ALGEBRAIC LOGIC AND LOGICALLY-GEOMETRIC TYPES IN VARIETIES OF ALGEBRAS

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The main objective of this paper is to show that the notion of type which was developed within the frames of logic and model theory has deep ties with geometric properties of algebras. These ties go back and forth from universal algebraic geometry to the model theory through the machinery of algebraic logic. We show that types appear naturally as logical kernels in the Galois correspondence between filters in the Halmos algebra of first order formulas with equalities and elementary sets in the corresponding affine space.

Keywords: Model theoretic type; logically geometric type; isotypic algebras; logical kernel of a point; elementary (definable) set; Galois correspondence; variety of algebras.

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1. Introduction

The main objective of the paper is to show that the notion of type which was developed within the frames of logic and model theory has deep ties with geometric properties of algebras. These ties go back and forth from universal algebraic geometry to model theory through the machinery of algebraic logic.

More precisely, we shall show that types appear naturally as logical kernels in the Galois correspondence between filters in the Halmos algebra of first order formulas over equalities and elementary sets in the corresponding affine space. Note that in our terminology the term "elementary set" has the meaning of "definable set" in the standard model theoretic terminology. This Galois correspondence generalizes classical Galois correspondence between ideals in the polynomial algebra and algebraic sets in the affine space. The sketch of the ideas of universal algebraic geometry can be found in [39], [41], [42], [44], [45], [3], [36], [29], [7] [8], [9],[10], [5], [6], [23], [24], [48], [49], etc. As for standard definitions of model theory, we refer to monographs [35], [47], [4], [18], etc. For the exposition of concepts and results of algebraic logic see [11] – [15], [16], [20], [21], [2], [1], etc.

Methodologically, in the paper we give a sketch of some ideas which provide interactions of algebraic logic with geometry, model theory and algebra. We believe that a development of the described approach can make benefits to each of these areas. We shall stress that the paper does not contain a bunch of new results. Its main duty is to specialize new problems and to underline common points of algebra, logic and geometry through the notion of the type.

The paper is organized as follows. Section 2 is devoted to structures of algebraic logic. We define here various kinds of Halmos algebras, consider the value homomorphism and provide the reader with the main examples of algebras under consideration. Section 3 deals with basic approaches of universal algebraic geometry. We define the general Galois correspondence which plays the important role in all considerations. The description of this correspondence starts from the classical case and extends to the case of multi-sorted logical geometry over an arbitrary variety of algebras. In Section 4 we recall the model theoretic notion of a type. In Section 5 we concentrate attention on types from the positions of one-sorted algebraic logic over given variety Θ. Section 6 deals with the ideas of universal logical geometry which give rise to LG-types and their geometric description. We finish the paper with the list of problems appearing in the context of previous considerations.

2. Structures of Algebraic Logic

2.1. *One-sorted case*

We consider algebra and logic with respect to a given variety of algebras Θ. This point of view (cf. [43]) implies some differences with the original notions introduced by P. Halmos ([11] – [15], see [31] for non-homogenious polyadic algebras). For the sake of convenience, in this section we provide the reader with all necessary definitions. It will be emphasized that the transition from pure logic to logic in Θ is caused by many reasons, and we would like to distinguish the needs of universal algebraic geometry among them.

Denote by Ω the signature of operations in algebras from Θ. Let $W(X)$ denote the free in Θ algebra over a non-empty set of variables X. In the meantime we assume that each X is a subset of some infinite set of variables X^0.

We shall recall the well-known definitions of the existential and universal quantifiers which are considered as new operations on Boolean algebras (see [11]).

Let B be a Boolean algebra. The mapping $\exists : B \to B$ is called an *existential* quantifier if

1. $\exists(0) = 0$,
2. $a \leq \exists(a)$,
3. $\exists(a \wedge \exists b) = \exists a \wedge \exists b$.

The *universal* quantifier $\forall : B \to B$ is defined dually:

1. $\forall(1) = 1$,
2. $a \geq \forall(a)$,
3. $\forall(a \vee \forall b) = \forall a \vee \forall b$.

Here the numerals 0 and 1 are zero and unit of the Boolean algebra B and a, b are arbitrary elements of B. Symbol $=$ means coincidence of elements in Boolean algebra, i.e., $a \leq b$ and $b \leq a$ is written as $a = b$, $a, b \in B$. The quantifiers $\exists$ and $\forall$ are coordinated by: $\neg(\exists a) = \forall(\neg a)$, i.e., $(\forall a) = \neg(\exists(\neg a))$.

A pair $(B, \exists)$, where B is a Boolean algebra and $\exists$ is the existential quantifier, is a *monadic* algebra (see [11]).

Definition 2.1. A Boolean algebra B is a *quantifier X-algebra* if a quantifier $\exists x \colon B \to B$ is defined for every variable $x \in X$, and

$$\exists x \exists y = \exists y \exists x,$$

for every $x, y \in X$.

Remark 2.1. See also the definition of diagonal-free cylindric algebras of Tarski e.a. [16].

Remark 2.2. According to [11], [43] a Boolean algebra B is a *quantifier X-algebra* if a quantifier $\exists(Y) \colon B \to B$ is defined for every subset $Y \subset X$, and

1. $\exists(\varnothing) = I_B$, the identity function on B,
2. $\exists(X_1 \bigcup X_2) = \exists(X_1)\exists(X_2)$, where X_1, X_2 are subsets in X.

If we restrict ourselves with finite nontrivial subsets of X, then these two definitions coincide, because condition 2) implies commutativity of quantifiers, and, conversely, one can define $\exists(Y) = \exists y_1 \cdots \exists y_k$, where $Y = \{y_1, \ldots, y_k\}$.

We shall consider also extended Boolean algebras over $W(X)$. We define an equality on a Boolean algebra B is symmetric, reflexive and transitive predicate $\equiv \colon W(X) \times W(X) \to B$ which takes a pair $w, w' \in W(X)$ to the constant in B denoted by $w \equiv w'$, subject to condition:

$$(w_1 \equiv w_1' \wedge \ldots \wedge w_n \equiv w_n') \leq (w_1 \ldots w_n \omega \equiv w_1' \ldots w_n' \omega),$$

where ω is an n-ary operation in Ω.

Definition 2.2. We call a Boolean algebra B an extended Boolean algebra over the free in Θ algebra $W(X)$, if

1. There are defined quantifiers $\exists x$ for all $x \in X$ in B with $\exists x \exists y = \exists y \exists x$ for all $x, y \in X$, i.e. B is a quantifier X-algebra.
2. To every pair $w, w' \in W(X)$ it corresponds a constant (called an equality) in B, denoted by $w \equiv w'$. Here,
   (a) $w_1 \equiv w_1' \leq w_1' \equiv w_1$.
   (b) $w \equiv w$ is the unit of the algebra B.
   (c) $w_1 \equiv w_2 \wedge w_2 \equiv w_3 \leq w_1 \equiv w_3$.
   (d) For every n-ary operation $\omega \in \Omega$, where Ω is a signature of the variety Θ, we have

$$w_1 \equiv w_1' \wedge \ldots \wedge w_n \equiv w_n' \leq w_1 \ldots w_n \omega \equiv w_1' \ldots w_n' \omega.$$

Remark 2.3. Endomorphisms of Boolean algebras leave constants $w \equiv w'$ unchanged.

Remark 2.4. Condition 2(d) means that equalities respect all operations on $W(X)$.

Definition 2.3. An algebra $\mathfrak{L} = \mathfrak{L}(X^0)$ is a Halmos algebra (one-sorted Halmos algebra) over $W(X^0)$, X^0 is infinite if:

1. $\mathfrak{L}$ is an extended Boolean algebra.
2. The action of the semigroup $End(W(X^0))$ is defined on $\mathfrak{L}$, so that for each $s \in End(W(X^0))$ there is the map $s_* : \mathfrak{L} \to \mathfrak{L}$ which preserves the Boolean structure of $\mathfrak{L}$.
3. The identities controlling the interaction of s_* with quantifiers are as follows:
   (a) $s_{1*} \exists x a = s_{2*} \exists x a,\ a \in \mathfrak{L}$, if $s_1(y) = s_2(y)$ for every $y \in X^0,\ y \neq x$.
   (b) $s_* \exists x a = \exists (s(x))(s_* a),\ a \in \mathfrak{L}$, if $s(x) = y$ and y is a variable which does not belong to the support of $s(x')$, for every $x' \in X^0$, and $x' \neq x$.
   This condition means that y does not participate in the shortest expression of the element $s(x') \in W(X^0)$ through the elements of X^0.
4. The identities controlling the interaction of s_* with equalities are as follows:
   (a) $s_*(w \equiv w') = (s(w) \equiv s(w'))$.
   (b) $(s_w^x)_* a \wedge (w \equiv w') \leq (s_{w'}^x)_* a$, where $a \in \mathfrak{L}$, and $s_w^x \in End(W(X^0))$ is defined by $s_w^x(x) = w$, and $s_w^x(x') = x'$, for $x' \neq x$.

For the definition of support see [43], Chapter 9, Section 1.

Remark 2.5. The set X^0 in the definition 2.3 must be infinite because otherwise $End(W(X^0))$ does not act on B (see [43], Chapter 8, Section 2 for the details) in the case of free Halmos algebras. In general this condition is superfluous since we require the action of the semigroup $End(W(X^0))$ on the algebra $\mathfrak{L}$.

Remark 2.6. Definition 2.3 introduces algebras which are very close to polyadic algebras of Halmos (see [11]) defined over an infinite set of variables X^0. The main difference between these classes comes from the desire to specialize an algebraization of first order logic to an arbitrary variety of algebras Θ. This means that instead of action of the semigroup of transformations $End(X^0)$ of the set X^0, we consider the action of the bigger semigroup $End(W(X^0))$ as the semigroup of Boolean endomorphisms. We also consider equalities of the type $w \equiv w'$ instead of the ones $x \equiv y$ for polyadic algebras.

Remark 2.7. Axioms 3(a) and 3(b) which look messy, are grounded on major examples of Halmos algebras. In particular, we will see that Halmos algebras of the kind $Hal_\Theta(H)$ (see Example 1) satisfy these identities. If instead of $Hal_\Theta(H)$ we consider the Halmos algebra of formulas $\widetilde{\Phi}$ (see below), then the identity 3(a) corresponds to the well-known fact that it is possible to replace a quantified variable in a formula by another one. The identity 3(b) has a similar explanation (see [11]).

Remark 2.8. In [11], [43] an equality in Halmos algebras is defined as a reflexive binary predicate which satisfies conditions 4(a) and 4(b). Then, it can be checked [43], that this predicate is automatically symmetric and transitive.

Example 1. We give an example of a Halmos algebra which plays a crucial role in further considerations.

Let X be any set (finite or infinite), H an algebra in Θ. Consider the set $Hom(W(X), H)$ of all homomorphisms from $W(X)$ to H. Let $Bool(W(X), H)$ be the Boolean algebra of all subsets A in $Hom(W(X), H)$. Our aim is to make it an extended Boolean algebra.

Define, first, quantifiers $\exists x$, $x \in X$ on $Bool(W(X), H)$. We set $\mu \in \exists x A$ if and only if there exists $\nu \in A$ such that $\mu(y) = \nu(y)$ for every $y \in X$, $y \neq x$. It can be checked that $\exists x$ defined in such a way is, indeed, an existential quantifier.

According to Definition 2.2, for each pair $w, w' \in W(X)$ we shall define a constant (equality) in $Bool(W(X), H)$. For some reasons for this particular algebra we denote it by $Val^X_H(w \equiv w')$. We define this constant as follows:

$$Val^X_H(w \equiv w') = \{\mu \mid \mu(w) = \mu(w')\}.$$

Thus, the algebra $Bool(W(X), H)$ is equipped with the structure of an extended Boolean algebra (we omit verification of the necessary axioms).

Let X^0 now be an infinite set. Define the action of the semigroup $End(W(X^0))$ in $Bool(W(X^0), H)$. Every homomorphism $s \in End(W(X^0))$ gives rise to a Boolean homomorphism

$$s_* : Bool(W(X^0), H) \to Bool(W(X^0), H),$$

defined by the rule: for each $A \subset Hom(W(X^0), H)$ the point μ belongs to s_*A if $\mu s \in A$.

The signature of a Halmos algebra for $Bool(W(X^0), H)$ is now completed. Denote it by $Hal^{X^0}_\Theta(H)$. The algebra $Hal^{X^0}_\Theta(H)$ is, indeed, a Halmos algebra, (it is proved in [39], that all algebras $Hal_{\Theta}^{X^0}(H)$ for different $H \in \Theta$ generate the variety of Halmos algebras).

2.2. *Multi-sorted case*

Our next aim is to define multi-sorted Halmos algebras. There are many reasons to do that. Some of them are related to potential applications of algebraic logic in computer science, but some have purely algebraic nature. For instance, we need multi-sorted variant of Halmos algebras in order to work with finite dimensional affine spaces and to construct geometry related to first order calculus in arbitrary Θ.

Every multi-sorted algebra D can be written as $D = (D_i, i \in \Gamma)$, where Γ is a set of sorts, which can be infinite, and D_i is a domain of the sort i. We can regard domains D_i as algebras from some variety (for definitions see [30], [33]).

Every operation ω in D has a specific type $\tau = \tau(\omega)$. This notion generalizes the notion of the arity of an operation. In the multi-sorted case an operation ω of the type $\tau = (i_1, \dots, i_n; j)$ operates as a mapping $\omega : D_{i_1} \times \dots \times D_{i_n} \to D_j$. Homomorphisms of multi-sorted algebras act component-wise and have the form $\mu = (\mu_i, i \in \Gamma) : D \to D'$, where $\mu_i : D_i \to D'_i$ are homomorphisms of algebras and, besides that, every μ is naturally correlated with the operations ω.

Subalgebras, quotient algebras, and cartesian products of multi-sorted algebras are defined in the usual way. Hence, one can define varieties of multi-sorted algebras. In every such a variety there exist free algebras over multi-sorted sets, determined by multi-sorted identities.

It is worth noting that categories and multi-sorted algebras are tightly connected [17], [32]. So, define, first, Halmos categories.

Let Θ^0 be the category of free algebras of the variety Θ, and X, Y be (finite or infinite) subsets of X^0.

Definition 2.4. A category Υ is a Halmos category if:

1. Every its object has the form $\Upsilon(X)$, where $\Upsilon(X)$ is an extended Boolean algebra in Θ over $W(X)$.
2. Morphisms are of the form $s_* : \Upsilon(X) \to \Upsilon(Y)$, where every $s : W(X) \to W(Y)$ is a homomorphism in Θ^0, s_* is the homomorphism of Boolean algebras and the correspondence: $W(X) \to \Upsilon(X)$ and $s \to s_*$ determines a covariant functor $\Theta^0 \to \Upsilon$.
3. The identities controlling the interaction of morphisms with quantifiers and equalities repeat the ones from Definition 2.3, where the endomorphisms s from $End(W(X))$ are replaced by homomorphisms $s : W(X) \to W(Y)$.

Now we are able to define multi-sorted Halmos algebras associated with Halmos categories. For the aims of logical geometry we assume that the set of sorts Γ is the set of all finite subsets of the infinite set X^0.

Let X be a finite subset of X^0, $x \in X$. Consider an arbitrary $W(X)$ in Θ, and take the signature

$$L_X = \{\vee, \wedge, \neg, \exists x, M_X\}, \text{ for all } x \in X.$$

Here M_X is the set of all equalities $w \equiv w'$, $w, w' \in W(X)$ over the algebra $W(X)$. We treat equalities from M_X as nullary operations. Consider the signature

$$L_\Theta = \{\vee, \wedge, \neg, \exists x, M_X, s_*\}, \text{ for all } x \in X, \text{ and various } X \in \Gamma.$$

In the signature L_Θ the symbol s_* is reserved for operations of the type $\tau = (X; Y)$, where $X, Y \in \Gamma$. Each homomorphism $s : W(X) \to W(Y)$ induces the operation s_* of the type $\tau = (X; Y)$.

Remark 2.9. The condition on Γ is not necessary for the definition of Halsmos algebras and made for the needs of universal algebraic geometry. Halmos algebras can be defined for various choices of Γ. For example, the one-sorted Halmos algebra from Definition 2.3 corresponds to the signature

$$L_\Theta = \{\vee, \wedge, \neg, \exists x, M_{X^0}, s_*\}, \text{ for all } x \in X^0,$$

where the operations s_* is induced by all possible homomorphisms $s : W(X^0) \to W(X^0)$, and Γ consists of only one set X^0.

Definition 2.5. We call an algebra $\Upsilon = (\Upsilon_X, X \in \Gamma)$ in the signature L_Θ a Halmos algebra, if

1. Every domain Υ_X is an extended Boolean algebra in the signature L_X.
2. Every mapping $s_* : \Upsilon_X \to \Upsilon_Y$ is a homomorphism of Boolean algebras. Let $s : W(X) \to W(Y)$, $s' : W(Y) \to W(Z)$, and let $u \in \Upsilon_X$. Then $s'_*(s_*(u)) = (s's)_*(u)$.
3. The identities, controlling interaction of operations s_* with quantifiers and equalities are the same as in the definition of Halmos categories.

Each Halmos category Υ can be viewed as a Halmos algebra and vice versa.

Remark 2.10. The choice of Θ gives rise to some conditions all s_* have to satisfy.

Now we shall construct two major examples of multi-sorted Halmos algebras. The first one mimics the construction of one-sorted Halmos algebra from Example 1.

Example 2. Our aim is to define the Halmos category $Hal_\Theta(H)$. Assume that we have a class of sets X, $X \in \Gamma$. Objects of this category are extended Boolean algebras $Bool(W(X), H)$ from Example 1, for various $X \in \Gamma$. Morphisms

$$s_* : Bool(W(X), H) \to Bool(W(Y), H),$$

are defined as follows:

$$\mu \in s_* A \Leftrightarrow \widetilde{s}(\mu) = \mu s \in A,$$

where $\mu : W(Y) \to H$, $A \subset Hom(W(X), H)$, $s : W(X) \to W(Y)$, and $\widetilde{s} : Hom(W(Y), H) \to Hom(W(X), H)$. Here $\widetilde{s}$ is viewed as a morphism of the category of affine spaces (for the definition see [46]). In other words, $s_*A = (\widetilde{s})^{-1}A$. A morphism s_* is automatically a homomorphism of Boolean algebras. The maps s_* are correlated with quantifiers and equalities, see [39] for details. Moreover, there is a covariant functor: $\Theta^0 \to Hal_\Theta(H)$. Hence, $Hal_\Theta(H)$ is a Halmos category.

The category $Hal_\Theta(H)$ gives rise to a multi-sorted (Γ-sorted) Halmos algebra, denoted by

$$Hal_\Theta(H) = (Bool(W(X), H), X \in \Gamma).$$

Each component (domain) here is the extended Boolean algebra. The operations in $Hal_\Theta(H)$ are presented by the operations in each component $Bool(W(X), H)$ and operations corresponding to morphisms

$$s_* : Bool(W(X), H) \to Bool(W(Y), H).$$

Example 3. Another important example of multi-sorted Halmos algebra is presented by algebra $\widetilde{\Phi} = (\Phi(X), X \in \Gamma)$ of first order formulas over equalities. It turns out that geometrical aims forces to consider multi-sorted variant of algebraization of first order calculus.

Consider once again the signature

$$L_\Theta = \{\vee, \wedge, \neg, \exists x, M_X, s_*\},\ x \in X,\ X \in \Gamma.$$

We construct the algebra $\widetilde{\Phi}$ in an explicit way. Denote by $M = (M_X, X \in \Gamma)$ the multi-sorted set of equalities with the components M_X.

Each equality $w \equiv w'$ is a formula of the length zero, and of the sort X if $w \equiv w' \in M_X$. Let u be a formula of the length n and the sort X. Then the formulas $\neg u$ and $\exists xu$ are the formulas of the same sort X and the length $(n + 1)$. Further, for the given $s : W(X) \to W(Y)$ we have the formula s_*u with the length $(n + 1)$ and the sort Y. Let now u_1 and u_2 be formulas of the same sort X and the length n_1 and n_2 accordingly. Then the formulas $u_1 \vee u_2$ and $u_1 \wedge u_2$ have the length $(n_1 + n_2 + 1)$ and the sort X. In such a way, by induction, we define lengths and sorts of arbitrary formulas.

Let $\mathfrak{L}^0_X$ be the set of all formulas of the sort X. Each $\mathfrak{L}^0_X$ is an algebra in the signature L_X and

$$\mathfrak{L}^0 = (\mathfrak{L}^0_X, X \in \Gamma)$$

is an algebra in the signature L_Θ. By construction, algebra $\mathfrak{L}^0$ is the absolutely free algebra of formulas over equalities (i.e. over nullary operations) concerned with the variety of algebras Θ.

Denote by $\tilde{\pi}$ the congruence in $\mathfrak{L}^0$ generated by the identities of Halmos algebras from Definition 2.5 (see also their list in Definition 2.3) and define the Halmos algebra of formulas as

$$\widetilde{\Phi} = \mathfrak{L}^0/\tilde{\pi}.$$

It can be written as $\widetilde{\Phi} = (\Phi(X), X \in \Gamma)$, where

$$\Phi(X) = \mathfrak{L}^0_X/\tilde{\pi}_X,$$

where each $\Phi(X)$ is an extended Boolean algebra of the sort X in the signature L_X. The algebra $\widetilde{\Phi}$ is the free algebra in the variety of all multi-sorted Halmos algebras associated with the variety of algebras Θ, with the set of free generators $M = (M_X, X \in \Gamma)$. Denote this variety by Hal_Θ.

Remark 2.11. One can show [40], that if we factor out component-wisely the algebra $\mathfrak{L}^0$ by the many-sorted Lindenbaum-Tarski congruence, then we get the same algebra $\widetilde{\Phi}$. This observation provides a bridge between syntactical and semantical description of the free multi-sorted Halmos algebra.

Remark 2.12. If an element u belongs to the domain $\Phi(X)$, and s_* corresponds to $s : \Phi(X) \to \Phi(Y)$, then we cannot represent the formula s_*u from $\Phi(Y)$ in terms of equalities, connectives, and quantifiers in $\Phi(Y)$. The formulas from $\Phi(X)$ may contain free generators from different X, $X \in \Gamma$.

For example, the formula

$$u = s_*(y_1 \equiv y_2) \vee (x_3 \equiv x_4),$$

where $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2\}$ and $s(y_1) = x_1$, $s(y_2) = x_2$, belongs to $\Phi(X)$.

Theorem 2.1 ([39]). *The variety Hal_Θ of multi-sorted Halmos algebras is generated by all algebras $Hal_\Theta(H)$, where $H \in \Theta$.*

Theorem 2.1 implies that $\widetilde{\Phi}$ is the free algebra in Hal_Θ. This allows us to study properties of $\widetilde{\Phi}$ using the very concrete algebra

$$Hal_\Theta(H) = (Bool(W(X), H), X \in \Gamma)$$

as a model. Recall that we have defined the image of equalities from M_X in $Bool(W(X), H)$ by:

$$Val^X_H(w \equiv w') = \{\mu \mid \mu(w) = \mu(w')\}.$$

This means that there is the map

$$Val_H : M \to Hal_\Theta(H).$$

Since equalities $M = (M_X, X \in \Gamma)$ generate freely the free multi-sorted Halmos algebra $\widetilde{\Phi}$, the map Val_H can be extended from generators to the homomorphism of multi-sorted Halmos algebras

$$Val_H : \widetilde{\Phi} \to Hal_\Theta(H).$$

Since $\widetilde{\Phi} = (\Phi(X), X \in \Gamma)$, where each component $\Phi(X)$ is an extended Boolean algebra, the homomorphism Val_H induces homomorphisms

$$Val^X_H : \Phi(X) \to Bool(W(X), H),$$

of the one-sorted extended Boolean algebras. This allows us to calculate the value of each element from $\Phi(X)$ in $Bool(W(X), H)$. Note that the values of elements of the form s_*u are calculated as follows. Take $s : W(X) \to W(Y)$ and consider the formula s_*u, where $u \in \Phi(X)$. By definition, s_*u belongs to $\Phi(Y)$. Since Val_H is a homomorphism, then

$$Val_H^Y(s_*u) = s_*(Val_H^X u).$$

In the next sections we shall put all this staff in the context of affine spaces in arbitrary varieties. Replacing usual equations by logical formulas we arrive at the field of logical geometry.

3. Structures of Universal Algebraic Geometry

Let us begin with the very classical setting (cf. [58]). Let K be a field and $T = \{f_1, \ldots, f_m\}$ be a set polynomials in the polynomial algebra $K[X] = K[x_1, \ldots, x_n]$. Consider the affine space K^n with points $\bar{a} = (a_1, \ldots, a_n)$, $a_i \in K$ and define the Galois correspondence between ideals T in $K[X]$ and algebraic sets A in K^n:

$$T'_K = A = \{\bar{a} \mid f_i(\bar{a}) = 0, \text{ for all } f_i \in T\},$$

$$A'_K = T = \{f_i \in K[X] \mid f_i(\bar{a}) = 0, \text{ for all } \bar{a} \in A\},$$

In this correspondence geometric objects: curves, surfaces, general algebraic sets appear as zero loci of polynomials in the algebra $K[X]$.

In order to generalize this situation to arbitrary varieties of algebras, consider the variety $Com - K$ of commutative, associative algebras with unit over the field K. Then the algebra $K[X]$ is the free algebra in this variety and polynomials f_i are just elements of free algebra. Consider the field K and its extensions as algebras in this variety. Consider elements $\bar{a} = (a_1, \ldots, a_n)$ of the affine space K^n as functions $\bar{a} : K[X] \to K$ defined by $\bar{a}(x_i) = a_i$, $i = 1, \ldots, n$. Using this vocabulary we can define the Galois correspondence and geometric objects not in $Com - P$ but in arbitrary Θ.

Let Θ be an arbitrary variety and H be an algebra in Θ. This algebra takes the role of the field K, hence the affine space has to be of the form H^n. Let $W(X)$ be the free algebra over X, $X = \{x_1, \ldots, x_n\}$. This is the place were equations are situated and thus it plays the role of $K[x_1, \ldots, x_n]$. The natural bijection $\alpha : Hom(W(X), H) \to H^n$ allows us to consider the set of homomorphisms $Hom(W(X), H)$ as the affine space and its elements as the points of the affine space. Let the point $\mu \in Hom(W(X), H)$ be induced by a map $\mu : X \to H$. Then it corresponds the point $\bar{a} = (a_1, \ldots, a_n)$ in H^n, where $a_i = \mu(x_i)$. This correspondence gives rise to kernels of points μ of the affine space. We define the kernel $Ker(\mu)$ of the point μ as the kernel of the homomorphism $\mu : W(X) \to H$.

Let T be a system of equations of the form $w \equiv w'$, $w, w' \in W(X)$ which we treat as a system of formulas of the form $w \equiv w'$ on $W(X)$. Since w and w' are elements in $W(X)$, then $w = w(x_1, \dots, x_n)$, $w' = w'(x_1, \dots, x_n)$.

Definition 3.1. A point $\bar{a} = (a_1, \dots, a_n) \in H^n$ is a solution of $w \equiv w'$ in the algebra H if $w(a_1, \dots, a_n) = w'(a_1, \dots, a_n)$. A point $\mu \in Hom(W(X), H)$ is a solution of $w \equiv w'$ if $w(\mu(x_1), \dots, \mu(x_n)) = w'(\mu(x_1), \dots, \mu(x_n))$.

The equality $w(\mu(x_1), \dots, \mu(x_n)) = w'(\mu(x_1), \dots, \mu(x_n))$ means that the pair (w, w') belongs to $Ker(\mu)$. Identifying pairs (w, w') with the elements $w \equiv w'$ one can say that the equation $w \equiv w'$ has a solution at the point μ if $w \equiv w'$ belongs to the kernel of the point μ. The kernel $Ker(\mu)$ is a congruence of the algebra $W(X)$, and thus the quotient algebra $W(X)/Ker(\mu)$ is defined.

Let now T be a system of equations in $W(X)$ and A a set of points in $Hom(W(X), H)$. Set the Galois correspondence by

$$T'_H = A = \{\mu : W(X) \to H \mid T \subset Ker(\mu)\}$$

$$A'_H = T = \{(w \equiv w') \mid (w, w') \in \bigcap_{\mu \in A} Ker(\mu)\}.$$

Definition 3.2. A set A in the affine space $Hom(W(X), H)$ is called an algebraic set if there exists a system of equations T in $W(X)$ such that each point μ of A satisfies all equations from T. A congruence T in $W(X)$ is called H-closed if there exists A such that $A'_H = T$.

We can rewrite the Galois correspondence through the values of formulas:

$$T'_H = A = \bigcap_{(w,w') \in T} Val^X_H(w \equiv w').$$

$$A'_H = T = \{w \equiv w' \mid A \subset Val^X_H(w \equiv w')\}.$$

The geometry obtained via this correspondence is an equational geometry grounded on algebra H in Θ. However, there are no reasons to restrict ourselves with equational predicates looking at the images of the formulas in the affine space. We can look at arbitrary first order formulas as at equations, and since arbitrary formulas are the elements of $\widetilde{\Phi} = (\Phi(X), X \in \Gamma)$, we shall replace in all consideration the free algebra $W(X)$ by the extended Boolean algebra $\Phi(X)$.

The sets of equations are defined as arbitrary subsets in $\Phi(X)$, the finite dimensional affine space H^n is the same as in equational case, and it remains to define the geometric objects, that is the images of the formulas $u \in \Phi(X)$ in the Galois correspondence. This can be done because, as we know, the equalities M_X, $X \in \Gamma$ represent the free generators of $\widetilde{\Phi}$ and, thus the value homomorphism Val^X_H can be extended from equalities to arbitrary formulas $u \in \Phi(X)$.

Let $\mu : W(X) \to H$ be a point. Along with the classical kernel $Ker(\mu)$ we define its logical kernel.

Definition 3.3. A formula $u \in \Phi(X)$ belongs to the logical kernel $LKer(\mu)$ of a point μ if and only if $\mu \in Val_H^X(u)$.

It can be verified that the logical kernel $LKer(\mu)$ is always a Boolean ultrafilter of $\Phi(X)$ [40].

Since we consider each formula $u \in \Phi(X)$ as an "equation" and $Val_H^X(u)$ as a value of the formula u in the algebra $Bool(W(X), H)$, then $Val_H^X(u)$ is a set of points $\mu : W(X) \to H$ satisfying the "equation" u. We call $Val_H^X(u)$ *solutions of the equation* u. We also say that the formula u holds true in the algebra H at the point μ.

We call the obtained geometry associated to an arbitrary variety Θ *and* $H \in \Theta$ *the logical geometry.*

In order to establish in this case the Galois correspondence we shall replace the kernel $Ker(\mu)$ by the logical kernel $LKer(\mu)$. Let T be a set of formulas in $\Phi(X)$ and A a set of elements in $Bool(W(X), H)$. Define

$$T_H^L = A = \{\mu : W(X) \to H \mid T \subset LKer(\mu)\},$$

$$A_H^L = T = \bigcap_{\mu \in A} LKer(\mu)$$

The same Galois correspondence can be rewritten as

$$T_H^L = A = \bigcap_{u \in T} Val_H^X(u).$$

$$A_H^L = T = \{u \in \Phi(X) \mid A \subset Val_H^X(u)\}.$$

Definition 3.4. A set A in the affine space $Hom(W(X), H)$ is called an elementary set if there exists a system of formulas T in $\Phi(X)$ such that each point μ of A satisfies all formulas from T. In other words, $A = A_H^{LL} = T_H^L$ is fulfilled for elementary sets.

Definition 3.5. A set of formulas $T \subset \Phi(X)$ such that $T = T_H^{LL} = A_H^L$ is called *an* H*-closed Boolean filter* in $\Phi(X)$.

Remark 3.1. The set of formulas T which defines an elementary set A can be infinite.

Remark 3.2. Elementary sets in the model theory are usually called definable sets. Since in the geometrical approach they are tightly connected with elementary theories, we use the term "elementary set" instead of "definable set".

4. Model Theoretic Types

In this small section we recall the well-known definitions from model theory. Exposition mainly follows the standard model theory course by [35], see also [34], [47], etc. We assume that the precise definition of an $\mathbb{L}$-structure is known. Basically, an $\mathbb{L}$-structure is a pair $(\mathbb{L}, M)$, where $\mathbb{L}$ is a language and M is a set, called the domain of the structure. Any language may contain functional symbols, symbols of relations, and special symbols called constants. Given an $\mathbb{L}$-structure, all these symbols are interpreted (realized) on the domain M. So any $\mathbb{L}$-structure can be considered as a triple $(\mathbb{L}, M, f)$, where f is an interpretation function.

Formulas of $\mathbb{L}$ are built inductively from atomic formulas, using the symbols of $\mathbb{L}$, symbols of variables $x_1, x_2, \dots$, the equality symbol $\equiv$, the Boolean connectives $\wedge$, $\vee$, $\neg$, the quantifiers $\exists$ and $\forall$, and the parentheses (,). We suppose that the interpretation of symbol $\equiv$ is always the equality on M.

A variable x *occurs freely* in a formula u if it is not bound by quantifiers $\exists x$ or $\forall x$. A formula u is called *a sentence* (or a closed formula) if it has no free variables. If $u(x_1, \dots, x_n)$ is a formula in free variables $x_1, \dots, x_n$, then its closure $\bar{u}$ is any sentence produced from u by binding all free variables by quantifiers.

Let $\mathbb{M}$ be an $\mathbb{L}$-structure. Let $u(x_1, \dots, x_n)$ be a formula in free variables $x_1, \dots, x_n$ which means that all occurrences of other variables in this formula are bound. The value ("true" or "false") of a formula $u = u(x_1, \dots, x_m)$ in the point $\bar{a} = (a_1, \dots, a_n) \in M^n$ is defined inductively, using the scheme of Tarski. Each $\mathbb{L}$-sentence is either true or false on the whole $\mathbb{M}$. The notation $\mathbb{M} \models u(\bar{a})$ means that $u(\bar{a})$ is true on $\mathbb{M}$. In this case we say that u is satisfiable on $\mathbb{M}$.

Definition 4.1. A set T of $\mathbb{L}$-sentences is called an $\mathbb{L}$-theory. $\mathbb{M}$ is a model of the theory T if $\mathbb{M} \models u$ for all $u \in T$. A theory is satisfiable if it has a model.

Suppose that $\mathbb{M}$ is an $\mathbb{L}$-structure and $A \subseteq M$. Let $\mathbb{L}_A$ be the language obtained by adding to $\mathbb{L}$ constant symbols for each $a \in A$. We can naturally view $\mathbb{M}$ as an $\mathbb{L}_A$-structure by interpreting the new symbols in the obvious way. Let $Th_A(\mathbb{M})$ be the set of all $\mathbb{L}_A$-sentences true in $\mathbb{M}$, that is the $\mathbb{L}_A$-*theory* of the model $\mathbb{M}$.

Definition 4.2. If $\mathbb{L}$ is a first order language, then $Th_A(\mathbb{M})$ is called the *elementary theory* of M.

Definition 4.3. Let $P = \{u_i(x_1, \dots, x_n)\}$ be a set of $\mathbb{L}_A$-formulas in free variables $x_1, \dots, x_n$. We call P an n-type (partial n-type) if $P \cup Th_A(\mathbb{M})$ is satisfiable. We say that P is a complete n-type if $u \in P$ or $\neg u \in P$ for all $\mathbb{L}_A$-formulas u with free variables from $x_1, \dots, x_n$.

So, the data for a type P is a structure $\mathbb{M}$ and a subset of constants $A \subseteq M$. If $\mathbb{M}$ is any $\mathbb{L}$-structure, $A \subseteq M$, and $\overline{a} = (a_1, \dots, a_n) \in M^n$, denote by $tp^{\mathbb{M}}(\overline{a}/A) = \{u(x_1, \dots, x_n) \in \mathbb{L}_A$ such that $\mathbb{M} \models u(a_1, \dots, a_n)\}$. Then, $tp^{\mathbb{M}}(\overline{a}/A)$ is a complete n-type.

Definition 4.4. We say that a complete n-type P is realizable in $\mathbb{M}$ if there is $\overline{a} = (a_1, \dots, a_n) \in M^n$ such that $P = tp^{\mathbb{M}}(\overline{a}/A)$.

Denote the sets of all complete realizable n-types over M by $S^n_A(\mathbb{M})$. In the case $A = M$ we denote this set by $S^n(\mathbb{M})$.

5. Algebraization of Model Theoretic Types

Define an algebraization of the notion of type. Let X^0 be an infinite set of variables. Let H be an algebra from a variety of algebras Θ and also the set of constants. That is we consider algebras G from the variety Θ^H of H-algebras. For example, if Θ is the variety of commutative and associative rings with the unit and K is a field, then Θ^K is the variety of algebras over the field K.

The free algebras in Θ^H have the form $W(X^0) = W'(X^0) * H$, where $W'(X^0)$ is the free algebra in Θ and $*$ stands for the free product in Θ.

Let $\Phi(X^0)$ be the one-sorted Halmos algebra of formulas associated with the variety Θ^H.

Let us consider the Galois correspondence for the one-sorted Halmos algebra $\Phi(X^0)$, X^0 is infinite. Let T be a set of formulas in $\Phi(X^0)$. We have

$$T^L_H = A = \{\mu : W(X^0) \to H \mid T \subset LKer(\mu)\},$$

$$A^L_H = T = \bigcap_{\mu \in A} LKer(\mu)$$

Let $X = \{x_1, \dots, x_n\}$ be a finite subset in X^0. We shall define X-MT-type (MT-type for short) of the point $\mu \in Hom(W(X), H) \cong H^n$.

For each point $\mu : W(X) \to H$ consider the set of points A_μ defined by: a point $\nu : W(X^0) \to H$ belongs to A_μ if $\nu(x) = \mu(x)$ for $x \in X$ and $\nu(y)$, where $y \in X^0 \setminus X$, is an arbitrary element in H. Define

$$T_\mu = (A_\mu)^L_H = \bigcap_{\nu \in A_\mu} LKer(\nu).$$

In other words T_μ is the set of all formulas $u \in \Phi(X^0)$ which hold on the points from A_μ. This means that $u \in T_\mu$ if and only if $A_\mu \subset Val^{X^0}_H(u)$. Since every logical kernel is an ultrafilter, the set T_μ is a filter in $\Phi(X^0)$.

Definition 5.1. We call the filter T_μ an MT-type of the point μ.

Remark 5.1. Let us compare Definitions 4.1 – 4.4 and Definition 5.1. In the definition 5.1 we consider an MT-type of the point $\bar{a} = (a_1, \dots, a_n)$, where $\mu(x_i) = a_i$, $a_i \in H$ for $x_i \in X$, as the set of all formulas u which hold true on the point μ (i.e., on the point $\bar{a}$). Therefore, the type of a point in our definition is always a filter.

On the other hand, by the definition 4.4 the type of the point $tp^{\mathbb{H}}(\bar{a}) = tp^{\mathbb{H}}(\mu)$, where $\mu(x_i) = a_i$, $i = 1, \dots, n$ is the set of the satisfiable in the point μ formulas of the form $u = u(x_1, \dots, x_n, y_1, \dots, y_k)$, where only x_i are free variables. This is a subset of T_μ and thus an MT-type T_μ is somewhat bigger than the corresponding $tp^{\mathbb{H}}(\mu)$.

Remark 5.2. The similar situation holds with the definition of the elementary theory of an algebra H.

We will consider elementary theory of H as the set of all formulas u true in every point μ in $Hom(W(X), H)$.

On the other side, according to the model-theoretic Definition 4.1 the elementary theory of H is smaller and consists of closed formulas true in H. Since every formula u true in H is equivalent to its closure $\bar{u}$, then by abuse of language we use the same notation $Th(H)$ for the elementary theory of H in both cases. So,

$$Th(H) = \bigcap_{\mu} T_\mu,$$

where $\mu \in Hom(W(X), H)$.

This situation is typical for algebraic logic and geometry where the free variables do not play the same role as in logic and model theory.

Denote the system of all MT-types T_μ of the algebra H by S_H^X. Here, $\mu : W(X) \to H$, and X runs all finite subsets of X^0.

Given finite subset $X \subset X^0$ and a point $\mu : W(X) \to H$, define $s = s^\mu : W(X^0) \to W(X^0)$, where $W(X^0) = W'(X^0) * H$, by letting $s(x_i) = \mu(x_i)$, if $x_i \in X$, and $s(y) = y$ for $y \in Y^0 = X^0 \setminus X$. Let $s_*^\mu : \Phi(X^0) \to \Phi(X^0)$ be the corresponding map of Halmos algebras.

Proposition 5.1. *A formula $u \in \Phi(X^0)$ belongs to T_μ if and only if $s_*^\mu u$ belongs to the elementary theory $Th(H)$.*

Proof. Let $s_*^\mu u$ belong to the elementary theory $Th(H)$. We shall prove that $u \in T_\mu$. Thus, we shall check that $A_\mu \subset Val_H^{X^0}(u)$. Let $\nu \in A_\mu$. Let $\delta : W(X^0) \to H$ be an arbitrary point in $Hom(W(X^0), H)$. Then, for $x_i \in X$, we have $\delta s^\mu(x_i) = \delta(\mu(x_i)) = \mu(x_i)$ since δ fixes constants. Correspondingly, $\delta s^\mu(y_i) = \delta(y_i)$. Thus we can choose δ such that $\delta s^\mu = \nu$ for any $\nu \in A_\mu$. Since $s_*^\mu u \in Th(H)$, then δ lies in $Val_H^{X^0}(s_*^\mu u) = s_*^\mu Val_H^{X^0}(u)$. The latter equality means, by definition, that δs^μ lies in $Val_H^{X^0}(u)$. Hence, $A_\mu \subset Val_H^{X^0}(u)$.

Conversely, let $u \in T_\mu$. We shall prove that $s_*^\mu u$ belongs to the elementary theory $Th(H)$. So we have to check that any point δ satisfies $s_*^\mu u$. Consider δs^μ. This point belongs to A_μ. Hence δs^μ lies in $Val_H^{X^0}(u)$. This means that δ lies in $s_*^\mu Val_H^{X^0}(u) = Val_H^{X^0}(s_*^\mu u)$. Thus, an arbitrary point δ belongs to $Val_H^{X^0}(s_*^\mu u)$ and $s_*^\mu u$ lies in $Th(H)$. □

Let $u = u(x_1, \ldots, x_n, y_1, \ldots, y_k)$ be a formula in $\Phi(X^0)$ such that $x_i \in X$, $y_i \in Y$, and all occurrences of x_i are free, all occurrences of y_i are bound. We call such a formula special.

Let u be a special formula. It can be seen that $s^{\mu}_{*}u$ replaces all occurrences of free variables x_i by the their images $h_i \in H$ under the homomorphism s^{μ}. Hence $s^{\mu}_{*}u$ has all variables bound, i.e., $s^{\mu}_{*}u$ is a sentence.

Any MT-type is complete with respect to special formulas. Indeed, let u be a special formula and let $u \notin T_{\mu}$. Consider $\neg u$. We have $s^{\mu}_{*}(\neg u) = \neg s^{\mu}_{*}(u)$. By Proposition 5.1, $s^{\mu}_{*}(u)$ does not hold in H. Since $s^{\mu}_{*}u$ is a sentence, the formula $\neg s^{\mu}_{*}(u)$ holds in H. Hence, $s^{\mu}_{*}(\neg u)$ holds in H and thus belongs to $Th(H)$. Then $\neg u \in T_{\mu}$ according to Proposition 5.1.

From now on, one can build the type theory from the positions of one-sorted algebraic logic. In the next section we consider a more geometric approach, related to multi-sorted logic and multi-sorted Halmos algebras.

6. Logically Geometric Types

Let us take the free multi-sorted Halmos algebra of formulas $\widetilde{\Phi} = (\Phi(X), X \in \Gamma)$, where all X are finite. Recall the necessary facts from the previous sections.

There is the value homomorphism of multi-sorted Halmos algebras $Val_H : \widetilde{\Phi} \to Hal_{\Theta}(H)$, which induces homomorphisms of extended Boolean algebras $Val^X_H : \Phi(X) \to Bool(W(X), H)$, where $Hal_{\Theta}(H) = (Bool(W(X), H), X \in \Gamma)$. We can write $Val_H = (Val^X_H, X \in \Gamma)$. For every X, the homomorphism Val^X_H gives rise to a major Galois correspondence of logical geometry between H-closed congruences in $\Phi(X)$ and elementary sets in finite dimensional affine spaces $Hom(W(X), H)$:

$$T^L_H = A = \{\mu : W(X) \to H \mid T \subset LKer(\mu)\},$$

$$A^L_H = T = \bigcap_{\mu \in A} LKer(\mu).$$

Let $Th(H) = (Th^X(H), X \in \Gamma)$ be the multi-sorted representation of the elementary theory of H. We call its component $Th^X(H)$ the *X-theory of the algebra H*. We have:

$$Ker(Val_H) = Th(H),$$

$$Ker(Val^X_H) = Th^X(H).$$

The key diagram which relates logic of different sorts in multi-sorted case is as follows:

$$\begin{array}{ccc} \Phi(X) & \xrightarrow{s_*} & \Phi(Y) \\ {\scriptstyle Val^X_H}\downarrow & & \downarrow{\scriptstyle Val^Y_H} \\ Bool(W(X), H) & \xrightarrow{s_*} & Bool(W(Y), H) \end{array}$$

Here the upper arrow represent the syntactical transitions in the category Hal_Θ, the lower level does the same with the respect to semantics in Hal_Θ, and the correlation is provided by the value homomorphism.

Recall that a formula $u \in \Phi(X)$ belongs to the logical kernel $LKer(\mu)$ of a point μ if and only if $\mu \in Val_H^X(u)$, that is u lies in $LKer(\mu)$ if a point μ satisfies the "equation" u. This is the Boolean ultrafilter, which contains $Th^X(H)$. Indeed, if $u \in Th^X(H)$ then $Val_X^H(u) = Hom(W(X), H)$. In particular, $\mu \in Val_H^X(u)$ and $u \in LKer(\mu)$. Thus $Th^X(H) \subset LKer(\mu)$. Moreover,

$$Th^X(H) = \bigcap_{\mu} LKer\mu.$$

Define now the concept of an *LG*-type.

Definition 6.1. Every ultrafilter T in the algebra $\Phi(X)$ containing $Th^X(H)$ is called X-*LG*-type.

Definition 6.2. An ultrafilter T is called X-*LG*-type of the algebra H, if there is a point $\mu : W(X) \to H$ such that $T = LKer(\mu)$.

In the latter case we also say that the type T is realizable in H. Since the elementary X-theory is contained in each $LKer(\mu)$ then the elementary X-theory $Th^X(H)$ is contained in each X-*LG*-type of H. Denote the system of all X-*LG*-types of the algebra H by $S^X(H)$.

Now we want to explore the geometrical nature of the Galois correspondence. In algebraic geometry, the category of all algebraic sets is an important invariant of the the algebra H. In most cases, this category is dual to the category of coordinated algebras. We want to use similar ideas in the case of logical geometry. The logical kernels take the role played by the radical ideals in classical geometry and the roles of closed congruences in the universal one. So, the types of the points represented by the logical kernels may have similar impact to logical geometry and may be involved in the similar algebraically-geometric ideas.

Two algebras H_1 and H_2 are called *geometrically equivalent* (*AG*-equivalent for short) (see [40], [41]) if for every finite X and T in $W(X)$ we have

$$T''_{H_1} = T''_{H_2}.$$

Definition 6.3 ([46]). Algebras H_1 and H_2 are called logically equivalent (*LG*-equivalent for short) if for every finite X and T in $\Phi(X)$ we have

$$T^{LL}_{H_1} = T^{LL}_{H_2}.$$

It can be seen (see [46]), that if two algebras H_1 and H_2 are logically equivalent then they are *elementary equivalent* (i.e., $Th(H_1) = Th(H_2)$). The converse statement is not true.

Definition 6.4 ([46]). Algebras H_1 and H_2 in Θ are called *LG*-isotypic, if for any finite X, every X-*LG*-type of the algebra H_1 is an X-*LG*-type of the algebra H_2 and vice versa.

Thus, the algebras H_1 and H_2 are LG-isotypic if $S^X(H_1) = S^X(H_2)$ for every $X \in \Gamma$. This coincidence clearly implies that they are elementary equivalent.

So, we have the geometric notion of logical equivalence of algebras which generalizes geometric equivalence, and the model theoretic notion of LG-isotypeness. Both of them imply elementary equivalence. The following theorem shows that these two notions coincide.

Theorem 6.1 ([40]). *Algebras H_1 and H_2 are LG-equivalent if and only if they are LG-isotypic.*

One can define the category of algebraic sets $K_\Theta(H)$ and the category of elementary sets $LK_\Theta(H)$. The objects of $K_\Theta(H)$ are of the form (X, A), where A is an algebraic set in $Hom(W(X), H)$. If we take for A the elementary sets, then we are getting to the category of elementary sets $LK_\Theta(H)$. The morphisms are of the form

$$[s] : (X, A) \to (Y, B).$$

Here $s : W(Y) \to W(X)$ is a morphism in the category Θ^0. The corresponding $\tilde{s} : Hom(W(X), H) \to Hom(W(Y), H)$ should be coordinated with A and B by the condition: if $\nu \in A \subset Hom(W(X), H)$, then $\tilde{s}(\nu) \in B \subset Hom(W(Y), H)$. Then the induced mapping $[s] : A \to B$ we consider as a morphism $(X, A) \to (Y, B)$.

The category $K_\Theta(H)$ is a full subcategory in $LK_\Theta(H)$. It is known that if two algebras H_1 and H_2 are geometrically equivalent, then the categories of algebraic sets $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are isomorphic. A similar fact is valid with respect to categories of elementary sets. Namely,

Theorem 6.2 ([46]). *If the algebras H_1 and H_2 are LG-isotypic then the categories $LK_\Theta(H_1)$ and $LK_\Theta(H_2)$ are isomorphic.*

7. Problems

In Sections 5 and 6 we described MT-types and LG-types. Now we want to compare these notions.

Recall that MT-types are defined for points $\mu : W(X) \to H$ of the affine space $Hom(W(X), H)$. However, the formulas from any MT-type T_μ lie in the algebra of formulas $\Phi(X^0)$, where X^0 is an infinite set.

In the case of LG-types, we consider finite sets X in X^0 and the multi-sorted algebra of formulas $\widetilde{\Phi} = (\Phi(X), X \in \Gamma)$, where all X are finite. The X-LG-type of the point $\mu : W(X) \to H$ is $LKer(\mu)$, which is calculated in the algebra $\Phi(X)$. This is one of the differences in two approaches. We shall also remember that the formulas from $T \subset \Phi(X)$ may contain free generators from different X, where $X \in \Gamma$ (see Remark 2.12). The following conjecture compares MT-isotypeness and LG-isotypeness.

Conjecture 7.1. *Two algebras H_1 and H_2 from the variety Θ are MT-isotypic if and only if they are LG-isotypic.*

This result will imply the general line which looks as follows: we work in the frames of logical geometry and apply the results and corresponding notions to model theoretic types.

For instance the notion of isotypeness of algebras is consistent with geometrical ideas and extends the notion of geometrical equivalence of algebras. However, from the point of view of model theory isotypeness is ultimately connected with the classical questions about elementary equivalence of algebras. As soon as both isotypenesses give the same we come out with one more bridge between logic and geometry.

Problems 7.1 and 7.2 are formulated for LG-types, but the above said makes them relevant for MT-types as well.

Problem 7.1. *Let F_n be a free group of the rank $n > 1$ and H be a group. Is it true that if F_n and H are LG-isotypic then they are isomorphic?*

This problem lies, in fact, in the mainstream of Tarski's problem which asks if two finitely generated non abelian free groups are elementary equivalent. The latter problem stimulated the development of the algebraic geometry over free groups and hyperbolic groups described in a series of brilliant papers (see [23]–[28], [49]–[57] and many others). In particular there is a description of all groups elementary equivalent to a given free group ([54], [26]).

Problem 7.2. *Are there LG-isotypic groups H_1 and H_2 such that H_1 is finitely generated and H_2 is an arbitrary non finitely generated group?*

C. Perin and R. Sklinos [38] (see also A. Ould Houcine [37]) proved that if for a non-abelian free group H there is the equality $T_\mu = T_\nu$ then $\mu = \sigma\nu$ for some automorphism σ of H.

Problem 7.3. *What are the varieties Θ such that for arbitrary free algebra $H = W(X)$ from Θ the equality $T_\mu = T_\nu$ implies $\mu = \sigma\nu$?*

Similar question for LG-types and free groups is of great interest.

Problem 7.4. *Is it true that for a given free non-abelian group the equality $LKer(\mu) = LKer(\nu)$ implies $\mu = \sigma\nu$?*

Note that the group of automorphisms of an algebra H acts on the affine space $Hom(W(X), H)$, and each elementary set is invariant under this action. If for the algebra H there are only a finite number of $Aut(H)$-orbits in $Hom(W(X), H)$ for every X, then there are only finite number of realizable LG-types in $\Phi(X)$. It can be shown that for free abelian groups of the exponent p this property is satisfied. It would be interesting to look for non-abelian examples.

Problem 7.5. *Find examples of algebras H such that for every X there are only a finite number of $Aut(H)$-orbits in $Hom(W(X), H)$.*

In [19], S. Feferman and R. Vaught studied the operations on algebras that preserve elementary equivalence. The remaining open question weather the free products of pair-wise elementarily equivalent groups are elementarily equivalent was solved recently by Z. Sela affirmatively ([57], see also [22]). Namely, he proved that if a group A_1 is elementarily equivalent to a group A_2, and a group B_1 is elementarily equivalent to B_2 then $A_1 * B_1$ is elementarily equivalent to $A_2 * B_2$.

Problem 7.6. *Let A_1, A_2, B_1, B_2 be groups and let A_1 be isotypic to A_2, and B_1 be isotypic to B_2. Is it true that the free product $A_1 * B_1$ is isotypic to $A_2 * B_2$?*

8. Recent results. Appendix

A bunch of results related to problems pointed above was obtained very recently by Z.Sela, R.Sklinos, and G.Zhitomirski. After the paper had been finished we were informed that Conjecture 7.1 is true (G. Zhitomirski [59]). Hence, the result of C. Perin and R. Sklinos solves Problem 7.4. Problems 7.3 and 7.4 have positive solutions for the varieties of abelian groups and nilpotent of class c groups (G. Zhitomirski, [59], R. Sklinos for abelian groups, unpublished). The question

Problem 8.1. *Is it true that for a given free solvable group of the derived length $c > 1$ the equality $LKer(\mu) = LKer(\nu)$ implies $\mu = \sigma\nu$?*

remains open.

Problem 7.1 is solved for the case of a finitely generated group H (R. Sklinos, unpublished), i.e., if a free group F_n of the rank $n > 1$ is isotypic to a finitely generated group H, then F_n and H are isomorphic. G.Zhitomirski ([59]) proved that if two abelian groups are isotypic and one of them is free and finitely generated then they are isomorphic.

At last, using the technique of [57], Z. Sela solved positively Problem 7.6 (unpublished).

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