla_\psi \in \mathrm{End}(V) \otimes \mathcal{O}_Y \otimes_{\mathcal{O}_X} \Omega_X.$$

Now set $Y = \mathcal{T}$, and let $\psi \in \mathcal{T}(Y)$ be the point corresponding to the identity map. Then ∇_ψ is a connection on $\mathcal{T} \times V$. If $g \in G$, it is easy to see that $\nabla_{\psi g} = (g^{-1} \otimes 1) \circ \nabla_\psi \circ g$. Hence ∇_ψ descends to a connection $\nabla_{\mathcal{V}}$ on the amalgamated product $\mathcal{V}$, and it retains the integrability and quasi-nilpotence of the original Gauss-Manin connection ∇.

Let st denote the standard n-dimensional representation of G. Then we see that $\mathcal{V}_{\mathrm{st}} = (R^1\pi_*\Omega^\bullet_{A/X})^\vee$, via the evaluation map on the amalgamated product. Define $s = a_1 + a_2 + \cdots + a_n$, and observe that the s-fold tensor product $V_{\mathrm{st}}^{\otimes s}$ contains an irreducible G-module of highest weight λ as a direct summand.

Similarly, let A^s be the s-fold fiber product $A \times_X A \times_X \cdots \times_X A \xrightarrow{\pi_s} X$. We can see as in Appendix II of [MT] that $\mathcal{V}_\lambda^\vee$ is a direct summand of the coherent sheaf $\mathcal{G} = R^s\pi_{s,*}\Omega^\bullet_{A^s/X}$. Furthermore, the assumption that λ is p-minute implies that $s < p-1$. Therefore $\mathcal{G}$ is locally free by [Ill], Corollaire 2.4.

If $C^\bullet$ is a complex and $j \in \mathbb{Z}$, let $(C^{\geq j})^\bullet$ denote the subcomplex such that

$$(C^{\geq j})^i = \begin{cases} C^i & : i \geq j \\ 0 & : i < j. \end{cases}$$

The Hodge filtration on $\mathcal{G}$ can now be defined by setting

$$\mathrm{Fil}^j\mathcal{G} = \mathrm{im}\left(R^s\pi_{s,*}(\Omega^{\geq j}_{A^s/X})^\bullet \to \mathcal{G}\right).$$

Finally, we define the filtration on $\mathcal{V}_\lambda^\vee$ by $\mathrm{Fil}^j\mathcal{V}_\lambda^\vee = \mathcal{V}_\lambda^\vee \cap \mathrm{Fil}^j\mathcal{G}$. The only reason we require λ to be p-minute, rather than just p-small, is that this hypothesis is needed to define the Hodge filtration on $\mathcal{V}_\lambda^\vee$. We have thus obtained an $\mathcal{O}_X$-module with filtration and connection. We pull it back to a sheaf on the special fiber X_0, which we continue to denote $\mathcal{V}_\lambda^\vee$.

2.3. **The H-filtration.** We would like to find a simpler way to compute the Hodge filtration. Let $H \in \mathfrak{p}$ be the element

$$H = \begin{pmatrix} 0 & & & 0 \\ & \ddots & & \\ & & 0 & \\ 0 & & & -1 \end{pmatrix},$$

i.e. the matrix with -1 in the bottom right corner and zeroes everywhere else. If $P \to \mathrm{GL}(W)$ is any $\mathbb{Z}_p$-representation of P, then by differentiating it we can realize H as an endomorphism of W. Define a decreasing filtration on W by setting W^i, for any integer i, to be the sum of the generalized H-eigenspaces with eigenvalues at least i.

We will now produce a filtered vector bundle on X associated to W, following section 5.3.2 of [MT]. We put the standard H-filtration on $\mathcal{O}_X^{\oplus n} = \mathcal{O}_X \otimes V_{st}$. Here $(\mathcal{O}_X^n)^1 = 0$ and $(\mathcal{O}_X^n)^{-1} = (\mathcal{O}_X^n)$, while $(\mathcal{O}_X^n)^0$ is a subsheaf of rank $n-1$. Then define $\mathcal{T}_H = \underline{Isom}^{fil}(\mathcal{O}_X^n, \mathcal{A}^+)$ to be the sheaf of isomorphisms compatible with the filtrations, where the filtration on the right-hand side is the Hodge filtration defined above. Now let

$$\mathcal{W} = \mathcal{T} \overset{P}{\times} W$$

be the amalgamated product as before. This is a vector bundle on X. We will denote its pull-back to the special fiber X_0 by $\mathcal{F}_P(W)$.

If the elements of the H-filtration on W are P-invariant subspaces, then they induce an H-filtration on $\mathcal{F}_P(W)$. Since H is not in the center of $\mathfrak{p}$, it is not true in general that the H-filtration is P-invariant. However, if $W = V_{\mathrm{st}}$ is the standard representation, then the filtration has only two steps, as above, and is manifestly P-invariant. The resulting H-filtration on $\mathcal{F}_P(V_{\mathrm{st}})$ induces one on $\mathcal{F}_P(V_\lambda)$.

Given an H-filtration on $\mathcal{F}_P(W)$, we define the dual H-filtration on $\mathcal{F}_P(W)^\vee$ as follows. Endow $\mathcal{O}_X$ with the trivial filtration: $\mathrm{Fil}^0(\mathcal{O}_X) = \mathcal{O}_X$ and $\mathrm{Fil}^j(\mathcal{O}_X) = 0$ for $j > 0$. Then for all i we set

$$\mathrm{Fil}_H^j(\mathcal{F}_P(W)^\vee) = \left\{\phi : \mathcal{F}_P(W) \to \mathcal{O}_X : \phi(\mathrm{Fil}_H^i \mathcal{F}_P(W)) \subset \mathrm{Fil}^{i+j}\mathcal{O}_X\right\}.$$

Lemma 2.1. *If V is the restriction to P of an irreducible G-representation of highest weight λ, then the H-filtration on $\mathcal{V}_\lambda^\vee$ coincides with the Hodge filtration.*

Proof. Recall that $V = V_{\mathrm{st}}$ is the standard representation of G. Its H-filtration is the following: $V_{\mathrm{st}}^1 = 0$, whereas V_{st}^0 is the P-invariant $(n-1)$-dimensional subspace and $V_{\mathrm{st}}^{-1} = V_{\mathrm{st}}$. Hence the induced H-filtration on $\mathcal{V}_{\mathrm{st}}^\vee = \mathcal{F}_P(V_{\mathrm{st}})^\vee$ is given by $(\mathcal{V}_{\mathrm{st}}^\vee)^0 = \mathcal{V}_{\mathrm{st}}^\vee$, $(\mathcal{V}_{\mathrm{st}}^\vee)^2 = 0$, and $(\mathcal{V}_{\mathrm{st}}^\vee)^1$ consists of the functions that annihilate $(\mathcal{V}_{\mathrm{st}})^0$. This can be seen to coincide with $\pi_*\Omega^1_{A/X}$. Therefore the H-filtration on $\mathcal{V}_{\mathrm{st}}^\vee$ is identical to the classical Hodge filtration.

Both filtrations are compatible with tensor products, so they coincide on $R^s\pi_{s,*}\Omega^\bullet_{A^s/X}$ and hence on the direct summand $\mathcal{V}_\lambda^\vee$. □

Lemma 2.2. *If $\lambda = (a_1, \dots a_n)$ is a p-minute weight with $a_n = 0$, then the sheaf $\mathcal{V}_\lambda^\vee$, equipped with the connection and filtration defined above, is an object of the category $\mathcal{MF}^\nabla_{[0,a_1]}$.*

Proof. The module V_λ has a basis consisting of weight vectors. A weight vector of weight $\mu = (m_1, \dots, m_n)$ is clearly an eigenvector for the H-action with eigenvalue $\mu(H) = -m_n$. Since λ is a dominant weight, we have $-a_1 \leq \mu(H) \leq 0$ for all weights μ appearing in V_λ. Hence the jumps of the filtration on $\mathcal{V}_\lambda^\vee$ are in the required range.

The maps ϕ^j are then the obvious ones, and it can be checked as in [MT] that the axioms defining $\mathcal{MF}^\nabla_{[0,a_1]}$ are satisfied. □

In section 2 of [Fal], Faltings constructs a functor $\mathbf{D}$ from crystals in $\mathcal{MF}^\nabla_{[0,p-2]}$ to étale sheaves on $X \otimes E_v$. As in [FC], Theorem VI.6.2(iii), we have

Lemma 2.3. *The crystal $\mathcal{V}_\lambda^\vee$ constructed above is associated to the étale sheaf $\mathcal{F}_\lambda$, in the sense that $\mathbf{D}(\mathcal{V}_\lambda^\vee) = \mathcal{F}_\lambda$.*

3. The Bernstein-Gelfand-Gelfand complex

3.1. **Weyl groups.** Let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be the standard basis of $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$, and consider the root system $R = \{\varepsilon_i - \varepsilon_j : 1 \leq i, j \leq n\}$ of G. The positive roots are $R^+ = \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq n\}$, and the simple roots are $\{\varepsilon_i - \varepsilon_{i+1} : 1 \leq i \leq n-1\}$. The Weyl group W of G is naturally identified with the symmetric group S_n; the reflection $s_{\varepsilon_i - \varepsilon_{i+1}}$ corresponds to the transposition $(i\ i+1)$. We view W as a Coxeter group in the generators $\{(i\ i+1) : 1 \leq i \leq n-1\}$, which allows us to define the length $l(w)$ of $w \in W$ as the length of the shortest expression of w as a word in these generators.

Recall that $L = \mathrm{GL}_{n-1} \times \mathrm{GL}_1$ is the Levi subgroup of the parabolic P. Its root system is $R_L = \{\varepsilon_i - \varepsilon_j : 1 \leq i, j \leq n-1\}$. Let $R_L^+ = R^+ \cap R_L$.

Our basis of $X(T)$ provides a natural identification of $X(T)$ with n-tuples $(a_1, \dots, a_n) \in \mathbb{Z}^n$. Let X^+ (resp. X_L^+) be the set of $\lambda \in X(T)$ such that $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in R^+$ (resp. for all $\alpha \in R_L^+$). Then,

$$\begin{aligned} X^+ &= \{(a_1, \dots, a_n) : a_1 \geq a_2 \geq \cdots \geq a_n\} \\ X_L^+ &= \{(a_1, \dots, a_n) : a_1 \geq a_2 \geq \cdots \geq a_{n-1}\}. \end{aligned}$$

The Weyl group W acts on $X(T)$ by permuting the components:

$$w(a_1, \dots, a_n) = (a_{w^{-1}(1)}, \dots, a_{w^{-1}(n)}).$$

If $\rho = (n-1, n-2, \dots, 1, 0) \in X^+$, then we define the usual dot Weyl action:

$$\begin{aligned} w \cdot \lambda &= w(\lambda + \rho) - \rho \\ w \cdot (a_1, \dots, a_n) &= (a_{w^{-1}(1)} + n - w^{-1}(1) - (n-1), \dots, a_{w^{-1}(n)} + n - w^{-1}(n)). \end{aligned}$$

Finally, define $W_L = \left\{ w \in W : w(X^+) \subset X_L^+ \right\}$, and set $W_L(i) = \{w \in W_L : l(w) = i\}$. Clearly W_L consists of the cycles $c_i = (i\ n\ n-1\ \cdots\ i+1)$ for $1 \leq i \leq n$, and $l(c_i) = n - i$.

3.2. **The BGG complex for distribution algebras.** In this section we present the BGG complex relevant to our problem. The necessary results were proved in greater generality by Polo and Tilouine in [PT]; here we present their work in the special case we need.

Consider the Koszul complex

$$\cdots \to U_{\mathbb{Z}_p}(\mathfrak{g}) \otimes_{\mathbb{Z}_p} \bigwedge_{\mathbb{Z}_p}^{2} \mathfrak{g} \to U_{\mathbb{Z}_p}(\mathfrak{g}) \otimes_{\mathbb{Z}_p} \mathfrak{g} \to \mathbb{Z}_p \to 0.$$

Since $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}_P^-$, we see that the $\mathbb{Z}_p[\mathfrak{p}]$-module $\mathfrak{g}/\mathfrak{p}$ is a direct factor in $\mathfrak{g}$. If V is any $\mathfrak{g}$-module, we obtain the following resolution of it by $U(\mathfrak{g})$-modules:

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \left(\bigwedge_{\mathbb{Z}_p}^{\bullet} (\mathfrak{g}/\mathfrak{p}) \otimes V|_{\mathfrak{p}} \right) \to V \to 0.$$

If $V = V_\lambda$, we denote this complex by $S_\bullet^{\mathbb{Z}_p}(\mathfrak{g}, \mathfrak{p}, \lambda)$.

Now let $\mathrm{Dist}(G)$ and $\mathrm{Dist}(P)$ be the distribution $\mathbb{Z}_p$-algebras over G and P, respectively. Analogously to the above we have a complex

$$\mathrm{Dist}(G) \otimes_{\mathrm{Dist}(P)} \left(\bigwedge_{\mathbb{Z}_p}^{\bullet} (\mathfrak{g}/\mathfrak{p}) \otimes V|_{\mathfrak{p}} \right) \to V \to 0,$$

which we denote by $S_\bullet^{\mathbb{Z}_p}(G, P, \lambda)$ in the event that $V = V_\lambda$. If $\lambda \in X_L^+$, then we define the generalized Verma module

$$M_P^{\mathbb{Z}_p}(\lambda) = \mathrm{Dist}(G) \otimes_{\mathrm{Dist}(P)} V_L^{\mathbb{Z}_p}(\lambda),$$

where $V_L^{\mathbb{Z}_p}(\lambda)$ is the Weyl module of L over $\mathbb{Z}_p$ with highest weight λ. Finally observe that $\mathfrak{n}_P^-$ is abelian; hence, N_P acts trivially on $\mathfrak{g}/\mathfrak{p}$. The hypotheses of Theorem D of [PT] are therefore satisfied, and we obtain:

Proposition 3.1 (Polo-Tilouine)**.** *Let $\lambda = (a_1, \ldots, a_n)$ be a p-small weight with $a_n = 0$. Then the standard complex $S_\bullet^{\mathbb{Z}_p}(G, P, \lambda)$ contains the complex $C_\bullet^{\mathbb{Z}_p}(G, P, \lambda)$ as a direct summand, where for each $i \geq 0$,*

$$C_i^{\mathbb{Z}_p}(G, P, \lambda) \simeq \bigoplus_{w \in W_L(i)} M_P^{\mathbb{Z}_p}(w \cdot \lambda).$$

3.3. **The BGG complex for crystals.** In this section we will derive a statement about the cohomology of crystals from Proposition 3.1. We follow [MT]; see also [Dim] §5.3.

Recall that the Shimura variety X is defined over $\mathcal{O}_v = \mathbb{Z}_p$. For every $r \geq 0$, set $S_r = \mathrm{Spec}\ (\mathbb{Z}_p/p^{r+1})$ and $X_r = X \times S_r$. The category $\mathcal{C}_r$ of crystals over $(X_0/S_r)^{cris}$ is equivalent to that of locally free $\mathcal{O}_{X_r}$-modules endowed with an integrable quasi-nilpotent connection. Let $\mathcal{L}$ be Grothendieck's linearization functor from the category of sheaves of $\mathcal{O}_{X_r}$-modules to $\mathcal{C}_r$. We emphasize that $\mathcal{L}$ is a covariant functor. If $\mathcal{M}$ is an $\mathcal{O}_{X_r}$-module, then by the crystalline Poincaré lemma (see, for instance, [BO] 6.13) it has an exact resolution

$$0 \to \mathcal{M} \to \mathcal{L}(\mathcal{M} \otimes_{\mathcal{O}_{X_r}} \Omega^\bullet_{X_r/S_r}).$$

Lemma 3.2. *Let λ be a p-small weight with $a_n = 0$, and denote $\mathcal{M}_\mu = \mathcal{F}_P(V_L^{\mathbb{Z}_p}(\mu))$ for any weight μ. The crystal $\mathcal{V}_\lambda^\vee$ has the following resolution in the category $\mathcal{C}_r$, for $r \geq 0$:*

$$0 \to \mathcal{V}_\lambda^\vee \to \mathcal{L}(\mathcal{K}_\lambda^0) \to \mathcal{L}(\mathcal{K}_\lambda^1) \to \mathcal{L}(\mathcal{K}_\lambda^2) \to \cdots$$

where

$$\mathcal{K}_\lambda^i = \bigoplus_{w \in W_L(i)} \mathcal{M}_{w\cdot\lambda}^\vee.$$

Proof. The proof is analogous to that of Proposition 4 of [MT]. If W_1 and W_2 are P-modules with p-small highest weights and $\mathcal{W}_i = \mathcal{F}_P(W_i)$, then as in [MT], Lemma 11 a morphism $\mathrm{Dist}(G) \otimes_{\mathrm{Dist}(P)} W_1 \to \mathrm{Dist}(G) \otimes_{\mathrm{Dist}(P)} W_2$ induces a PD differential operator $\mathcal{W}_2^\vee \to \mathcal{W}_1^\vee$, which is an ordinary differential operator since the weights are p-small. Applying the linearization functor, we get a morphism $\mathcal{L}(\mathcal{W}_2^\vee) \to \mathcal{L}(\mathcal{W}_1^\vee)$ of crystals in $\mathcal{C}_r$. Hence from Proposition 3.1 we obtain a complex as follows:

$$0 \to \mathcal{V}_\lambda^\vee \to \mathcal{L}(\mathcal{K}_\lambda^0) \to \mathcal{L}(\mathcal{K}_\lambda^1) \to \cdots$$

Applying the same construction to the retraction of complexes $S_\bullet^{\mathbb{Z}_p}(G, P, \lambda) \to C_\bullet^{\mathbb{Z}_p}(G, P, \lambda)$, we obtain an injective map of complexes

$$\mathcal{L}(\mathcal{K}_\lambda^\bullet) \hookrightarrow \mathcal{L}(\mathcal{V}_\lambda^\vee \otimes_{\mathcal{O}_{X_r}} \Omega^\bullet_{X_r/S_r}).$$

In fact, the image of this map is a direct summand. Since $\mathcal{L}(\mathcal{V}_\lambda^\vee \otimes_{\mathcal{O}_{X_r}} \Omega^\bullet_{X_r/S_r})$ is an exact resolution of $\mathcal{V}_\lambda^\vee$ by the crystalline Poincaré lemma, it follows that $\mathcal{L}(\mathcal{K}_\lambda^\bullet)$ is also an exact resolution of $\mathcal{V}_\lambda^\vee$. ∎

As in [MT] 5.3.4, the H-filtration of the complex $\mathcal{K}_\lambda^\bullet$ is given by

$$\mathrm{Fil}^l(\mathcal{K}_\lambda^i) = \bigoplus_{\substack{w \in W_L(i) \\ (w\cdot\lambda)(H) \leq -l}} \mathcal{M}_{w\cdot\lambda}^\vee.$$

It follows from the much more general statement of [Ill], Corollaire 4.13 that the Hodge to de Rham spectral sequence in this situation degenerates at E_1.

Proposition 3.3. *Let $F(\lambda)$ be a p-minute weight with $a_n = 0$. The spectral sequence given by the Hodge filtration*

$$E_1^{i,j} = \bigoplus_{\substack{w \in W_L \\ -(w\cdot\lambda)(H) = i}} H^{i+j-l(w)}(X_0, \mathcal{M}_{w\cdot\lambda}) \Rightarrow H_{dR}^{i+j}(X_0, \mathcal{V}_\lambda)$$

degenerates at E_1:

$$gr^i H^j_{dR}(X_0, \mathcal{V}_\lambda) = \bigoplus_{\substack{w \in W_L \\ l(w) \leq j \\ -(w\cdot\lambda)(H) = i}} H^{j-l(w)}(X_0, \mathcal{M}_{w\cdot\lambda}).$$

4. The proof of Theorem 1.2

We are now ready to put together the results of the previous sections to prove Theorem 1.2. Before doing that, we prove the following result, which allows us to restrict to the case of $a_n = 0$. This statement seems to be well-known, but since we have not found a reference in the literature, we will give its proof.

Proposition 4.1. *Let $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{F}_p^* \subset \overline{\mathbb{F}}_p^*$ be the mod p cyclotomic character, and let $b \in \mathbb{Z}$. Then a Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is modular of weight $F(\lambda)$ if and only if $\rho \otimes \chi^b$ is modular of weight $\det^b \otimes F(\lambda)$.*

Proof. Let $\eta = F(1, 1, \dots, 1) : \mathrm{GL}_n(\mathbb{F}_p) \to \overline{\mathbb{F}}_p^*$ be the one-dimensional representation given by $\eta(g) = \det g$. It is easy to see that $\mathcal{F}_{\lambda\otimes\eta} = \mathcal{F}_\lambda(1)$. Hence

$$H^m(X_U \otimes \overline{E}, \mathcal{F}_{\lambda\otimes\eta}) \simeq H^m(X_U \otimes \overline{E}, \mathcal{F}_\lambda) \otimes \chi|_{\mathrm{Gal}(\overline{E}/E)}$$

as $\mathrm{Gal}(\overline{E}/E)$-modules. Alternatively, we can see this as follows:

Composing χ with the reciprocity map of global class field theory, we obtain a character $\chi : (\mathbb{A}^\infty)^*/\mathbb{Q}^* \to \overline{\mathbb{F}}_p^*$; since this is a continuous map with finite image, it is locally constant. We normalize the reciprocity map so that the geometric Frobenius Frob_l^{-1} at a place l of $\mathbb{Q}$ is the image of a uniformizer π_l in $\mathbb{Q}_l$.

Let $\Gamma_1 \subset \Gamma$ be the subgroup denoted G_1 in [HT1] I.7; it is the derived subgroup of Γ and the kernel of the map $\nu : \Gamma \to \mathbb{G}_m$. Let T denote the quotient Γ/Γ_1. Then, by [Del] 2.5 the map ν induces a bijection

$$\pi_0(X_{U_1}) \simeq \pi_0\left(T(\mathbb{Q})\backslash T(\mathbb{A}^\infty)/\nu(U_1)\right).$$

Decreasing U^p at finitely many bad places if necessary, we may assume that $\nu(U_1) \subset \ker\chi$. Hence we obtain a function $\chi : \pi_0(X_{U_1}) \to \overline{\mathbb{F}}_p^*$. The group $\Gamma(\mathbb{A}^\infty)$ acts on X_{U_1} by right multiplication, and hence $g \in U/U_1 = \mathrm{Gal}(X_{U_1}/X_U)$ acts on $\pi_0(X_{U_1})$ by multiplication by $\nu(g) = \det(g)$. Hence the function $\chi : \pi_0(X_{U_1}) \to \overline{\mathbb{F}}_p^*$ induces a cohomology class $c_\chi \in H^0(X_U \otimes \overline{E}, \mathcal{F}_\eta)$.

Let w be a place of E dividing a rational prime that splits in E; note that the set of such places has Dirichlet density 1 in E. Moreover, for almost all such w at which D splits, we see from the congruence relation of [HT2], Proposition 4.2.6 that the geometric Frobenius Frob_w^{-1} acts on X_{U_1} as a matrix in $\Gamma(\mathbb{A}^\infty)$ with determinant $\sigma(\pi_w)^{-1}$, where $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ is non-trivial. Hence Frob_w^{-1} acts on $\pi_0(X_{U_1})$ by $(Nw)^{-1} = \chi(\mathrm{Frob}_w^{-1})$. Thus the Galois representation generated by c_χ is one-dimensional and isomorphic to the character $\chi|_{\mathrm{Gal}(\overline{E}/E)}$ by the Chebotarev density theorem.

For a subspace $V \subset H^m(X_U \otimes \overline{E}, \mathcal{F}_\lambda)$, let $V \cup c_\chi = \{v \cup c_\chi : v \in V\} \subset H^m(X_U \otimes \overline{E}, \mathcal{F}_{\lambda\otimes\eta})$. If $V' \subset V \subset H^m(X_U \otimes \overline{E}, \mathcal{F}_\lambda)$ are such that $V/V' \simeq \rho|_{\mathrm{Gal}(\overline{E}/E)}$, then it is easy to see that $(V \cup c_\chi)/(V' \cup c_\chi) \simeq (\rho \otimes \chi)|_{\mathrm{Gal}(\overline{E}/E)}$. □

Observe that if $\lambda = (a_1, \ldots, a_n)$, then $F(\lambda) = \det^{a_n} \otimes F(\lambda')$, where $\lambda' = (a_1 - a_n, \ldots, a_{n-1} - a_n, 0)$. Also note that if $\rho|_{G_p}$ is in the image of $\mathbf{T}^*_x$ with Fontaine-Laffaille numbers $b_1, \ldots, b_n$, then $(\rho \otimes \chi^b)$ is in the image of $\mathbf{T}^*_{x+b}$ with Fontaine-Laffaille numbers $b_1 + b, \ldots, b_n + b$. Therefore it suffices to prove Theorem 1.2 for p-minute weights $F(\lambda) = F(a_1, \ldots, a_n)$ with $a_n = 0$.

So let $F(\lambda)$ be p-minute, where $\lambda = (a_1, \ldots, a_n)$ and $a_n = 0$. Then, by Lemma 2.2, $\mathcal{V}^\vee_\lambda$ is an object of the category $\mathcal{MF}^\nabla_{[0,a_1]}$. Since the crystal $\mathcal{V}^\vee_\lambda$ is associated to the étale sheaf $\mathcal{F}_\lambda$ by Lemma 2.3, in the sense that $\mathcal{F}_\lambda = \mathbf{D}(\mathcal{V}^\vee_\lambda)$, and since $a_1 + (n-1) < p - 1$, the hypotheses of Faltings' comparison theorem (Theorem 5.3 of [Fal]) are satisfied. It follows that

$$\mathbf{D}^*_{cris}(H^*_{\acute{e}t}(X \otimes_{\mathcal{O}_v} \overline{E}_v, \mathcal{F}_\lambda)) \simeq H^*_{cris}(X \otimes_{\mathcal{O}_v} \mathbb{F}_p, \mathcal{V}^\vee_\lambda).$$

Here $\mathbf{D}^*_{cris}$ is the functor defined by Fontaine from the category of crystalline Galois representations to that of filtered modules. The Fontaine-Laffaille functor $\mathbf{T}^*_0$ is an inverse of $\mathbf{D}^*_{cris}$. Moreover, the isomorphism above is compatible with the Galois actions and the filtrations on both sides. Recall that the jumps of the filtration on the left-hand side are the Fontaine-Laffaille numbers of $H^*_{\acute{e}t}(X \otimes_{\mathcal{O}_v} \overline{E}_v, \mathcal{F}_\lambda)$, viewed as a Galois representation. Hence they are the possible Fontaine-Laffaille numbers of the restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of a Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ that is modular of weight $F(\lambda)$.

But we can read off the jumps of the filtration on the right-hand side from Proposition 3.3. There it was proved that $gr^k H^*(X \otimes_{\mathcal{O}_v} \mathbb{F}_p, \mathcal{V}^\vee_\lambda) \neq 0$ only if there exists $w \in W_L$ such that $-(w \cdot \lambda)(H) = k$. We saw in section 3.1 that $W_L = \{c_i : 1 \leq i \leq n\}$. Moreover, $-(c_i \cdot \lambda)(H) = a_i + (n - i)$. This completes the proof of Theorem 1.2.

References

[ADP] Avner Ash, Darrin Doud, and David Pollack. Galois representations with conjectural connections to arithmetic cohomology. *Duke Math. J.* **112**(2002), 521–579.

[AS] Avner Ash and Warren Sinnott. An analogue of Serre's conjecture for Galois representations and Hecke eigenclasses in the mod p cohomology of $\mathrm{GL}(n, \mathbf{Z})$. *Duke Math. J.* **105**(2000), 1–24.

[BO] Pierre Berthelot and Arthur Ogus. *Notes on crystalline cohomology.* Princeton University Press, Princeton, N.J., 1978.

[Bre] Christophe Breuil. Construction de représentations p-adiques semi-stables. *Ann. Sci. École Norm. Sup. (4)* **31**(1998), 281–327.

[Del] Pierre Deligne. Travaux de Shimura. In *Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389*, pages 123–165. Lecture Notes in Math., Vol. 244. Springer, Berlin, 1971.

[Dim] Mladen Dimitrov. Galois representations modulo p and cohomology of Hilbert modular varieties. *Ann. Sci. École Norm. Sup. (4)* **38**(2005), 505–551.

[Fal] Gerd Faltings. Crystalline cohomology and p-adic Galois-representations. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 25–80. Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[FC] Gerd Faltings and Ching-Li Chai. *Degeneration of abelian varieties*, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990. With an appendix by David Mumford.

[HT1] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[HT2] Michael Harris and Richard Taylor. Regular models of certain Shimura varieties. *Asian J. Math.* **6**(2002), 61–94.

[Her] Florian Herzig. The weight in a Serre-type conjecture for tame n-dimensional Galois representations. *Preprint* (2007).

[Ill] Luc Illusie. Réduction semi-stable et décomposition de complexes de de Rham à coefficients. *Duke Math. J.* **60**(1990), 139–185.

[Jan] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.

[Kot] Robert E. Kottwitz. Points on some Shimura varieties over finite fields. *J. Amer. Math. Soc.* **5**(1992), 373–444.

[MT] Abdellah Mokrane and Jacques Tilouine. Cohomology of Siegel varieties with p-adic integral coefficients and applications. *Astérisque* **280**(2002), 1–95. Cohomology of Siegel varieties.

[PT] Patrick Polo and Jacques Tilouine. Bernstein-Gelfand-Gelfand complexes and cohomology of nilpotent groups over $\mathbb{Z}_{(p)}$ for representations with p-small weights. *Astérisque* (2002), 97–135. Cohomology of Siegel varieties.

Einstein Institute of Mathematics, Hebrew University of Jerusalem, Givat Ram, Jerusalem 91904, Israel

E-mail address: mschein@math.huji.ac.il