WEIGHTS IN SERRE'S CONJECTURE FOR HILBERT MODULAR FORMS: THE RAMIFIED CASE

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Abstract. Let F be a totally real field and $p \geq 3$ a prime. If $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is continuous, semisimple, totally odd, and tamely ramified at all places of F dividing p, then we formulate a conjecture specifying the weights for which ρ is modular. This extends the conjecture of Diamond, Buzzard, and Jarvis, which required p to be unramified in F. We also prove a theorem that verifies one half of the conjecture in many cases and use Dembélé's computations of Hilbert modular forms over $\mathbb{Q}(\sqrt{5})$ to provide evidence in support of the conjecture.

1. Introduction

Let F be a totally real field and $p \geq 3$ a rational prime. For any place v of F, we write $\mathcal{O}_v$ for the completion of $\mathcal{O}_F$ at v and k_v for the residue field. Let $p\mathcal{O}_F = \prod_{v|p} v^{e_v}$ be the factorization of p into prime ideals of F, so that e_v is the ramification index of F_v over $\mathbb{Q}_p$. The purpose of this paper is to formulate, and prove some cases of, a Serre-type "epsilon conjecture" for mod p Hilbert modular forms over F. Previously this has been done only in the case of p unramified in F, i.e. $e_v = 1$ for all $v|p$.

Definition 1.1. A *(Serre) weight* is an irreducible $\overline{\mathbb{F}}_p$-representation of the group $\mathrm{GL}_2(\mathcal{O}_F/p) = \prod_{v|p} \mathrm{GL}_2(\mathcal{O}_F/v^{e_v})$.

Any irreducible mod p representation of $\mathrm{GL}_2(\mathcal{O}_F/v^{e_v})$ factors through the natural surjection $\mathrm{GL}_2(\mathcal{O}_F/v^{e_v}) \to \mathrm{GL}_2(k_v)$; indeed, the kernel is a p-group and hence acts trivially (see [Edi2] for a proof of this). By Proposition 1 of [BL], the irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(k_v)$ are:

$$\sigma_v = \bigotimes_{\tau : k_v \hookrightarrow \overline{\mathbb{F}}_p} (\det{}^{w_\tau} \mathrm{Sym}^{k_\tau - 2} k_v^2) \otimes_{k_v, \tau} \overline{\mathbb{F}}_p,$$

where $2 \leq k_\tau \leq p+1$ and $0 \leq w_\tau \leq p-1$, and the w_τ are not all $p-1$. Let $\Gamma = \prod_{v|p} \mathrm{GL}_2(k_v)$. Then the irreducible $\overline{\mathbb{F}}_p$-representations of Γ are $\sigma = \otimes_{v|p} \sigma_v$ with σ_v as above, and every weight factors through Γ. We call the irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(k_v)$ *Serre weights at v*.

Buzzard, Diamond, and Jarvis in [BDJ] formulated a Serre-type conjecture for Hilbert modular forms in the case where p is unramified in F. We would like to have a conjecture in the general case. We may assume that $F \neq \mathbb{Q}$, as otherwise the conjecture is well-known (and mostly proved!).

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Given a continuous, irreducible, totally odd Galois representation $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, let $W(\rho)$ denote the set of weights for which it is modular; we explain below what is meant by "modular." For each $v|p$ we will construct a set $W_v^?(\rho)$ of Serre weights at v and conjecture that

$$W(\rho) = \left\{ \sigma = \bigotimes_{v|p} \sigma_v : \forall v, \sigma_v \in W_v^?(\rho) \right\}.$$

This allows us to treat each $v|p$ separately.

In the next section we will state the conjecture in two equivalent forms, very much in the spirit of Florian Herzig's reformulation of the [BDJ] conjecture. The proof that they are equivalent (Theorems 2.4 and 2.5) relies heavily on Herzig's ideas in [Her], §14. In the third section we prove a theorem towards our conjecture; it shows, in many cases when the restriction of ρ to a decomposition group at a place $v|p$ is irreducible, that the v-component of any modular weight does indeed lie in $W_v^?(\rho)$. This statement, Theorem 3.4, generalizes the main result of [Sch] and is proved by a similar argument; it was proved before the conjecture was formulated and played an important role in motivating it. Finally, in the last section we use Dembélé's computations of Hilbert modular forms over $\mathbb{Q}(\sqrt{5})$ and their weights to obtain some computational evidence in support of the conjecture.

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2. A conjecture

First we introduce the notion of modularity. Let D be a quaternion algebra over F which is split at exactly one real place of F and at all places over p. Let $G = \mathrm{Res}_{F/\mathbb{Q}}(D^*)$ be the associated reductive group; for an open compact subgroup $U \subset G(\mathbb{A}^\infty)$ we have a Shimura curve M_U/F whose complex points are

$$M_U(\mathbb{C}) = G(\mathbb{Q}) \backslash G(\mathbb{A}^\infty) \times (\mathbb{C} - \mathbb{R})/U.$$

The M_U are not in general geometrically connected. Let the abelian variety $\mathrm{Pic}^0(M_U)/F$ be the identity component of the relative Picard scheme of M_U, which parametrizes line bundles locally of degree zero.

Let $U' = \ker((D \otimes \hat{\mathbb{Z}})_p^* = \prod_{v|p} \mathrm{GL}_2(\mathcal{O}_v) \to \mathrm{GL}_2(\mathcal{O}_F/p))$, and let $U'' = \ker(\prod_{v|p} \mathrm{GL}_2(\mathcal{O}_v) \to \prod_{v|p} \mathrm{GL}_2(k_v))$. Clearly $U' \subset U''$. We say that an open compact $U \subset G(\mathbb{A}^\infty)$ is *of type* $(*)$ if $U = U' \times U^p$, where $U^p \subset G(\mathbb{A}^{\infty,p})$. Let $V = \prod_{v|p} \mathrm{GL}_2(\mathcal{O}_v) \times U^p$. If U^p is sufficiently small as in section 3.1 of [Sch], then M_U/M_V is a Galois cover with group $V/U = \mathrm{GL}_2(\mathcal{O}_F/p)$. Hence we have an action of V/U on $\mathrm{Pic}^0(M_U)$.

Definition 2.1. An irreducible Galois representation $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is *modular of weight* σ if there exists a quaternion algebra D/F as above and an open compact $U \subset (D \otimes \hat{\mathbb{Z}})^* \subset G(\mathbb{A}^\infty)$

of type $(*)$, such that $(\mathrm{Pic}^0(M_U)[p]\otimes_{\overline{\mathbb{F}}_p}\sigma)^{\mathrm{GL}_2(\mathcal{O}_F/p)} = (\mathrm{Pic}^0(M_{U''\times U^p})[p]\otimes_{\overline{\mathbb{F}}_p}\sigma)^{\Gamma}$ has ρ as a Jordan-Hölder constituent.

Fix a place $\mathfrak{p}|p$ of F; we will now define $W^?_{\mathfrak{p}}(\rho)$. Choose a decomposition subgroup $G_{\mathfrak{p}}\subset \mathrm{Gal}(\overline{F}/F)$ at $\mathfrak{p}$, and let $I_{\mathfrak{p}}$ and $I'_{\mathfrak{p}}$ be the corresponding inertia and wild inertia subgroups. Denote by $I_{t,\mathfrak{p}} = I_{\mathfrak{p}}/I'_{\mathfrak{p}}$ the tame inertia, and let the residue field $k_{\mathfrak{p}}$ have cardinality $q = p^s$.

We will state our conjecture in the language of Herzig's reformulation of the [BDJ] conjecture. Let I be the set of embeddings $k_{\mathfrak{p}}\hookrightarrow\overline{\mathbb{F}}_p$, and as in [Sch], let $\tau_0,\dots,\tau_{s-1}$ be a labeling of its elements such that $\tau_{j-1}=\tau_j^p$ for all $j\in\mathbb{Z}/s\mathbb{Z}$. Similarly, let $k'_{\mathfrak{p}}$ be a quadratic extension of $k_{\mathfrak{p}}$ and fix a labeling $\sigma_0,\dots,\sigma_{2s-1}$ of the embeddings $k'_{\mathfrak{p}}\hookrightarrow\overline{\mathbb{F}}_p$ such that $\sigma_{i-1}=\sigma_i^p$ for all $i\in\mathbb{Z}/2s\mathbb{Z}$ and such that $\sigma_i|_{k_{\mathfrak{p}}}=\tau_{\pi(i)}$, where $\pi:\mathbb{Z}/2s\mathbb{Z}\to\mathbb{Z}/s\mathbb{Z}$ is the natural projection. Given such an embedding $\tau\in I$ (resp. $\sigma: k'_{\mathfrak{p}}\hookrightarrow\overline{\mathbb{F}}_p$), let $\lambda_\tau: I_{t,\mathfrak{p}}\simeq\varprojlim\mathbb{F}^*_{p^n}\to\overline{\mathbb{F}}_p$ (resp. $\psi_\sigma: I_{t,\mathfrak{p}}\to\overline{\mathbb{F}}_p$) be the corresponding fundamental character of level s (resp. $2s$). Often we write λ_j,ψ_i for $\lambda_{\tau_j},\psi_{\sigma_i}$. Note that Herzig's convention is $\psi_{i+1}=\psi_i^p$; the reader should bear this in mind when comparing our work with his.

If $b=\sum_{j=0}^{s-1} w_j p^{s-j}$ and $a-b=\sum_{j=0}^{s-1}(k_j-2)p^{s-j}$ for $0\le w_j\le p-1$ and $2\le k_j\le p+1$, then we denote

$$F(a,b)=\bigotimes_{j\in\mathbb{Z}/s\mathbb{Z}}(\det{}^{w_j}\,\mathrm{Sym}^{k_j-2}k^2_{\mathfrak{p}})\otimes_{k_{\mathfrak{p}},\tau_j}\overline{\mathbb{F}}_p.$$

Of course this notation comes from the theory of Weyl modules, but for the purposes of this article we may take the expression above as a definition.

Given $\rho|_{I_{\mathfrak{p}}}$, we first associate to it a characteristic zero representation of $\mathrm{GL}_2(k_{\mathfrak{p}})$ as in [Her], Def. 14.1. Here $I(\chi_1,\chi_2)$ are the usual principal series, while the $\Theta(\xi)$, for $\xi: k'_{\mathfrak{p}}\hookrightarrow\overline{\mathbb{F}}_p$, are the cuspidal representations (see, for instance, [DL]).

Definition 2.2. (1) If $\rho|_{I_{\mathfrak{p}}}\sim\begin{pmatrix}\prod_{j\in\mathbb{Z}/s\mathbb{Z}}\lambda_j^{m_j} & 0\\ 0 & \prod_j\lambda_j^{n_j}\end{pmatrix}$, then $V_{\mathfrak{p}}(\rho)=I(\prod\tau_j^{m_j},\prod\tau_j^{n_j})$.

(2) If $\rho|_{I_{\mathfrak{p}}}\sim\begin{pmatrix}\prod_{i\in\mathbb{Z}/2s\mathbb{Z}}\psi_i^{m_i} & 0\\ 0 & \prod_i\psi_i^{m_{i+s}}\end{pmatrix}\prod_{j\in\mathbb{Z}/s\mathbb{Z}}\lambda_j^{w_j}$, then $V_{\mathfrak{p}}(\rho)=\Theta(\prod\sigma_i^{m_i})\otimes\prod_j\tau_j^{w_j}$.

Since $V_{\mathfrak{p}}(\rho)$ can be realized over $\overline{\mathbb{Z}}_p$, we may consider its reduction modulo p, denoted $\overline{V_{\mathfrak{p}}(\rho)}$. For any representation V, we write $JH(V)$ for the set of its Jordan-Hölder constituents. The sets $JH(\overline{V_{\mathfrak{p}}(\rho)})$ are computed in [Dia].

In Lemma 3.1 we compute the determinant of $\rho|_{I_{\mathfrak{p}}}$, and hence the central character of any modular weight. If $e\ge p$, we conjecture that *all* weights with this central character are modular. Indeed, this is suggested by the fact that we already conjecture this "maximal" set of weights when $e=p-1$, as can be seen from Theorems 2.4 and 2.5, and by the observation that the number of conjectured modular weights increases with e for $e\le p-1$.

Let $Y_{\mathfrak{p}}$ be the set of Serre weights at $\mathfrak{p}$. If $e\le p-1$, let $\delta\in\Delta=[0,e-1]^I$ be a vector whose components are choices of an integer $0\le\delta_\tau\le e-1$ for each $\tau\in I$. Given δ, we will define a multi-valued function $\mathcal{R}^\delta_{\mathfrak{p}}: Y_{\mathfrak{p}}\to Y_{\mathfrak{p}}$ for which we conjecture the following:

Conjecture 1. Let $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be continuous, irreducible, totally odd, and tame at $\mathfrak{p}$. Then

(1) $W^?_{\mathfrak{p}}(\rho) = \bigcup_{\delta\in\Delta} \mathcal{R}^\delta_{\mathfrak{p}}(JH(\overline{V_{\mathfrak{p}}(\rho)}))$ if $e \leq p-1$.
(2) $W^?_{\mathfrak{p}}(\rho) = \left\{ F(a,b) : \det\rho|_{I_{\mathfrak{p}}} = \lambda_0^{a+b+\sum_{j=0}^{s-1} ep^j} \right\}$ if $e \geq p$.

We will now assume $e \leq p-1$, fix $\delta \in \Delta$, and construct the map $\mathcal{R}^\delta_{\mathfrak{p}}$. Given $F(a,b)$, define $\alpha(j) = p+1-k_j \in [0, p-1]$ for every $j \in \mathbb{Z}/s\mathbb{Z}$. Define x_j to be the integer such that $\alpha(j) + x_j p \in [1+2\delta_j - (e-1), p+2\delta_j-(e-1)]$. Under the assumption that $e \leq p-1$, we have $x_j \in \{-1,0,1\}$ for all j. We say that $F(a,b)$ is a δ-*regular* Serre weight at $\mathfrak{p}$ if the x_j are all zero. If $F(a,b)$ is not δ-regular, then for every $j \in \mathbb{Z}/s\mathbb{Z}$ we define $\theta_j = x_{j+n}$, where n is the smallest positive integer such that $x_{j+n} \neq 0$.

Suppose first that $F(a,b)$ is a δ-regular Serre weight. Then we define

$$\mathcal{R}^\delta_{\mathfrak{p}}(F(a,b)) = \left\{ F(c,d) : \begin{array}{c} c \equiv b - \sum_{j=0}^{s-1}(1+\delta_j)p^{s-j} \mod p^s - 1 \\ d \equiv a - \sum_{j=0}^{s-1}(e-1-\delta_j)p^{s-j} \mod p^s - 1 \end{array} \right\}.$$

If $F(a,b)$ is irregular, things become more complicated. We define a collection $\mathcal{S}^\delta(F(a,b))$ of subsets of $\mathbb{Z}/s\mathbb{Z}$ as follows. Let $S \subset \mathbb{Z}/s\mathbb{Z}$. Then $S \in \mathcal{S}^\delta(F(a,b))$ if and only if for every $j \in S$ the following two conditions hold:

(1) One of the following two conditions holds:
   (a) $x_j = -1$ or $\alpha(j) \in [2\delta_j - (e-1), p-1+2\delta_j-(e-1)] \cap [0,p-1]$, and there is an integer $n \geq 0$ such that $x_{j+m} = 1 + 2\delta_{j+m} - (e-1)$ for $1 \leq m \leq n$ (if any such m exists) and $x_{j+n+1} = 1$.
   (b) $x_j = 1$ or $\alpha(j) \in [2+2\delta_j-(e-1), p+1+2\delta_j-(e-1)] \cap [0,p-1]$, and there is an $n \geq 0$ such that $x_{j+m} = p + 2\delta_{j+m} - (e-1)$ for all $1 \leq m \leq n$ and $x_{j+n+1} = -1$.
(2) In either of the cases above, $j+m \notin S$ for $1 \leq m \leq n$.

We emphasize that $\mathcal{S}^\delta(F(a,b))$ depends only on $a-b$. Finally, writing $F = F(a,b)$, we can give the general definition:

$$\mathcal{R}^\delta_{\mathfrak{p}}(F) = \left\{ F(c,d) : \begin{array}{c} c \equiv b + \sum_{j\in S}\theta_j p^{s-j} - \sum_{j=0}^{s-1}(1+\delta_j)p^{s-j} \mod p^s-1 \\ d \equiv a - \sum_{j=0}^{s-1}(e-1-\delta_j)p^{s-j} - \sum_{j\in S}\theta_j p^{s-j} \mod p^s - 1 \end{array} : S \in \mathcal{S}^\delta(F) \right\}.$$

Lemma 2.3 ([Her], Lemma 14.3)**.** *If ρ is of level $2s$, then $\sigma_{\mathfrak{p}} = \bigotimes_{\tau\in I}(\det^{w_\tau}\mathrm{Sym}^{k_\tau-2}k_{\mathfrak{p}}^2) \otimes_{k_{\mathfrak{p}},\tau} \overline{\mathbb{F}}_p$ is a Jordan-Hölder constituent of $\overline{V_{\mathfrak{p}}(\rho)}$ if and only if for each $\tau \in I$ there is a labeling $\{\tilde{\tau}, \tilde{\tau}'\}$ of its two lifts to $k'_{\mathfrak{p}}$ such that*

$$\rho|_{I_{\mathfrak{p}}} \sim \prod_\tau \lambda_\tau^{w_\tau+k_\tau-2} \begin{pmatrix} \prod_\tau \psi_{\tilde{\tau}}^{p+1-k_\tau} & 0 \\ 0 & \prod_\tau \psi_{\tilde{\tau}'}^{p+1-k_\tau} \end{pmatrix}.$$

Theorem 2.4. *Suppose that $\rho|_{I_\mathfrak{p}}$ is of level $2s$. Then $W^?_\mathfrak{p}(\rho)$ consists precisely of those Serre weights at $\mathfrak{p}$*

$$\sigma_\mathfrak{p} = \bigotimes_{\tau \in I} (\det{}^{w_\tau} \operatorname{Sym}^{k_\tau - 2} k_\mathfrak{p}^2) \otimes_{k_\mathfrak{p},\tau} \overline{\mathbb{F}}_p \tag{1}$$

such that for each $\tau \in I$ there exists a labeling $\{\tilde\tau, \tilde\tau'\}$ of its two lifts to $k'_\mathfrak{p}$ and an integer $0 \le \delta_\tau \le e-1$ such that

$$\rho|_{I_\mathfrak{p}} \sim \prod_{\tau\in I} \lambda_\tau^{w_\tau} \begin{pmatrix} \prod_\tau \psi_{\tilde\tau}^{k_\tau - 1 + \delta_\tau} \psi_{\tilde\tau'}^{e-1-\delta_\tau} & 0 \\ 0 & \prod_\tau \psi_{\tilde\tau}^{e-1-\delta_\tau} \psi_{\tilde\tau'}^{k_\tau - 1 + \delta_\tau} \end{pmatrix}.$$

Proof. If $e \ge p$ the theorem is evident, so we assume from now on that $e \le p-1$. Let $L^\delta_\mathfrak{p}(\rho)$ be the set of weights satisfying the condition in the statement above for a given δ. We claim that $\mathcal{R}^\delta_\mathfrak{p}(JH(\overline{V_\mathfrak{p}(\rho)})) = L^\delta_\mathfrak{p}(\rho)$ for every choice of δ.

Fix δ, and suppose that $F = F(a,b)$ is a Jordan-Hölder constituent of $\overline{V_\mathfrak{p}(\rho)}$. Without loss of generality, we may assume that $b = 0$, and we write $a = \sum_{j=0}^{s-1} a_j p^j$ with $0 \le a_j \le p-1$. Set $\alpha(j) = p - 1 - a_j$. Then by Lemma 2.3 we have

$$\begin{aligned} \rho|_{I_\mathfrak{p}} \quad \sim \quad & \prod_{j\in\mathbb{Z}/s\mathbb{Z}} \lambda_j^{a_j} \begin{pmatrix} \prod_{i\in J} \psi_i^{\alpha(i)} & 0 \\ 0 & \prod_{i\in J^c} \psi_i^{\alpha(i)} \end{pmatrix} = \\ & \prod_{j\in\mathbb{Z}/s\mathbb{Z}} \lambda_j^{a_j + p - e + \delta_j} \begin{pmatrix} \prod_{i\in J} \psi_i^{\alpha(i)+e-1-\delta_i} \psi_{i+s}^{e-1-\delta_i} & 0 \\ 0 & \prod_{i\in J^c} \psi_i^{\alpha(i)+e-1-\delta_j} \psi_{i+s}^{e-1-\delta_j} \end{pmatrix}, \end{aligned} \tag{2}$$

where $J \subset \mathbb{Z}/2s\mathbb{Z}$ is a subset such that $|J| = s$ and $\pi(J) = \mathbb{Z}/s\mathbb{Z}$ and we write $\alpha(i)$ for $\alpha(\pi(i))$ and δ_i for $\delta_{\pi(i)}$. Our goal now is to write $\prod_{i\in J} \psi_i^{\alpha(i)}$ in the form $\eta \prod_{i\in J} \psi_i^{\beta(i)}$, where $\beta(i) \in [1 + 2\delta_i - (e-1), p + 2\delta_i - (e-1)]$ and η is a character of level s; from such an expression we will read off a weight in $L^\delta_\mathfrak{p}(\rho)$.

Observe first that such an expression is unique if it exists. Indeed, suppose that $\eta \prod_J \psi_i^{\beta(i)}$ and $\eta' \prod_J \psi_i^{\beta(i)'}$ are two such expressions. Then $\psi = \prod_{i\in J} \psi_i^{\beta(i)-\beta(i)'}$ is a character of level s, whence

$$\psi^{1-p^s} = \prod_{i\in J} \psi_i^{\beta(i)-\beta(i)'} \psi_{i+s}^{\beta(i)'-\beta(i)} = 1.$$

Since $|\beta(i) - \beta(i)'| \le p-1$ for all $i \in J$, it is evident that we must have $\beta(i) = \beta(i)'$ for all i.

Two issues must be dealt with in obtaining the desired expression. First, J is not specified uniquely by (2). Indeed, if $\alpha(j) = 0$ for some $j \in \mathbb{Z}/s\mathbb{Z}$, then we can choose either of the elements of $\pi^{-1}(j)$ to lie in J. In this case, $\overline{V_\mathfrak{p}(\rho)}$ has fewer constituents than usual, but the ambiguity in J allows us to produce several modular weights from each constituent. Second, the $\alpha(j)$ need not lie in the range $[1 + 2\delta_j - (e-1), p + 2\delta_j - (e-1)]$, so we must "carry" exponents. If $e = 1$, this problem occurs only when $\alpha(j) = 0$, which is exactly when the first problem arises. In general we do not have this coincidence, whence the relative complexity of our construction to the analogous one in [Her], §14. If $e = 1$, then the argument below reduces precisely to Herzig's argument.

For each $j \in \mathbb{Z}/s\mathbb{Z}$, let $x_j \in \mathbb{Z}$ be such that $\alpha(j) + x_j p \in [1 + 2\delta_j - (e-1), p + 2\delta_j - (e-1)]$. For instance, if $e = 1$, then $x_j = 1$ if $\alpha(j) = 0$ and $x_j = 0$ otherwise. Observe that if $e \leq p-1$, then $x_j \in \{-1, 0, 1\}$; moreover, $\alpha(j) + x_j(p-1)$ also lies in the specified range.

As in [Her], we consider an interval in $\mathbb{Z}/s\mathbb{Z}$ to be a sequence $[[j,n]] = \{j, j+1, \ldots, n\}$. The predecessor of $[[j,n]]$ is $j-1$ (these correspond, of course, to Herzig's successors; the difference is a consequence of our opposite conventions). The terminus of $[[j,n]]$ is n. We define $\mathcal{L}_\delta$ as the set of all pairs $(\alpha, \mathcal{I})$, where $\alpha : \mathbb{Z}/s\mathbb{Z} \to [0, p-1]$ is a map, $\mathcal{I}$ is a collection of disjoint intervals, each labeled with a sign, and the following axioms are satisfied. For each $j \in \mathbb{Z}/s\mathbb{Z}$, given α, we can formally define x_j as above. If $j-1$ is the predecessor of an interval and n is its terminus, and $x_n \neq 0$, then define $z_{j-1} = x_n$. Otherwise it will follow from the third axiom that $n+1 \in \bigcup \mathcal{I}$ and we define $z_{j-1} = z_n$. Thus $z_j = \pm 1$. Finally, let $C^\pm = \{j \in \mathbb{Z}/s\mathbb{Z} : x_j = \pm 1\}$. The axioms are:

(1) For each interval $I \in \mathcal{I}$, either $I \subset C^+ \cup \{j : \alpha(j) = 1 + 2\delta_j - (e-1)\}$ or $I \subset C^- \cup \{j : \alpha(j) = p + 2\delta_j - (e-1)\}$.
(2) If $j \in \bigcup \mathcal{I}$ and $\alpha(j) \neq 0$, then j is the terminus of its interval.
(3) If $j \in \bigcup \mathcal{I}$, then $\alpha(j) \in \{1 + 2\delta_j - (e-1), p + 2\delta_j - (e-1)\}$ if and only if j is the terminus of an $\mathcal{I}$-interval and the predecessor of a negative $\mathcal{I}$-interval.
(4) If $j \notin \bigcup \mathcal{I}$ and $x_j \neq 0$, then $j+1 \in \bigcup \mathcal{I}$.
(5) If a positive $\mathcal{I}$-interval has predecessor j, then either j lies in an $\mathcal{I}$-interval and satisfies $z_j \neq x_j$, or j does not lie in any interval and $\alpha(j) \in [1 - z_j + 2\delta_j - (e-1), p - z_j + 2\delta_j - (e-1)]$.
(6) If a negative $\mathcal{I}$-interval I has predecessor j, then either j lies in an $\mathcal{I}$-interval and satisfies $z_j = x_j$, or j lies in an $\mathcal{I}$-interval and $\alpha(j) = 1 + 2\delta_j - (e-1)$ (resp. $p + 2\delta_j - (e-1)$) if $z_j = 1$ (resp. $z_j = -1$), or else j does not lie in any interval and $\alpha(j) \in [1 + z_j + 2\delta_j - (e-1), p + z_j + 2\delta_j - (e-1)]$.

Similarly, let $\mathcal{M}_\delta$ be the set of pairs $(\beta, \mathcal{I})$, where $\mathcal{I}$ as before is a collection of signed intervals and $\beta : \mathbb{Z}/s\mathbb{Z} \to \mathbb{Z}$ is a map such that for every $j \in \mathbb{Z}/s\mathbb{Z}$, we have $\beta(j) \in [1+2\delta_j-(e-1), p+2\delta_j-(e-1)]$. Let y_j be the integer such that $\beta(j) - y_j p \in [0, p-1]$. Let $D^\pm = \{j \in \mathbb{Z}/s\mathbb{Z} : y_j = \pm 1\}$. If j is the predecessor of an interval, let u_j be the number defined in the same way as z_j, but with x_j replaced by y_j in the definition. We require that $(\beta, \mathcal{I})$ satisfy the following axioms:

(1) For each interval $I \in \mathcal{I}$, either $I \subset D^+ \cup \{j : \beta(j) = p-1\}$ or $I \subset D^-$.
(2) The set of termini of $\mathcal{I}$-intervals is $D^+ \cup D^-$.
(3) If a positive $\mathcal{I}$-interval has predecessor j, then either j lies in an $\mathcal{I}$-interval and satisfies $u_j \neq y_j$, or j does not lie in any interval and $\beta(j) \in [u_j, p-1+u_j]$.
(4) If a negative $\mathcal{I}$-interval has predecessor j, then either j lies in an $\mathcal{I}$-interval and satisfies $u_j = y_j$, or j does not lie in any interval and $\beta(j) \in [-u_j, p-1-u_j]$.

There is a bijection $\xi : \mathcal{L}_\delta \to \mathcal{M}_\delta$ which can be written down as follows. Like Herzig, we represent the function α by the string of numbers $\alpha(0), \alpha(1), \ldots, \alpha(s-1)$. We underline each $\mathcal{I}$-interval and put its sign after its last entry. Pairs $(\beta, \mathcal{I})$ are written similarly. Then ξ acts as follows, where j is always the predecessor of the last interval, in the third line k is the predecessor of the first interval, and we assume $x_j = 0$ in the second line and $x_j \neq 0, x_k \neq 0$ in the third:

$$\begin{array}{rcl} x', \underline{(0,\ldots,0,)x}_{\pm} & \mapsto & x' \pm z_j, \underline{(z_j(p-1),\ldots,z_j(p-1),)x + z_j p}_{\pm} \\ y', \underline{(0\ldots0,)y,}_{\pm}\underline{(0\ldots0,)y''}_{-} & \mapsto & y' \pm z_j, \underline{(z_j(p-1)\ldots z_j(p-1)), y + z_j(p-1),}_{\pm}\underline{(z_j(p-1)\ldots,}_{-} \\ w', \underline{(0\ldots0)w,}_{\pm}\underline{(0\ldots0)w''}_{\pm'} & \mapsto & w' \pm z_k, \underline{(z_k(p-1),\ldots), w \pm z_k p \pm' z_j}_{\pm}\underline{(z_j(p-1),\ldots), w'' + z_j p}_{\pm'} \end{array}$$

All other entries are unchanged by ξ. The reader may verify that ξ is indeed a bijection between $\mathcal{L}_\delta$ and $\mathcal{M}_\delta$. It does not affect the collection $\mathcal{I}$ of signed intervals. We will prove below (Lemma 2.6) that if $S \subset \mathbb{Z}/s\mathbb{Z}$, then $S \in \mathcal{S}^\delta(F(a,0))$ if and only if S is the set of predecessors of positive intervals in $\mathcal{I}$ for some $(\alpha,\mathcal{I}) \in \mathcal{L}_\delta$ (hence for some $(\beta,\mathcal{I}) \in \mathcal{M}_\delta$), where α is derived from a as above.

Given $S \in \mathcal{S}^\delta(F)$, let $(\alpha,\mathcal{I}) \in \mathcal{L}_\delta$ be such that S is the set of predecessors of positive intervals. Let J_+ (resp. J_-) be the elements of J whose projections to $\mathbb{Z}/s\mathbb{Z}$ are predecessors of positive (resp. negative) intervals, and similarly for J^c. Let $\tilde{\mathcal{I}}$ be the collection of intervals in $\mathbb{Z}/2s\mathbb{Z}$ that project to $\mathcal{I}$-intervals, and let J_0 be the elements of J that do not lie in any $\tilde{\mathcal{I}}$-intervals. Then as in [Her] we observe that $\prod_{i\in J}\psi_i^{\alpha(i)} = \chi\prod_{i\in J_+\cup J_+^c}\psi_i^{-z_i}$, where

$$\chi = \prod_{i\in J_+}\psi_i^{\alpha(i)+z_i}\prod_{J_0\backslash(J_+\cup J_-)}\psi_i^{\alpha(i)}\prod_{J_-}\psi_i^{\alpha(i)-z_i}\prod_{\substack{i-1\in J_-\cap J_+^c\\ [[i,n]]\in\tilde{\mathcal{I}}}}(\psi_i^{p-1}\psi_{i+1}^{p-1}\cdots\psi_n^{p})^{z_{i-1}}\prod_{J\backslash(J_+\cup J_-\cup J_0)}\psi_i^{\alpha(i)}$$

and it is not hard to see that in this expression, one of every pair $\{\psi_i,\psi_{i+s}\}$ appears with exponent zero and the other appears with exponent in the range $[1+2\delta_j-(e-1), p+2\delta_j-(e-1)]$. Hence

$$\rho|_{I_\mathfrak{p}} \sim \prod_{j\in\mathbb{Z}/s\mathbb{Z}}\lambda_j^{a_j+p-e+\delta_j}\begin{pmatrix}\chi_1 & 0\\ 0 & \chi_2\end{pmatrix}\prod_{j\in S}\lambda_j^{-z_j},$$

where each ψ_i, $i\in\mathbb{Z}/2s\mathbb{Z}$, appears with exponent $e-1-\delta_i$ in one of χ_1,χ_2 and with some exponent $\beta(i)=\beta_{\pi(i)}$ in the range $[1+\delta_j, p+\delta_j]$ in the other. From such an expression we can read off a weight $F(A,B)\in L_\mathfrak{p}(\rho)$.

Now, from (2) we see that $\det\rho|_{I_\mathfrak{p}} = \prod_{j\in\mathbb{Z}/s\mathbb{Z}}\lambda_j^{a_j} = \lambda_0^{\sum_{m=0}^{s-1}a_{s-j}p^j}$. Let $1_S : I \to \{0,1\}$ be the characteristic function of S. Then from the displayed expressions above we find that

$$\det\rho|_{I_\mathfrak{p}} = \prod_{j\in\mathbb{Z}/s\mathbb{Z}}\lambda_j^{2a_j-(e-1)+\delta_j+\beta_j}\prod_{j\in S}\lambda_j^{-2z_j} = \lambda_0^{\sum_{m=0}^{s-1}(2a_{-m}-(e-1)+\delta_{-m}+\beta_{-m}-2\cdot 1_S(-m)z_{-m})p^j}.$$

Hence, noting that for $j\in S$ we have $w_j = z_j$, we find that

$$\begin{aligned} B &\equiv a + \sum_{m=0}^{s-1}(\delta_j-(e-1))p^{s-j} - \sum_{j\in S}w_j p^{s-j} \mod p^s-1 \\ A &\equiv \sum_{j\in S}w_j p^{s-j} - \sum_{j=0}^{s-1}(\delta_j+1)p^{s-j} \mod p^s-1 \end{aligned}$$

It remains to check that any other weight $F(\tilde{A},\tilde{B})$ satisfying the same congruences is also contained in $L^\delta_\mathfrak{p}(\rho)$. The only cases when more than one weight satisfies such a congruence are the pairs $F(b,b), F(p^s-1+b,b)$ for some b. But then it is obvious from the definition of $L^\delta_\mathfrak{p}(\rho)$ that one of these weights is contained there if and only if the other one is. Hence we have shown that $\mathcal{R}^\delta_\mathfrak{p}(F) \subset L^\delta_\mathfrak{p}(\rho)$.

Conversely, suppose that $F(a,b) \in L^\delta_\mathfrak{p}(\rho)$. We may assume without loss of generality that $b=0$, and as usual write $a = \sum_{j=0}^{s-1} a_j p^{s-j}$. Then,

$$\rho|_{I_\mathfrak{p}} \sim \begin{pmatrix} \prod_{i\in L} \psi_i^{\beta(i)} & 0 \\ 0 & \prod_{i \in L^c} \psi_i^{\beta(i)} \end{pmatrix} \prod_{i \in \mathbb{Z}/2s\mathbb{Z}} \psi_i^{e-1-\delta_i},$$

where $L \subset \mathbb{Z}/2s\mathbb{Z}$ is mapped bijectively to $\mathbb{Z}/s\mathbb{Z}$ by π and $\beta(i) = a_{\pi(i)} + 1 + 2\delta_i - (e-1) \in [1+2\delta_i-(e-1), p+2\delta_i-(e-1)]$. Let y_i be an integer such that $\beta(i) - y_i p \in [0,p-1]$; under our assumptions on e, we have $y_i \in \{-1,0,1\}$. Let $D^\pm = \{i \in \mathbb{Z}/2s\mathbb{Z} : y_i = \pm 1\}$. We now define a collection $\mathcal{I}$ of intervals in bijection with $D^+ \cup D^-$ as follows. If $i \in D^+$ and $i \in L$ (resp. $i \in L^c$), choose n such that $[[n,i]] \subset L$ (resp. L^c) and $\beta(m) = p-1$ for all $m \in [[n,i]]\backslash\{i\}$, and such that n is minimal for this property (i.e. $n-1$ will not work). Then $[[n,i]]$ is the interval corresponding to i, and we let it be negative if and only if $y_{n-1} = 1$ or $y_{n-1} = 0$ and $n-1 \in L$ (resp. L^c).

Similarly, if $i \in D^-$, then the corresponding interval is $[[i]]$. It is negative if and only if $y_{i-1} = -1$ or $y_{i-1} \in \{1+2\delta_{i-1}-(e-1), p+2\delta_{i-1}-(e-1), p-1\}$.

It is easy to see that $(\beta,\mathcal{I}) \in \mathcal{M}_\delta$. Let L_+, L_-, L_0 be defined as before, and let $S = L_+ \cup L_+^c$ be the set of predecessors of positive intervals. We invert the previous construction to find that

$$\rho|_{I_\mathfrak{p}} \sim \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} \prod_{j \in \mathbb{Z}/s\mathbb{Z}} \lambda_j^{e-1-\delta_j} \prod_{j \in S} \lambda_j^{u_j},$$

where

$$\chi_1 = \prod_{i \in L_+} \psi_i^{\beta(i)-u_i} \prod_{L_0 \backslash (L_+ \cup L_-)}^{\beta(i)} \prod_{L_-} \psi_i^{\beta(i)+u_i} \prod_{\substack{i-1 \in L_- \cup L_+^c \\ [[i,n]] \in \tilde{\mathcal{I}}}} (\psi_i^{p-1} \cdots \psi_n^p)^{-u_{i-1}} \prod_{L \backslash (L_+ \cup L_- \cup L_0)} \psi_i^{\beta(i)},$$

and χ_2 is the same but with the roles of L and L^c reversed. Each ψ_i appears with non-zero exponent in at most one of χ_1, χ_2, and this exponent always lies in the range $[1, p-1]$. Thus we have obtained an expression of the form

$$\rho|_{I_\mathfrak{p}} \sim \begin{pmatrix} \prod_{i\in L} \psi_i^{\alpha(i)} & 0 \\ 0 & \prod_{i \in L^c} \psi_i^{\alpha(i)} \end{pmatrix} \prod_{j \in \mathbb{Z}/s\mathbb{Z}} \lambda_j^{e-1-\delta_j} \prod_{j \in S} \lambda_j^{u_j}.$$

Using Lemma 2.3 we can read off a weight $F(A,B) \in JH(\overline{V_{\mathfrak{p}}(\rho)})$. Moreover, clearly $(\alpha,\mathcal{I}) = \xi^{-1}(\beta,\mathcal{I})$, whence $S \in \mathcal{S}^{\delta}(F(A,B))$. Comparing two expressions for $\det\rho|_{I_{\mathfrak{p}}}$ as before, we find that

$$\sum_{j=0}^{s-1}(a_j+e)p^{s-j} \equiv \sum_{j=0}^{s-1}(\alpha(j)+2[(e-1-\delta_j)+1_S(j)u_j])p^{s-j} \mod p^s-1.$$

Clearly $u_j = w_j$ for $j \in S$. Also we see that

$$\begin{aligned} B &\equiv \sum_{j=0}^{s-1}(e-1-\delta_j+1_S(j)u_j+\alpha(j))p^{s-j} \equiv \sum_{j=0}^{s-1}(a_j+1+\delta_j)p^{s-j} - \sum_{j\in S}u_jp^{s-j} \mod p^s-1 \\ A &\equiv \sum_{j=0}^{s-1}(e-1-\delta_j+1_S(j)u_j)p^{s-j} \mod p^s-1 \end{aligned}$$

Hence, $F(a,b) \in \mathcal{R}^{\delta}_{\mathfrak{p}}(F(A,B))$. This completes the proof that $\mathcal{R}^{\delta}_{\mathfrak{p}}(JH(\overline{V_{\mathfrak{p}}(\rho)})) = L^{\delta}_{\mathfrak{p}}(\rho)$. □

A very similar argument establishes an analogous statement in the level s case:

Theorem 2.5. *Suppose that $\rho|_{I_{\mathfrak{p}}}$ is of level s and, as always, tame at $\mathfrak{p}$. Then $W^?_{\mathfrak{p}}(\rho)$ consists precisely of the Serre weights at $\mathfrak{p}$ as in* (1) *for which there exist a set $J \subset I$ and an integer $0 \le \delta_\tau \le e-1$ for each $\tau \in I$ such that*

$$\rho|_{I_{\mathfrak{p}}} \sim \prod_{\tau\in I}\lambda_\tau^{w_\tau}\begin{pmatrix} \prod_{\tau\in J}\lambda_\tau^{k_\tau-1+\delta_\tau}\prod_{\tau\notin J}\lambda_\tau^{e-1-\delta_\tau} & 0 \\ 0 & \prod_{\tau\in J}\lambda_\tau^{e-1-\delta_\tau}\prod_{\tau\notin J}\lambda_\tau^{k_\tau-1+\delta_\tau}\end{pmatrix}.$$

Finally we establish a lemma that was needed in the proof of Theorem 2.4.

Lemma 2.6. *Let $\alpha : \mathbb{Z}/s\mathbb{Z} \to [0,p-1]$ be a function, and let $S \subset \mathbb{Z}/s\mathbb{Z}$. Then $S \in \mathcal{S}(F(a,b))$ for some (hence all) weights $F(a,b)$ such that $a-b = \sum_{j=0}^{s-1}(p-1-\alpha(j))p^{s-j}$ if and only if S is the set of predecessors of positive $\mathcal{I}$-intervals for some $(\alpha,\mathcal{I}) \in \mathcal{L}_\delta$.*

Proof. It is easy to see from the axioms of $\mathcal{L}_\delta$ that the set of predecessors of positive intervals of any $(\alpha,\mathcal{I})$ lies in $\mathcal{S}^{\delta}(F(a,b))$.

Conversely, suppose $S \in \mathcal{S}^{\delta}(F(a,b))$; we will construct an appropriate $\mathcal{I}$. We let $j \in \bigcup\mathcal{I}$ if and only if there exists $n \ge 0$ such that $x_{j+n+1} = 0$ and for all $1 \le m \le n$ we have $j+m \notin S$ and either $x_{j+m} = 1+2\delta_{j+m}-(e-1)$ for all m or $x_{j+m} = p+2\delta_{j+m}-(e-1)$ for all m. We let $j \in \bigcup\mathcal{I}$ be the terminus of an $\mathcal{I}$-interval if and only if $j+1 \notin \bigcup\mathcal{I}$, or if $j+1 \in \bigcup\mathcal{I}$ and $\alpha(j) \neq 0$, or $\alpha(j) = 0 \in \{1+2\delta_j-(e-1), p+2\delta_j-(e-1)\}$. This specifies $\mathcal{I}$, and we define an $\mathcal{I}$-interval to be positive if and only if its predecessor is contained in S. The reader may verify that $(\alpha,\mathcal{I}) \in \mathcal{L}_\delta$. □

3. A theorem towards the conjecture

As before, let I be the set of embeddings $\tau : k_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{F}}_p$. Suppose the Galois representation $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular of a weight σ whose $\mathfrak{p}$-component is

$$\sigma_{\mathfrak{p}} = \bigotimes_{\tau\in I}(\det{}^{w_\tau}\mathrm{Sym}^{k_\tau-2}k_{\mathfrak{p}}^2)\otimes_{k_{\mathfrak{p}},\tau}\overline{\mathbb{F}}_p. \tag{3}$$

Suppose that the restriction of ρ to the decomposition subgroup $G_{\mathfrak{p}}$ is irreducible. Then as in [Sch] we have

$$\rho|_{I_{\mathfrak{p}}}^{ss} \sim \begin{pmatrix} \phi & 0 \\ 0 & \phi^q \end{pmatrix},$$

where $\phi : I_{t,\mathfrak{p}} = I_{\mathfrak{p}}/I'_{\mathfrak{p}} \to \overline{\mathbb{F}}_p^*$ is a character of level $2s$. Let K be the maximal unramified extension of $F_{\mathfrak{p}}$, and let K'/K be the totally ramified extension such that $\mathrm{Gal}(K'/K) \simeq k_{\mathfrak{p}}^*$.

The present argument is very similar to the one in [Sch], so we refer the reader to that article and only indicate the differences. In particular, the first four sections of [Sch] do not depend on the assumption that p is unramified in F, so they hold in our case as well. Suppose that ρ is modular of weight $\sigma = \sigma_{\mathfrak{p}} \otimes (\otimes_{v \neq \mathfrak{p}} \sigma_v)$ and that $\sigma_{\mathfrak{p}}$ is a Jordan-Hölder constituent of $Ind_B^{\mathrm{GL}_2(k_{\mathfrak{p}})}\theta$, where $B \subset \mathrm{GL}_2(k_{\mathfrak{p}})$ is the subgroup of upper triangular matrices and $\theta : B \to \overline{\mathbb{F}}_p^*$ is given by

$$\theta : \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \prod_{\tau : k_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{F}}_p} \tau(ad)^{w_\tau} \tau(d)^{k_\tau - 2}. \tag{4}$$

Lemma 3.1. *Write k_j for k_{τ_j}. Then,*

$$\phi^{q+1} = \prod_{i \in \mathbb{Z}/2s\mathbb{Z}} \psi_i^{2w_{\pi(i)} + k_{\pi(i)} - 2 + e}.$$

Proof. By [Sch], Prop. 3.19, for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$ we have $\det \rho(\sigma) = \chi(\sigma)\langle\sigma\rangle^{-1}$, where χ is the mod p cyclotomic character and $\langle\cdot\rangle$ is the diamond operator map. If $\sigma \in \mathrm{Gal}(\overline{K}/K) = I_{\mathfrak{p}}$, suppose its image in $\mathrm{Gal}(K'/K)$ is sent by the Artin reciprocity map to $j(\sigma) \in \mathcal{O}_{\mathfrak{p}}^*/(1 + \mathfrak{p})$. Then we have

$$\phi^{q+1}(\sigma) = \det \rho(\sigma) = \chi(\sigma)\langle\sigma\rangle^{-1} = \prod_{\tau : k_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{F}}_p} \tau(j(\sigma))^{k_\tau - 2} \tau(j(\sigma))^e = \prod_{i \in \mathbb{Z}/2s\mathbb{Z}} \psi_i(\sigma)^{k_{\pi(i)} - 2 + e},$$

just as in the proof of [Sch], Lemma 5.1. □

Assume from now on that $e \leq p - 1$. Let $\boldsymbol{\mu} \in \mathbb{Z}^s$ be the vector whose components are given by $\mu_i = a_i + a_{i+s} - (k_{i+1} - 2 + e)$. By the previous lemma $\boldsymbol{\mu}$ lies in the lattice

$$\Lambda = \mathbb{Z}(p, 0, \ldots, 0, -1) \oplus \mathbb{Z}(-1, p, 0, \ldots, 0) \oplus \cdots \oplus \mathbb{Z}(0, \ldots, 0, -1, p).$$

By [Sch], Corollary 3.21, we may assume that $w_\tau = 0$ for all τ. For $j \in \mathbb{Z}/s\mathbb{Z}$, let $c_j = k_j - 2 + p(k_{j-1} - 2) + \cdots + p^{s-1}(k_{j+1} - 2)$. Assume first that θ is non-trivial; then $0 < c_j < p^s - 1$. Let H be an $\mathbb{F}_{p^{2s}}$-vector space scheme over D' defined just as in [Sch]; it satisfies the condition $(**)$ of [Ray]. Let a_i, a'_i, and b_i, for $i \in \mathbb{Z}/2s\mathbb{Z}$, be parameters defined as in [Edi1], §5 or [Sch], 4.1. The relevant facts about them are that $0 \leq a'_i \leq e(p^s - 1)$, that $b_i \in \{c_{\pi(i)}, 0\}$ (just as in [Sch], Lemma 5.3), and that they satisfy the relation

$$a'_i = b_{i+1} - pb_i + (p^s - 1)a_i. \tag{5}$$

We apply this relation to determine the a_i. As in section 5.1 of [Sch], we consider four cases:

Case 1. $b_i = 0, b_{i+1} = c_{i+1}$. Then by (5) we have

$$a'_i - (p^s - 1)a_i = b_{i+1} - pb_i = c_{i+1}.$$

By virtue of the bound on a'_i, this equation admits e solutions:

$$\begin{aligned} a'_i &= c_{i+1} & a_i &= 0 \\ a'_i &= c_{i+1} + p^s - 1 & a_i &= 1 \\ &\cdots & &\cdots \\ a'_i &= c_{i+1} + (e-1)(p^s - 1) & a_i &= e - 1 \end{aligned}$$

Case 2. $b_i = c_i, b_{i+1} = 0$. Then (5) says that

$$a'_i - (p^s - 1)a_i = -pc_i = \beta - (p^s - 1)(k_{i+1} - 1),$$

where $\beta = (p + 1 - k_{i+1}) + p(p + 1 - k_i) + \cdots + p^{s-1}(p + 1 - k_{i+2})$. Since $0 < \beta < p^s - 1$, we again have e solutions:

$$\begin{aligned} a'_i &= \beta & a_i &= k_{i+1} - 1 \\ a'_i &= \beta + p^s - 1 & a_i &= k_{i+1} \\ &\cdots & &\cdots \\ a'_i &= \beta + (e-1)(p^s - 1) & a_i &= k_{i+1} - 1 + (e - 1) \end{aligned}$$

Case 3. $b_i = 0, b_{i+1} = 0$. Then $a'_i - (p^s - 1)a_i = 0$, which has $e + 1$ solutions:

$$\begin{aligned} a'_i &= 0 & a_i &= 0 \\ a'_i &= p^s - 1 & a_i &= 1 \\ &\cdots & &\cdots \\ a'_i &= e(p^s - 1) & a_i &= e \end{aligned}$$

Case 4. $b_i = c_i, b_{i+1} = c_{i+1}$. Then $a'_i - (p^s - 1)a_i = c_{i+1} - pc_i = -(p^s - 1)(k_{i+1} - 2)$, and there are $e + 1$ solutions:

$$\begin{aligned} a'_i &= 0 & a_i &= k_{i+1} - 2 \\ a'_i &= p^s - 1 & a_i &= k_{i+1} - 1 \\ &\cdots & &\cdots \\ a'_i &= e(p^s - 1) & a_i &= k_{i+1} - 2 + e \end{aligned}$$

Lemma 3.2. *We may assume without loss of generality that* $\{b_i, b_{i+s}\} = \{0, c_i\}$ *for each* $i \in \mathbb{Z}/2s\mathbb{Z}$.

Proof. We sketch the proof, using the notions and notations of [Sch] without comment. Recall that $H \subset \mathrm{Pic}^0(\mathbf{M}^{bal}_{U_1(\mathfrak{p}),U})[p^\infty]$, where $U \subset G(\mathbb{A}^{\infty,\mathfrak{p}})$ is an appropriate open compact subgroup and $\mathbf{M}^{bal}_{U_1(\mathfrak{p}),U} \to \mathrm{Spec}\ D'$ is the semistable model of a Shimura curve as described there and in [Gee], Thm. 2.18. As in [Gee], $\mathbf{M}^{bal}_{U_1(\mathfrak{p}),U}$ represents the functor that associates to an $\mathbf{L}^*_{1,U}$-scheme

S the collection of canonical balanced $U_1(\mathfrak{p})$-structures on S. The scheme $\mathbf{M}^{bal}_{U_1(\mathfrak{p}),U}$ carries an "Atkin-Lehner" automorphism w that sends a canonical balanced $U_1(\mathfrak{p})$-structure $(P, P', \mathcal{K}, \mathcal{K}')$ to a structure $(Q, Q', \mathcal{L}, \mathcal{L}')$, where $\mathcal{L}$ is a lifting of $\mathcal{K}'$ to $\mathbf{E}_{1,U}|_S$ and Q' is the image of P in $\mathcal{L}'$. The map w interchanges the two components I and E of the special fiber of $\mathbf{M}^{bal}_{U_1(\mathfrak{p}),U}$.

By the arguments of [Car] §10 we see that $\mathrm{Frob}_{\mathfrak{p}}$ preserves $H \oplus w(H)$. Hence $w(H)$ is an $\mathbb{F}_{p^{2s}}$-vector space scheme over D' lifting the vector space scheme H_{ϕ^q} over K on which $\mathrm{Gal}(\overline{K}/K)$ acts via the character ϕ^q. Let $w(H)$ be defined by the parameters a_i^w, $(a_i')^w$, b_i^w. Then $a_i^w = a_{i+s}$ and as in [Sch], Lemma 5.3, we see that $b_i^w = 0$ (resp. $b_i^w = c_i$) if $b_i = c_i$ (resp. $b_i = 0$).

Now, in all the subscripts of the parameters defining $w(H)$, replace i by $i + s$. We get an $\mathbb{F}_{p^{2s}}$-vector space scheme $\tilde{H}$, defined by parameters $\tilde{a}_i$, $\tilde{a}_i'$, $\tilde{b}_i$, where $\tilde{a}_i = a_i$ and

$$\tilde{b}_i = \begin{cases} c_i & : b_{i+s} = 0 \\ 0 & : b_{i+s} = c_i \end{cases}$$

Let $N^+ \subset \mathbb{Z}/2s\mathbb{Z}$ (resp. N^-) be the set of i such that $b_i = b_{i+s} = c_i$ (resp. $b_i = b_{i+s} = 0$), and let $N = N^+ \cap N^-$. Suppose first that $N \neq \mathbb{Z}/2s\mathbb{Z}$. Then there exists an i such that $i \in N$ but $i + 1 \notin N$. Suppose that $i \in N^-$ (the case $i \in N^+$ is very similar), and let $n \geq 0$ be the largest integer such that $i - n' \in N^-$ for all $0 \leq n' \leq n$. It is easy to see that if $\alpha \in \{i, i + s\}$ is such that $b_{\alpha-n-1} = c_{\alpha-n-1}$, then we can switch $b_{i-n'}$ to $c_{i-n'}$ and still obtain the same set of a_i's as possible solutions. Note that the existence of $\tilde{H}$ guarantees that the $a_{i-n'}$ are in the range where this is possible. Iterating this procedure proves the lemma.

Finally suppose that $N = \mathbb{Z}/2s\mathbb{Z}$. Since ϕ is a character of level $2s$, there is some i such that $a_i \neq a_{i+s}$. We leave it as an exercise to the reader to show that, after possible replacing i with $i + s$, for all $0 \leq n' \leq s - 1$, if $b_{i-n'} = 0$ (resp. $b_{i-n'} = c_{i-n'}$) we may change it to $c_{i-n'}$ (resp. to 0), and still obtain the same set of a_i's as possible solutions. ☐

From the definition of $\boldsymbol{\mu}$ we see that $-e \leq \mu_i \leq e$ for all i and that for some i we have $-(e-1) \leq \mu_i \leq e - 1$. Since $\boldsymbol{\mu} \in \Lambda$, this implies $\boldsymbol{\mu} = 0$. Thus $a_i + a_{i+s} = k_{i+1} - 2 + e$ for all i.

Proposition 3.3. *Let $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be such that $\rho|_{G_{\mathfrak{p}}}$ is irreducible and ρ is modular of weight σ such that $\sigma_{\mathfrak{p}}$ is a constituent of $Ind_B^{\mathrm{GL}_2(k_{\mathfrak{p}})}\theta$, where $\theta : B \to \overline{\mathbb{F}}_p$ is non-trivial and has the form of (4) above. Then there exists a subset $S \subset I$ and a labeling $\{\tilde{\tau}, \tilde{\tau}'\}$ of the two liftings of $\tau : k_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{F}}_p$ to $\mathbb{F}_{p^{2s}}$ for each τ, such that*

$$\rho|_{I_{t,\mathfrak{p}}} \sim \begin{pmatrix} \phi & 0 \\ 0 & \phi^q \end{pmatrix},$$

where for each $\tau : k_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{F}}_p$ there is an integer $0 \leq \delta_\tau \leq e - 1$ such that

$$\phi = \prod_{\tau \in I} (\psi_{\tilde{\tau}}\psi_{\tilde{\tau}'})^{w_\tau} \prod_{\tau \in S} \psi_{\tilde{\tau}}^{k_\tau - 2 + \delta_\tau + \nu_S(\tau)} \psi_{\tilde{\tau}'}^{e-1-\delta_\tau} \prod_{\tau \notin S} \psi_{\tilde{\tau}}^{p+e-1-\delta_\tau} \psi_{\tilde{\tau}'}^{k_\tau - 2 + \delta_\tau + \nu_S(\tau)}.$$

Proof. This is analogous to Proposition 5.6 and Corollary 5.8 of [Sch]. As in that paper, we reduce to the case of $w_\tau = 0$ for all $\tau \in I$. Let $\Phi(\theta)$ be the set of all ϕ of the form in the statement. Any $\phi \in \Phi(\theta)$ is specified by the data $(S, \varepsilon_j, \delta_j)$, where $S \subset I$ and for any $j \in \mathbb{Z}/j\mathbb{Z}$ we have a bijection of two-element sets $\varepsilon_j : \pi^{-1}(j) = \{j, j+s\} \to \{\psi_{\tilde{\tau}_j}, \psi_{\tilde{\tau}'_j}\}$ and an integer $0 \leq \delta_j \leq e-1$. The character corresponding to $(S, \varepsilon_j, \delta_j)$ is $\phi = \prod_{i \in \mathbb{Z}/2s\mathbb{Z}} \psi_i^{m_i}$, where

$$m_i = \begin{cases} k_i - 2 + \nu_S(\tau_i) + \delta_i & : \tau_i \in S, \varepsilon_i(i) = \psi_{\tilde{\tau}_i} \\ e - 1 - \delta_i & : \tau_i \in S, \varepsilon_i(i) = \psi_{\tilde{\tau}'_i} \\ p + e - 1 - \delta_i & : \tau_i \notin S, \varepsilon_i(i) = \psi_{\tilde{\tau}_i} \\ k_i - 2 + \nu_S(\tau_i) + \delta_i & : \tau_i \notin S, \varepsilon_i(i) = \psi_{\tilde{\tau}'_i} \end{cases}.$$

Here we make the usual abuse of notation: $\tau_i = \tau_{\pi(i)}$, $\delta_i = \delta_{\pi(i)} = \delta_{\tau_i}$, etc. Clearly every $\phi \in \Phi(\theta)$ is described in this way, although possibly not uniquely.

Let $\Omega_e(\theta)$ be the set of all ϕ satisfying all the conditions emerging from the computations earlier in this section. Any $\phi \in \Omega_e(\theta)$ is specified by the data (S', r_j, δ'_j), where $S' \subset I$ and for every $j \in \mathbb{Z}/s\mathbb{Z}$ we have a bijection $r_j : \{j, j+s\} \to \{0, c_j\}$ and an integer $0 \leq \delta'_j \leq e-1$. The corresponding character is $\phi = \prod_{i \in \mathbb{Z}/2s\mathbb{Z}} \psi_i^{a_{i-1}}$, where

$$a_{i-1} = \begin{cases} e - 1 - \delta'_i & : r_i(i) = 0, r_{i+1}(i+1) = c_i \\ k_i - 1 + \delta'_i & : r_i(i) = c_{i-1}, r_{i+1}(i+1) = 0 \\ e - 1 - \delta'_i & : r_i(i) = r_{i+1}(i+1) = 0, \tau_{i+1} \in S' \\ e - \delta'_i & : r_i(i) = r_{i+1}(i+1) = 0, \tau_{i+1} \notin S' \\ k_i - 1 + \delta'_i & : r_i(i) = c_{i-1}, r_{i+1}(i+1) = c_i, \tau_{i+1} \in S' \\ k_i - 2 + \delta'_i & : r_i(i) = c_{i-1}, r_{i+1}(i+1) = c_i, \tau_{i+1} \notin S' \end{cases}.$$

Again it is easy to see that every $\phi \in \Omega_e(\theta)$ is described (non-uniquely) in this way. Here $r_i(i) = 0$ and $r_i(i) = c_{i-1}$ correspond to $b_{i-1} = 0$ and $b_{i-1} = c_{i-1}$, respectively, and S' accounts for the extra possibilities in Cases 3 and 4. As in [Sch], Prop. 5.6 one constructs a bijection between these two collections of data and deduces that $\Phi(\theta) = \Omega_e(\theta)$. □

Theorem 3.4. *Suppose that $e < p - 1$ and let $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be such that $\rho|_{G_\mathfrak{p}}$ is irreducible and ρ is modular of weight σ, where $\sigma_\mathfrak{p}$, written as in (3), satisfies $k_\tau - 2 + e \leq p - 1$ for all τ. Then there exists a labeling $\{\tilde{\tau}, \tilde{\tau}'\}$ of the two liftings of $\tau : k_\mathfrak{p} \hookrightarrow \overline{\mathbb{F}}_p$ to $\mathbb{F}_{p^{2s}}$ for each τ, such that*

$$\rho|_{I_{t,\mathfrak{p}}} \sim \begin{pmatrix} \phi & 0 \\ 0 & \phi^q \end{pmatrix},$$

where for each $\tau : k_\mathfrak{p} \hookrightarrow \overline{\mathbb{F}}_p$ there is an integer $0 \leq \delta_\tau \leq e - 1$ such that

$$\phi = \prod_{\tau \in I} (\psi_{\tilde{\tau}} \psi_{\tilde{\tau}'})^{w_\tau} \prod_{\tau \in I} \psi_{\tilde{\tau}}^{k_\tau - 1 + \delta_\tau} \psi_{\tilde{\tau}'}^{e - 1 - \delta_\tau}.$$

Proof. As in [Sch] we may assume that $w_\tau = 0$ for all τ. Assume first that $k_\tau \neq 2$ for some τ. Denote by $\Theta(\sigma_\mathfrak{p})$ the set of all characters $\theta : B \to \overline{\mathbb{F}}_p^*$ such that $\sigma_\mathfrak{p}$ is a constituent of $Ind_B^{\mathrm{GL}_2(k_\mathfrak{p})}\theta$; all these characters θ are non-trivial. The elements of $\Theta(\sigma_\mathfrak{p})$ are the following, where T runs over all $T \subset I$:

$$\theta_T : \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \prod_{\tau \in T} \tau(ad)^{p-1}\tau(d)^{k_\tau - 1 - \nu_T(\tau)} \prod_{\tau \notin T} \tau(ad)^{k_\tau - 2}\tau(d)^{p+1-k_\tau-\nu_T(\tau)}.$$

If ρ is modular of a weight whose $\mathfrak{p}$-component is $\sigma_\mathfrak{p}$, then $\phi \in \bigcap_{\theta\in\Theta(\sigma_\mathfrak{p})} \Phi(\theta)$, and we will compute this intersection. If $s = 1$, then the desired result is immediate from Proposition 3.3 by considering $\Phi(\theta_I)$. Otherwise, suppose that $\phi \in \bigcap_{\theta\in\Theta(\sigma_\mathfrak{p})} \Phi(\theta)$, but ϕ is not of the form specified in the statement of the theorem. Since $\phi \in \Phi(\theta_I)$, it is easy to see that $\phi = \prod_{j\in\mathbb{Z}/2s\mathbb{Z}} \psi_j^{m_j}$ where $\{m_i, m_{i+s}\} = \{\varepsilon_i, k_i - 2 + e - \varepsilon_i\}$, where $0 \leq \varepsilon_i \leq e$ and for some i we have $\varepsilon_i = e$. Moreover, we may assume that $k_i > e+1$, since otherwise $\{k_i - 2, e\} = \{k_i - 2 + e - \varepsilon_i, \varepsilon_i\}$ for some $0 \leq \varepsilon_i \leq e-1$.

If $s \geq 2$, then the elements of $\Phi(\theta_{T=\{\tau_i\}})$ are the following, as S runs over the subsets of I and each δ_τ runs over $\{0, 1, \ldots, e-1\}$:

$$\prod_{\substack{\tau\in S\\ \tau\neq\tau_i}} \psi_{\tilde\tau}^{k_\tau-2+e-\delta_\tau-\nu_S(\tau)} \psi_{\tilde\tau'}^{p+\delta_\tau-\nu_T(\tau)} \prod_{\substack{\tau\notin S\\ \tau\neq\tau_i}} \psi_{\tilde\tau}^{k_\tau-2+e-\delta_\tau-\nu_S(\tau)} \psi_{\tilde\tau'}^{\delta_\tau-\nu_T(\tau)}$$
$$\times \begin{cases} \psi_{\tilde\tau_i}^{k_i+p-1+\delta_i} \psi_{\tilde\tau_i'}^{p+e-1-\delta_i-\nu_S(\tau_i)} & : \tau_i \in S \\ \psi_{\tilde\tau_i}^{k_i-1+\delta_i} \psi_{\tilde\tau_i'}^{p+e-1-\delta_i-\nu_S(\tau_i)} & : \tau_i \notin S \end{cases}$$

Dividing this by the expression for ϕ found above, we see that for some $S \subset I$ we have

$$1 = \prod_{\substack{\tau\in S\\ \tau\neq\tau_i}} \psi_{\tilde\tau}^{\substack{\varepsilon_\tau-\delta_\tau-\nu_S(\tau) \\ k_\tau-2+e-\delta_\tau-\varepsilon_\tau-\nu_S(\tau)}} \psi_{\tilde\tau'}^{\substack{p+\delta_\tau-\varepsilon_\tau-\nu_T(\tau) \\ p+\delta_\tau-k_\tau+2-e+\varepsilon_\tau}} \prod_{\substack{\tau\notin S\\ \tau\neq\tau_i}} \psi_{\tilde\tau}^{\substack{\varepsilon_\tau-\delta_\tau-\nu_S(\tau) \\ k_\tau-2+e-\delta_\tau-\varepsilon_\tau-\nu_S(\tau)}} \psi_{\tilde\tau'}^{\substack{\delta_\tau-\varepsilon_\tau-\nu_T(\tau) \\ \delta_\tau-k_\tau+2-e+\varepsilon_\tau}}$$
$$\times \begin{cases} \psi_{\tilde\tau_i}^{\substack{p+1+\delta_i \\ p+k_i-1-e+\delta_i}} \psi_{\tilde\tau_i'}^{\substack{p-1-\delta_i-\nu_S(\tau_i) \\ p+1+e-k_i-\delta_i-\nu_S(\tau_i)}} & : \tau_i \in S \\ \psi_{\tilde\tau_i}^{\substack{1+\delta_i \\ k_i-1-e+\delta_i}} \psi_{\tilde\tau_i'}^{\substack{p-1-\delta_i-\nu_S(\tau_i) \\ p+1+e-k_i-\delta_i-\nu_S(\tau_i)}} & : \tau_i \notin S \end{cases}$$

Here for each pair $\psi_{\tilde\tau}, \psi_{\tilde\tau'}$ we choose either the top or the bottom exponent in both cases. If we rewrite this expression as $\prod_{i\in\mathbb{Z}/2s\mathbb{Z}} \psi_i^{r_i}$, then we must have $(r_0, \ldots, r_{2s-1}) \in \Lambda$. Under our hypotheses, all these exponents lie in the range $[-(p-1), 2p-2]$. However, they cannot all be $-(p-1)$, nor can they all be $2p-2$, and hence the only possible values of the r_i are $-1, 0, p-1$, and p. Consider now the exponent $r_{\tilde\tau_i}$ of $\psi_{\tilde\tau_i}$. Since $1 \leq 1+\delta_i \leq p-2$ and $1 \leq k_i - 1 - e + \delta_i \leq p-2$ (recall $k_i > e+1$), we see that $r_{\tilde\tau_i}$ cannot take any of the allowed values, whence we cannot have $\tau_i \notin S$. But similarly $\tau_i \in S$ is impossible. We obtain a contradiction, which proves that $\phi \notin \bigcap_{\theta\in\Theta(\sigma_\mathfrak{p})} \Phi(\theta)$.

Finally, suppose $k_\tau = 2$ for all τ. In this case (recall $w_\tau = 0$ for all τ) the only θ such that $\sigma_\mathfrak{p}$ is a constituent of $Ind_B^{\mathrm{GL}_2(k_\mathfrak{p})}\theta$ is the trivial character ([Dia], Prop. 1.1). Just as in [Sch], 5.4., we construct an $\mathbb{F}_{p^{2s}}$-vector space scheme V such that $\mathrm{Gal}(\overline{K}/K)$ acts on V_K by the character ϕ. Let

a_i, a'_i, b_i be the parameters associated to V. As in [Sch] we see that $b_i = 0$ for all i; hence, by (5), each a_i can take any value between 0 and e. Our claim now follows from Lemma 3.1. $\square$

Remark 3.5. As in [Sch] §5, it is possible to relax the hypothesis that $k_\tau - 2 + e \leq p - 1$ for all τ, at the price of obtaining a somewhat weaker result. In this case, the set $\bigcap_{\theta \in \Theta(\sigma_\mathfrak{p})} \Phi(\theta)$ will be larger than the conjectured set of ϕ's for representations modular of a weight with $\mathfrak{p}$-component $\sigma_\mathfrak{p}$. However, we can still assert that $\phi \in \bigcap_{\theta \in \Theta(\sigma_\mathfrak{p})} \Phi(\theta)$.

4. Examples

Let $F = \mathbb{Q}(\sqrt{5})$. Let $p = 5$; then $(p) = \mathfrak{p}^2$ in F, where $\mathfrak{p} = ((5+\sqrt{5})/2)$, and $k_\mathfrak{p} = \mathbb{F}_5$. Thus we have $e = 2$ and $s = 1$. The weights in this situation are $\det^w \mathrm{Sym}^{k-2}\mathbb{F}_5 \otimes \overline{\mathbb{F}}_5 = F(w+k-2, w)$, where $2 \leq k \leq 6$ and $0 \leq w \leq 3$. All our examples rely on Lassina Dembélé's computations of Hilbert modular forms (see [Dem]), which so far exist only for $\mathbb{Q}(\sqrt{5})$. For each Hilbert modular form, Dembélé computes the list of weights for which the associated mod 5 Galois representation $\overline{\rho}$ is modular. He also provides evidence for (but does not actually compute) the projective image of $\overline{\rho}^{ss}$; clearly $\overline{\rho}$ is reducible if and only if this projective image is cyclic.

We have used Magma to find (elliptic) modular newforms f with integer coefficients. Then $\overline{\rho}_f|_{I_p}$ is described by classical theorems of Deligne and Fontaine. We search for the base change of f to F in Dembélé's tables and obtain the weights for which $\overline{\rho}_f|_{\mathrm{Gal}(\overline{F}/F)}$ is modular. In all examples that we have computed the results are, fortunately, consistent with Conjecture 1.

4.1. **Non-ordinary forms.** If f is non-ordinary at 5 and has weight $2 \leq k \leq 6$, then $\overline{\rho}_f$ is tame at 5. By a result of Fontaine (see [Edi1], Thm. 2.6),

$$\overline{\rho}|_{I_5} \sim \begin{pmatrix} \psi^{k-1} & 0 \\ 0 & \psi^{5(k-1)} \end{pmatrix},$$

where ψ is a fundamental character of level 2. From the description of the isomorphism between $I_{t,\mathfrak{p}}$ and $\varprojlim \mathbb{F}^*_{p^n}$ (see, for instance, [Sch], 4.1) we see that

$$\overline{\rho}|_{I_\mathfrak{p}} \sim \begin{pmatrix} \psi^{2(k-1)} & 0 \\ 0 & \psi^{10(k-1)} \end{pmatrix}.$$

The weights predicted by our conjecture are the following:

$$\begin{aligned} F(0,0), F(3,1), F(4,0), F(5,3), &\qquad k = 2,6 \\ F(1,1), F(4,2), F(5,1), F(6,4), &\qquad k = 5 \\ F(2,0), F(3,3), F(4,2), F(7,3), &\qquad k = 3 \\ F(0,0), F(3,1), F(4,0), &\qquad k = 4 \end{aligned}$$

Here are some of the computational results. Observe that the form with $k = 4$ gives a tame example of level 1, where the associated local Galois representation at $\mathfrak{p}$ is scalar. In this case the global mod 5 Galois representation is reducible; hence this is not a counterexample to the conjecture, even though we obtain only two weights. The other representations in the list are irreducible.

weight	level	q-expansion of f	modular weights of $\overline{\rho}_f\vert_{\mathrm{Gal}(\overline{F}/F)}$
2	14	$q - q^2 - 2q^3 + q^4 + 2q^6 + q^7 + O(q^8)$	$F(0,0), F(3,1), F(5,3), F(4,0)$
3	7	$q - 3q^2 + 5q^4 - 7q^7 + O(q^8)$	$F(3,3), F(2,0), F(4,2), F(7,3)$
3	8	$q - 2q^2 - 2q^3 + 4q^4 + 4q^6 + O(q^8)$	$F(3,3), F(2,0), F(4,2), F(7,3)$
4	9	$q - 8q^4 + 20q^7 - 70q^{13} + O(q^{16})$	$F(0,0), F(4,0)$
6	14	$q + 4q^2 + 8q^3 + 16q^4 + 10q^5 + 32q^6 + O(q^7)$	$F(0,0), F(3,1), F(5,3), F(4,0)$

4.2. **Ordinary forms.** Elliptic modular newforms which are ordinary at 5 are much more plentiful than non-ordinary ones. In this case, $\overline{\rho}_f|_{\mathrm{Gal}(\overline{F}/F)}$ is not in general tame at $\mathfrak{p}$, and it is natural to expect that even when $\overline{\rho}_f|_{\mathrm{Gal}(\overline{F}/F)}$ is irreducible, the modular weights will be only a subset of those which are modular for the semisimplification. If f has weight $2 \leq k \leq 6$, then by a theorem of Deligne (see [Edi1], Thm. 2.5),

$$\overline{\rho}_f|_{I_{\mathfrak{p}}} \sim \begin{pmatrix} \psi^{2(k-1)} & * \\ 0 & 1 \end{pmatrix},$$

where ψ is a fundamental character of level 1. If $\overline{\rho}_f$ is tame, then Conjecture 1 predicts the following sets of weights:

$$\begin{aligned} F(0,0), F(2,2), F(3,1), F(5,3), F(4,0), F(6,2), \qquad & k = 2,4,6 \\ F(3,3), F(2,0), F(7,3), \qquad & k = 3,5 \end{aligned}$$

For most of the forms we found, Dembélé's computations suggest that the global Galois representation is reducible. We found only three irreducible examples, which are compatible with the conjecture:

wt.	level	q-expansion of f	modular weights
4	8	$q - 4q^3 - 2q^5 + 24q^7 + O(q^9)$	$F(0,0), F(4,0)$
6	8	$q + 20q^3 - 74q^5 + O(q^7)$	$F(4,0)$
6	9	$q + 6q^2 + 4q^4 - 6q^5 + O(q^7)$	$F(4,0)$

In the examples where the global Galois representation appears to be reducible, we can still apply our conjecture to $\overline{\rho}_f|_{I_{\mathfrak{p}}}$ to obtain a set $W^?_{\mathfrak{p}}(\overline{\rho}_f)$. The computed modular weights always lie inside this set.

wt.	level	q-expansion of f	modular weights
2	8	$q + 2q^2 + 2q^3 + 4q^4 + 4q^5 + 4q^6 + O(q^7)$	$F(3,1), F(5,3)$
2	9	$q - 3q^2 + 7q^4 - 6q^5 + O(q^7)$	$F(3,1), F(5,3)$
3	3	$q + 3q^2 + 9q^3 + 13q^4 + 24q^5 + 27q^6 + O(q^7)$	$F(3,3), F(7,3)$
3	4	$q + 4q^2 + 8q^3 + 16q^4 + 26q^5 + 32q^6 + O(q^7)$	$F(3,3), F(2,0), F(7,3)$
3	7	$q + 5q^2 + 8q^3 + 21q^4 + 24q^5 + 40q^6 + O(q^7)$	$F(3,3), F(7,3)$
3	8	$q + 4q^2 + 10q^3 + 16q^4 + 24q^5 + 40q^6 + O(q^7)$	$F(3,3), F(7,3)$
4	6	$q - 2q^2 - 3q^3 + 4q^4 + 6q^5 + 6q^6 + O(q^7)$	$F(2,2), F(6,2)$
4	8	$q - 4q^3 - 2q^5 + 24q^7 + O(q^9)$	$F(0,0), F(4,0)$
4	8	$q + 8q^2 + 26q^3 + 64q^4 + 124q^5 + 208q^6 + O(q^7)$	$F(3,1), F(5,3)$
4	9	$q - 9q^2 + 73q^4 - 126q^5 + O(q^7)$	$F(3,1), F(5,3)$
5	4	$q - 4q^2 + 16q^4 - 14q^5 + O(q^8)$	$F(3,3), F(2,0), F(7,3)$
5	7	$q + 17q^2 + 80q^3 + 273q^4 + 624q^5 + 1360q^6 + O(q^7)$	$F(3,3), F(7,3)$
5	8	$q + 16q^2 + 82q^3 + 256q^4 + 624q^5 + 1312q^6 + O(q^7)$	$F(3,3), F(7,3)$
6	8	$q + 20q^3 - 74q^5 + O(q^7)$	$F(4,0)$
6	8	$q + 32q^2 + 242q^3 + 1024q^4 + 3124q^5 + 7744q^6 + O(q^7)$	$F(3,1), F(5,3)$
6	9	$q + 6q^2 + 4q^4 - 6q^5 + O(q^7)$	$F(4,0)$
6	9	$q - 33q^2 + 1057q^4 - 3126q^5 + O(q^7)$	$F(3,1), F(5,3)$

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