REDUCTION MODULO p OF CUSPIDAL REPRESENTATIONS AND WEIGHTS IN SERRE'S CONJECTURE

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ABSTRACT. Let $\mathcal{O}$ be the ring of integers of a p-adic field and $\mathfrak{p}$ its maximal ideal. We compute the Jordan-Hölder decomposition of the reduction modulo p of the cuspidal representations of $\mathrm{GL}_2(\mathcal{O}/\mathfrak{p}^e)$ for $e \geq 1$. We also provide an alternative formulation of Serre's conjecture for Hilbert modular forms.

1. Cuspidal representations and weights

1.1. **Cuspidal representations.** Let $K/\mathbb{Q}_p$ be a local field, where p is a prime, and let $\mathcal{O}$ be the ring of integers and $\mathfrak{p}$ its maximal ideal. Let $R_e = \mathcal{O}/\mathfrak{p}^e$. In particular, $R_1 = \mathcal{O}/\mathfrak{p}$ is the residue field; let $q = p^f$ be its cardinality. Let $\tilde{K}$ be the unramified quadratic extension of K, and let $\tilde{\mathcal{O}}$ and $\tilde{\mathfrak{p}}$ be its ring of integers and maximal ideal.

The cuspidal complex representations of $\mathrm{GL}_2(R_e)$ are well known (see for instance [PS]) in the case $e = 1$ and have been constructed for general e, under various names, by several authors; see, for instance, [Shi], [Gér], [How], [Car], [BK], and [Hil]. Aubert, Onn, and Prasad proved ([AOP], Theorem B; note that the notions of cuspidal and strongly cuspidal representations coincide for GL_2 by Theorem A) that they are parametrized by $\mathrm{Gal}(\tilde{K}/K)$-orbits of strongly primitive characters $\xi : (\tilde{\mathcal{O}}/\tilde{\mathfrak{p}}^e)^* \to \mathbb{C}^*$. A strongly primitive character of $(\tilde{\mathcal{O}}/\tilde{\mathfrak{p}})^*$ is one that does not factor through the norm map $N : \tilde{\mathcal{O}}/\tilde{\mathfrak{p}} \to \mathcal{O}/\mathfrak{p}$. See [AOP], 5.2, for the definition of strongly primitive characters for general e. We denote by $\Theta_e(\xi)$ the cuspidal representation of $\mathrm{GL}_2(R_e)$ corresponding to ξ. Fix an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$, and from now on we view ξ and $\Theta_e(\xi)$ as p-adic representations.

In this note we compute the Jordan-Hölder constituents of $\overline{\Theta_e(\xi)}$, the reduction mod p of $\Theta_e(\xi)$, and use the notions introduced to reformulate the Serre-type conjecture for Hilbert modular forms of [Sch]. See the last section for some remarks about motivation.

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1.2. **Brauer characters.** Let $\Theta_e(\xi)$ be a cuspidal representation of $\mathrm{GL}_2(R_e)$. The Jordan-Hölder constituents of $\overline{\Theta_e(\xi)}$ are determined by its Brauer character, hence by the values of the character of $\Theta_e(\xi)$ at p-regular conjugacy classes. The p-regular conjugacy classes of $\mathrm{GL}_2(R_e)$ are sent by the natural surjection $\pi : \mathrm{GL}_2(R_e) \to \mathrm{GL}_2(R_1)$ to p-regular conjugacy classes of $\mathrm{GL}_2(R_1)$. Moreover,

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since the kernel of π is a p-group, irreducible mod p representations of $\mathrm{GL}_2(R_e)$ factor through π; see [Edi] for a proof of this. Thus the character of $\Theta_e(\xi)$ is constant on all p-regular conjugacy classes of $\mathrm{GL}_2(R_e)$ lying above a given conjugacy class of $\mathrm{GL}_2(R_1)$, and we will abusively write characters of $\mathrm{GL}_2(R_e)$-representations as though they were functions on conjugacy classes of $\mathrm{GL}_2(R_1)$.

Let $r \in R_1^*$ be a non-square element. Then we have an R_1-algebra embedding $i : \tilde{R}_1 \to \mathrm{M}_2(R_1)$ given by

$$a + b\sqrt{r} \mapsto \begin{pmatrix} a & rb \\ b & a \end{pmatrix}, a, b \in R_1.$$

Let $\mathcal{X}$ be a set of representatives of equivalence classes in $(R_1^*)^2$ under the equivalence relation $(x, y) \sim (y, x)$. Then the following is a list of representatives of p-regular conjugacy classes of $\mathrm{GL}_2(R_1)$:

$$\begin{aligned} m(x,y) &= \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad (x,y) \in \mathcal{X} \\ i(z), & \qquad\qquad z \in (\tilde{R}_1^* - R_1^*)/\mathrm{Gal}(\tilde{R}_1/R_1). \end{aligned}$$

Consider $\xi : \tilde{R}_e^* \to \overline{\mathbb{Q}}_p^*$. Its reduction modulo p is a character $\bar{\xi} : \tilde{R}_e^* \to \overline{\mathbb{F}}_p^*$ that factors through the map $\tilde{R}_e^* \to \tilde{R}_1^*$, whose kernel is a p-group. Let $\chi'_\xi : \tilde{R}_1^* \to \overline{\mathbb{Q}}_p^*$ be the canonical lift of the resulting character, and let $\chi_\xi = \chi'_\xi|_{R_1^*}$, where we use the embedding of $R_1 \hookrightarrow \tilde{R}_1$ that is implicit in i. It follows from [AOP], Theorem C, that the Brauer character β_ξ of $\overline{\Theta_e(\xi)}$ is:

$$\begin{aligned} \beta_\xi(m(x,y)) &= \begin{cases} q^{e-1}(q-1)\chi_\xi(x) & : x = y \\ 0 & : x \neq y \end{cases} \\ \beta_\xi(i(z)) &= (-1)^e(\chi'_\xi(z) + \chi'_\xi(z^q)). \end{aligned} \tag{1}$$

We note that the conjugacy classes of $\mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$ are computed in [CDSG]. Its character table is given in [DD]; the characters of the cuspidal representations are the family denoted $\chi_{ij}^{p(p-1)}$ there.

1.3. **Weights.** The notation in this section will mostly follow [Dia]. Recall ([BL], Prop. 1.1) that the distinct irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(\mathcal{O}/\mathfrak{p}) = \mathrm{GL}_2(R_1)$ are

$$\bigotimes_{\mu : R_1 \hookrightarrow \overline{\mathbb{F}}_p} \left(\left(\det{}^{m_\mu} \otimes \mathrm{Sym}^{n_\mu - 1} R_1^2 \right) \otimes_{R_1, \mu} \overline{\mathbb{F}}_p \right), \tag{2}$$

where $1 \leq n_\mu \leq p$ and $0 \leq m_\mu \leq p - 1$ for each μ, and the m_μ are not all $p - 1$. These are called weights. Let I be the set of field embeddings $R_1 \hookrightarrow \overline{\mathbb{F}}_p$, and fix a labeling $\mu_0, \mu_1, \ldots, \mu_{f-1}$ of its elements such that $\mu_i = (\mu_{i-1})^p$ for all i. We write the representation in (2) as $V_{m,\vec{n}}$, where $m = \sum_{i=0}^{f-1} m_{\mu_i} p^i$ and $\vec{n}$ is the vector $(n_{\mu_0}, n_{\mu_1}, \ldots, n_{\mu_{f-1}})$. Clearly $0 \leq m \leq q - 2$, and we can recover the m_{μ_i} by writing m in base p.

The Brauer character $\beta_{m,\vec{n}}$ of $V_{m,\vec{n}}$ is not hard to compute. We copy it from [Dia] for easy reference. For each $0 \leq i \leq f - 1$, let $\mu'_i : \tilde{R}_1 \hookrightarrow \overline{\mathbb{F}}_p$ be either of the two field embeddings lifting μ_i

to $\tilde{R}_1$. Denote the canonical lift of μ_i (resp. μ_i') to a character $R_1^* \to \overline{\mathbb{Q}}_p^*$ (resp. $\tilde{R}_1^* \to \overline{\mathbb{Q}}_p^*$), by τ_i (resp. τ_i'). Then,

$$\begin{aligned}
\beta_{m,\vec{n}}(m(x,y)) &= \prod_{i=0}^{f-1} \left(\tau_i(xy)^{m_{\mu_i}} \sum_{\nu=0}^{n_{\mu_i}-1} \tau_i(y)^{\nu} \tau_i(x)^{n_{\mu_i}-1-\nu} \right) \\
\beta_{m,\vec{n}}(i(z)) &= \prod_{i=0}^{f-1} \left(\tau_i'(z)^{(q+1)m_{\mu_i}} \sum_{\nu=0}^{n_{\mu_i}-1} \tau_i'(z)^{n_{\mu_i}-1+(q-1)\nu} \right).
\end{aligned}$$

The reader can easily check that this expression is independent of the choice of μ_i'.

Finally, suppose that the character $\xi : \tilde{R}_1^* \to \overline{\mathbb{Q}}_p^*$ factors through the norm $N : \tilde{R}_1 \to R_1$. We will define a mod p virtual representation $\overline{\Theta_1(\xi)}$, which will simplify our arguments in the next section. The equality $\xi = (\tau_0')^a$ holds for some a divisible by $q+1$. Denote by $\vec{p}$ the vector $(p, p, \ldots, p)$, and define $\vec{1}$ similarly. Let $a = m(q+1)$, where $0 \le m < q-1$, and set

$$\overline{\Theta_1(\xi)} = V_{m,\vec{p}} - V_{m,\vec{1}}. \tag{3}$$

It is easy to check that the Brauer character β_ξ of this virtual representation satisfies (1).

2. Jordan-Hölder constituents

Let Γ_e be the Grothendieck group of virtual $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(R_e)$. If $e' \ge e$, then inflation of representations from $\mathrm{GL}_2(R_e)$ to $\mathrm{GL}_2(R_{e'})$ induces a map $\Gamma_e \to \Gamma_{e'}$. In the sequel we will abusively consider elements of Γ_e as lying in $\Gamma_{e'}$ for $e' \ge e$. As an element of the Grothendieck group, $\overline{\Theta_e(\xi)}$ clearly depends only on χ_ξ'. We write $\overline{\Theta_e(a)}$ for $\overline{\Theta_e(\xi)}$, where a is such that $\chi_\xi' = (\tau_0')^a$. For any integers a and w, we define an element $P(a,w)$ of Γ_1, hence of any Γ_e, as follows:

$$P(a,w) = \overline{\Theta_1(a-(q-1)w)}.$$

The element $P(a,w)$ may be described explicitly. If $a-(q-1)w$ is divisible by $q+1$, then $P(a,w)$ is described by (3) above. Otherwise, we can write $a-(q-1)w = (q+1)r + b$, where $1 \le b \le q$. Now express $b = 1 + \sum_{i=0}^{f-1} b_i p^i$, where $0 \le b_i \le p-1$. Recall that I is the set of embeddings $R_1 \hookrightarrow \overline{\mathbb{F}}_p$. For any $S \subset I$ and $\mu_i \in I$ we define $\delta_S(\mu_i)$ to be 1 if $\mu_{i-1} \in S$ and 0 otherwise. Then $P(a,w) = \sum_{S \subset I} V_{m_S,\vec{n}_S}$, where for a subset $S \subset I$, we define m_S and $\vec{n}_S$ as follows (see [Dia], Prop.

1.3). Set $m_{S,0} = \delta_S(\mu_0)$ if $\mu_0 \in S$ and $m_{S,0} = b_0 + 1$ if $\mu_0 \notin S$. Then,

$$m_S \equiv m_{S,0} + \sum_{\substack{i=1 \\ \mu_i \notin S}}^{f-1} (b_i + \delta_S(\mu_i))p^i + r \mod q-1$$

$$n_{S,\mu_i} = \begin{cases} b_i + 1 - \delta_S(\mu_i) & : \mu_i = \mu_0 \in S \\ p - b_i - 1 + \delta_S(\mu_i) & : \mu_i = \mu_0 \notin S \\ b_i + \delta_S(\mu_i) & : \mu_i \neq \mu_0, \mu_i \in S \\ p - b_i - \delta_S(\mu_i) & : \mu_i \neq \mu_0, \mu_i \notin S. \end{cases}$$

Here we make the convention that if $n_{S,\mu_i} = 0$ for any i, then $V_{m_S,\vec{n}_S} = 0$. Generically $P(a,w)$ is a sum of 2^f weights, but it may have fewer summands. For instance, if $f = 1$ and $a - (p-1)w = (p+1)r + b$, then the set of Jordan-Hölder constituents of $\overline{\Theta_1(a-(p-1)w)}$ is $\{V_{1+r,b-1} + V_{b+r,p-b}\}$. Each constituent appears with multiplicity one. Note that if $b \in \{1,p\}$, then $P(a,w)$ is a single weight and not a sum of two weights.

Lemma 2.1. *The Brauer character of $P(a,w)$ is the following:*

$$\beta_{P(a,w)}\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right) = \begin{cases} (q-1)\tau_0(x)^a & : x = y \\ 0 & : x \neq y \end{cases}$$

$$\beta_{P(a,w)}(i(z)) = -\tau_0'(z)^a(\tau_0'(z)^{(q-1)(a+w)} + \tau_0'(z)^{-(q-1)w}).$$

Proof. This is immediate from (1) above. □

Lemma 2.2. *The following equality holds in the Grothendieck group of* $\mathrm{GL}_2(R_2)$*:*

$$\overline{\Theta_2(a)} = \sum_{w=1}^{q} P(a,w).$$

Proof. We need to show that the Brauer characters of the summands on the right-hand side of the formula above add up to the Brauer character of $\overline{\Theta_2(a)}$. This claim is obvious for the conjugacy classes $m(x,y)$. Set $\eta(z) = \tau_0'(z)^{q-1}$. By Lemma 2.1, we see that

$$\sum_{w=1}^{q} \beta_{P(a,w)}(i(z)) = -\tau_0'(z)^a \sum_{w=1}^{q} (\eta(z)^{a+w} + \eta(z)^{-w}).$$

Since $\eta(z) \neq 1$, we have $\sum_{y=0}^{q} \eta(z)^y = 0$, and therefore

$$\sum_{w=1}^{q} \beta_{P(a,w)}(i(z)) = \tau_0'(z)^a(\eta(z)^a + 1) = \tau_0'(z)^a + \tau_0'(z^q)^a,$$

which by (1) is the desired result. □

We will now give a recursive description of the Jordan-Hölder constituents of $\overline{\Theta_e(a)}$ for all $e \geq 1$ and all a. Given $e \geq 1$ and integers a and w, we define the following element $P_e(a, w)$ of the Grothendieck group of $\mathrm{GL}_2(R_e)$:

$$\begin{aligned} P_e(a,0) &= \begin{cases} P(a,0) & : e = 1 \\ \sum_{w=1}^{q} P_{e-1}(a,w) & : e > 1 \end{cases} \\ P_e(a,w) &= P_e(a-(q-1)w, 0) \end{aligned}$$

Theorem 2.3. *In the Grothendieck group of* $\mathrm{GL}_2(R_e)$*, for* $e > 1$*, we have the equality*

$$\overline{\Theta_e(a)} = P_e(a,0) = \sum_{w=1}^{q} P_{e-1}(a,w).$$

Proof. By the argument of Lemma 2.2 and induction on e, the Brauer character of $P_e(a, w)$ is:

$$\begin{aligned} \beta_{P_e(a,w)}(m(x,y)) &= \begin{cases} q^{e-1}(q-1)\tau_0(x)^a & : x = y \\ 0 & : x \neq y \end{cases} \\ \beta_{P_e(a,w)}(i(z)) &= (-1)^e \tau_0'(z)^a(\eta(z)^{a+w} + \eta(z)^{-w}), \end{aligned}$$

where $\eta(z) = \tau_0'(z)^{q-1}$ as before. The theorem follows by comparing Brauer characters. ☐

Remark 2.4. Observe that since $\overline{\Theta_e(a)}$ is an actual representation of $\mathrm{GL}_2(R_e)$ and not just a virtual representation, every irreducible mod p representation of $\mathrm{GL}_2(R_e)$ appears with non-negative multiplicity in $P_e(a, 0)$. We thus obtain a recursive formula for the Jordan-Hölder constituents of $\overline{\Theta_e(a)}$. Moreover, it follows from the definition of strongly primitive characters in [AOP] 5.2 that if $e \geq 2$, then $\overline{\Theta_e(a)}$ is the reduction modulo p of an irreducible cuspidal representation of $\mathrm{GL}_2(R_e)$ for all a.

3. Weights in Serre's conjecture

In this section we reformulate the Serre-type conjecture for Hilbert modular forms of [Sch], using the notions introduced earlier. First we recall the form of the conjecture.

Let F be a totally real field, p an odd rational prime, and $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ a continuous, irreducible, totally odd Galois representation. A weight is an irreducible $\overline{\mathbb{F}}_p$-representation of the finite group $\mathrm{GL}_2(\mathcal{O}_F/p)$. Any weight factors through the quotient $\prod_{v|p} \mathrm{GL}_2(\mathcal{O}_F/v)$, since the kernel of this quotient is a p-group. One can define what it means for ρ to be modular of a given weight; see, for instance, [Sch] §2. Serre's conjecture has been generalized to this situation; conjectures due to Buzzard, Diamond, and Jarvis [BDJ] when p is unramified in F and to the author [Sch] for general F specify a list $W(\rho)$ of modular weights for ρ. We note that the conjecture of [Sch] is formulated only when ρ is tamely ramified at all places dividing p. Moreover, there exist sets

$W_v(\rho)$ of irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(\mathcal{O}_F/v)$ for each prime v of F dividing p such that

$$W(\rho) = \left\{ \sigma = \otimes_{v|p} \sigma_v : \forall v, \sigma_v \in W_v(\rho) \right\}.$$

Let $\mathfrak{p}$ be a place of F dividing p, suppose that ρ is tame at $\mathfrak{p}$, let $G_{\mathfrak{p}} \subset \mathrm{Gal}(\overline{F}/F)$ be a decomposition subgroup at $\mathfrak{p}$, and let $I_{\mathfrak{p}} \subset G_{\mathfrak{p}}$ be the inertia. Let $K = F_{\mathfrak{p}}$, let π be a uniformizer, and write K^{nr} for the maximal unramified extension of K. As before let $q = p^f$ be the cardinality of the residue field k of K, and denote by $I = \{\mu_0, \ldots, \mu_{f-1}\}$ be the set of embeddings $k \hookrightarrow \overline{\mathbb{F}}_p$, where the labeling is chosen so that $\mu_i = \mu_{i-1}^p$. Let k' be a quadratic extension of k ($\tilde{R}_1$ in the notation of the previous sections), and let $\mu'_0, \mu'_1, \ldots, \mu'_{2f-1}$ be the collection of embeddings $k' \hookrightarrow \overline{\mathbb{F}}_p$, labeled so that $\mu'_i = (\mu'_{i-1})^p$ and so that, for $0 \leq i \leq f-1$, we have $(\mu'_i)|_k = \mu_i$.

Suppose that the restriction of ρ to $G_{\mathfrak{p}}$ is irreducible; it follows that the restriction of ρ to $I_{\mathfrak{p}} \simeq \mathrm{Gal}(\overline{K}/K^{nr})$ factors through $\mathrm{Gal}(L/K^{nr}) \simeq (k')^* \simeq \mathbb{F}_{q^2}^*$, where $L = K^{nr}(\pi^{1/(q^2-1)})$ is the totally tamely ramified extension of K^{nr} of degree $q^2 - 1$, and that

$$\rho|_{I_{\mathfrak{p}}} \sim \begin{pmatrix} \phi & 0 \\ 0 & \phi^q \end{pmatrix},$$

where $\phi : (k')^* \to \overline{\mathbb{F}}_p^*$ is a character such that $\phi \neq \phi^q$. Let $\Theta(\phi)$ be the cuspidal representation of $\mathrm{GL}_2(k)$ associated to the canonical lift of ϕ. We say that a character $\xi : (k')^* \to \overline{\mathbb{F}}_p^*$ is indecomposable if $\xi^q \neq \xi$.

If V is any $\overline{\mathbb{F}}_p$-representation, we write $JH(V)$ for the set of its Jordan-Hölder constituents. Let e be the ramification index of $K/\mathbb{Q}_p$, and let $\Delta \subset \mathbb{Z}^f$ be the collection of f-tuples $(\delta_{\mu_0}, \delta_{\mu_1}, \ldots, \delta_{\mu_{f-1}})$ such that $0 \leq \delta_\mu \leq e-1$ for each $\mu \in I$. Let $Y_{\mathfrak{p}}$ be the set of irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(k)$. Then for each $\delta \in \Delta$ we defined in [Sch] a multi-valued map $\mathcal{R}_e^\delta : Y_{\mathfrak{p}} \to Y_{\mathfrak{p}}$ and conjectured that the set of ($\mathfrak{p}$-components of) modular weights of ρ is

$$W_{\mathfrak{p}}(\rho) = \bigcup_{\delta \in \Delta} \mathcal{R}_e^\delta(JH(\overline{\Theta(\phi)})).$$

Herzig observed in [Her], §11 that the conjecture of Buzzard, Diamond, and Jarvis [BDJ], which addresses the unramified case $e = 1$, could be reformulated in this way. In that case, $\Delta = \{\vec{0}\}$, and the definition of $W_{\mathfrak{p}}(\rho)$ involves a single map $\mathcal{R} = \mathcal{R}_1^{\vec{0}}$. We will now reformulate Conjecture 1 of [Sch] so that it involves only the map $\mathcal{R}$, rather than a collection of maps that depends on the ramification of K.

Given an f-tuple $\delta \in \Delta$, we define an integer $w(\delta) = \sum_{i=0}^{f-1} \delta_{\mu_i} p^i$. Now set $d = \sum_{i=0}^{f-1} p^i = (q-1)/(p-1)$, and for $\delta \in \Delta$ let ξ_δ be the character $\phi \cdot \mu_0^{-w(\delta)} \cdot (\mu'_0)^{2w(\delta)-(e-1)d} : (k')^* \to \overline{\mathbb{F}}_p^*$. If $\mu_i \in I$, where $0 \leq i \leq f-1$, we write δ_{μ_i} for δ_i. Let Δ' be the set of $\delta \in \Delta$ such that ξ_δ does not factor through the norm map $N_{k'/k}$.

Proposition 3.1. *If the notation is as above and $\rho|_{G_\mathfrak{p}}$ is irreducible, then*

$$W_\mathfrak{p}(\rho) = \bigcup_{\delta\in\Delta'} \mathcal{R}(JH(\overline{\Theta(\xi_\delta)})).$$

Proof. Let $\sigma = \bigotimes_{\mu\in I}((\det^{m_\mu} \otimes \mathrm{Sym}^{k_\mu-2}k^2)\otimes_{k,\mu}\overline{\mathbb{F}}_p)$ be an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_2(k)$, where $2\le k_\mu \le p+1$ for every $\mu\in I$. Suppose that $\sigma\in W_\mathfrak{p}(\rho)$. Then by [Sch], Theorem 2.4, for each μ there exists a $\delta\in\Delta$ and a labeling $\{\alpha_\mu,\beta_\mu\}$ of the two embeddings $k'\hookrightarrow\overline{\mathbb{F}}_p$ lifting μ such that

$$\phi = \prod_{\mu\in I}\mu^{m_\mu}\prod_{\mu}\alpha_\mu^{k_\mu-1+\delta_\mu}\beta_\mu^{e-1-\delta_\mu}. \tag{4}$$

Let $T\subset I$ be the set of $\mu\in I$ such that $\alpha_\mu=\mu'_i$ with $0\le i\le f-1$. Define $\delta'\in\Delta$ by $\delta'_\mu = e-1-\delta_\mu$ for $\mu\in T$ and $\delta'_\mu=\delta_\mu$ for $\mu\notin T$. Then,

$$\begin{aligned}\phi\cdot(\mu'_0)^{2w(\delta')-(e-1)d} &= \prod_{\mu\in I}\mu^{m_\mu}\prod_{\mu\in T}\alpha_\mu^{k_\mu-2+e-\delta_\mu}\beta_\mu^{e-1-\delta_\mu}\prod_{\mu\notin T}\alpha_\mu^{k_\mu-1+\delta_\mu}\beta_\mu^{\delta_\mu} = \\ &\prod_{\mu\in I}\mu^{m_\mu}\prod_{\mu\in T}\mu^{e-1-\delta_\mu}\alpha_\mu^{k_\mu-1}\prod_{\mu\notin T}\mu^{\delta_\mu}\alpha_\mu^{k_\mu-1}. \\ \xi_{\delta'} &= \prod_{\mu\in I}\mu^{m_\mu}\alpha_\mu^{k_\mu-1}.\end{aligned}$$

If $\xi_{\delta'}$ is indecomposable, then it follows from [Her], Theorem 11.3, that $\sigma\in\mathcal{R}(JH(\overline{\Theta(\xi_{\delta'})}))$.

From the expression above it is easy to see that $\xi_{\delta'}$ is decomposable if and only if there exist numbers $-1=r_0<r_1<r_2<\cdots<r_s=f-1$ such that, possibly after a cyclic relabeling of the embeddings μ_i, the set I can be split into intervals $I=\{\mu_0,\mu_1,\ldots,\mu_{r_1}\}\cup\{\mu_{r_1+1},\mu_{r_1+2}\ldots,\mu_{r_2}\}\cup\cdots\cup\{\mu_{r_{s-1}+1},\ldots,\mu_{r_s}\}$ with the following properties. Each interval contains at least two elements, and for every such interval $I_i=\{\mu_{r_{i-1}+1},\ldots,\mu_{r_i}\}$ we have $k_{r_{i-1}+1}=p+1$ and $k_{r_{i-1}+2}=\cdots=k_{r_i-1}=p$ and $k_{r_i}=2$. Moreover, $T\subset I$ must be such that for each I_i we have either $T\cap I_i=\{\mu_{r_i}\}$ or $T\cap I_i = I_i-\{\mu_{r_i}\}$.

Consider the interval I_1, and suppose that $T\cap I_1=\{\mu_{r_1}\}$; the other case is analogous. We must have $e>1$, since otherwise it is easy to see that $\phi^q=\phi$. Then at least one of δ_{r_1} and $e-1-\delta_{r_1}$ must be non-zero. Suppose that $\delta_{r_1}\ne 0$. We write α_i for α_{μ_i} and similarly with β_i. Then we have $\alpha_i^p=\alpha_{i+1}$ for $0\le i\le r_1-1$ and $\alpha_{r_1-1}^p=\beta_{r_1}$. Hence the piece of ϕ corresponding to the elements of I_1 is:

$$\begin{aligned}&\left(\prod_{i=0}^{r_1}\mu_i^{m_{\mu_i}}\right)\alpha_0^{p+\delta_0}\beta_0^{e-1-\delta_0}\left(\prod_{i=1}^{r_1-1}\alpha_i^{p-1+\delta_i}\beta_i^{e-1-\delta_i}\right)\alpha_{r_1}^{1+\delta_{r_1}}\beta_{r_1}^{e-1-\delta_{r_1}} = \\ &\qquad\left(\prod_{i=0}^{r_1}\mu_i^{m_{\mu_i}}\right)\alpha_0^{\delta_0}\beta_0^{p+e-1-\delta_0}\left(\prod_{i=1}^{r_1-1}\alpha_i^{\delta_i}\beta_i^{p-1+(e-1-\delta_i)}\right)\alpha_{r_1}^{1+(\delta_{r_1}-1)}\beta_{r_1}^{e-1-(\delta_{r_1}-1)}\end{aligned}$$

Define an f-tuple $\tilde{\delta} \in \Delta$ as follows: if $0 \leq i \leq r_1 - 1$, then $\tilde{\delta}_i = e - 1 - \delta_i$. Also $\tilde{\delta}_{r_1} = \delta_{r_1} - 1$ and $\tilde{\delta}_i = \delta_i$ for $i > r_1$. Then the expression above shows that ϕ can be rewritten in the form of (4) for $\tilde{\delta}$ instead of δ. The corresponding subset $\tilde{T} \subset I$ satisfies $\tilde{T} \cap I_1 = I_1$ and $\tilde{T} \cap (I - I_1) = T \cap (I - I_1)$. Therefore, if $\tilde{\delta}' \in \Delta$ is obtained from $\tilde{\delta}$ in the same way that δ' was obtained from δ (in fact, $\tilde{\delta}'_{r_1} = \delta'_{r_1} + 1$ and $\tilde{\delta}'_i = \delta'_i$ for $i \neq r_1$), then we see as above that $\tilde{\delta}' \in \Delta'$ and $\sigma \in \mathcal{R}(JH(\overline{\Theta(\xi_{\tilde{\delta}'})}))$. The case of $e - 1 - \delta_{r_1} \neq 0$ is dealt with similarly.

Conversely, if $\sigma \in \mathcal{R}(JH(\overline{\Theta(\xi_\delta)}))$ for some $\delta \in \Delta'$, then the same argument in reverse shows that $\sigma \in W_{\mathfrak{p}}(\rho)$. □

Let $\chi, \chi' : k^* \to \overline{\mathbb{Q}}_p^*$ be two characters. If $B(k) \subset \mathrm{GL}_2(k)$ is the subgroup of upper triangular matrices, then we obtain a character $\chi \otimes \chi' : B(k) \to \overline{\mathbb{Q}}_p^*$ by setting

$$\chi \otimes \chi' : \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi(a)\chi'(d).$$

Then $I(\chi, \chi') = \mathrm{Ind}_{B(k)}^{\mathrm{GL}_2(k)}(\chi \otimes \chi')$ is a p-adic representation of $\mathrm{GL}_2(k)$ and is irreducible when $\chi \neq \chi'$. If $\rho|_{G_{\mathfrak{p}}}$ is reducible, then $\rho|_{I_{\mathfrak{p}}}$ factors through k^*, and, since ρ is assumed to be tame at $\mathfrak{p}$,

$$\rho|_{I_{\mathfrak{p}}} \sim \begin{pmatrix} \phi & 0 \\ 0 & \phi' \end{pmatrix}$$

for some characters $\phi, \phi' : k^* \to \overline{\mathbb{F}}_p^*$. In this case, a similar argument to the above, using [Sch], Theorem 2.5, proves the following:

Proposition 3.2. *If $\rho|_{G_{\mathfrak{p}}}$ is reducible and tamely ramified, then*

$$W_{\mathfrak{p}}(\rho) = \bigcup_{\delta \in \Delta} \mathcal{R}(JH(\overline{I(\phi \cdot \tau_0^{w(\delta)-(e-1)d}, \phi' \cdot \tau_0^{-w(\delta)})})).$$

In the unramified case $e = 1$, Herzig's restatement in [Her] §11 of the conjecture of [BDJ] discovered a remarkable correspondence between irreducible characteristic zero representations of $\mathrm{GL}_2(k) = \mathrm{GL}_2(R_1)$ and restrictions to inertia $I_{\mathfrak{p}}$ of mod p Galois representations that are tame at $\mathfrak{p}$. A Galois representation ρ corresponds to a representation $V(\rho)$ of $\mathrm{GL}_2(k)$ such that $W_{\mathfrak{p}}(\rho) = \mathcal{R}(JH(\overline{V(\rho)}))$. Locally irreducible (resp. reducible) Galois representations correspond to cuspidal representations (resp. principal series). Our motivation for computing the Jordan-Hölder constituents of the reductions modulo p of representations of $\mathrm{GL}_2(R_e)$ was a hope that this correspondence could be generalized to all e and still be characterized in a similar way using the conjectural sets of modular weights. This hope failed, as for $e \geq 2$ Theorem 2.3 shows that all weights with the appropriate central character appear as constituents of $\overline{\Theta(\xi)}$. However, Propositions 3.1 and 3.2 establish a correspondence between restrictions to inertia of tamely ramified $\overline{\mathbb{F}}_p$-representations of $\mathrm{Gal}(\overline{F_{\mathfrak{p}}}/F_{\mathfrak{p}})$ and *collections*, generically of cardinality e^f, of characteristic zero representations of $\mathrm{GL}_2(k)$.

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