UNIVERSAL ALGEBRAIC GEOMETRY: SYNTAX AND SEMANTICS

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Abstract. In this paper we give a general insight on the ideas which make ground for the developing of universal algebraic geometry and logical geometry. We specify the role of the algebraic logic as one of the major instruments of the whole theory. The problem of the sameness of geometries of algebraic and definable sets for different algebras is considered as the illuminating example how algebra, geometry, model theory and algebraic logic work together.

1. Informal introduction of B.Plotkin

The whole story began for me in the 80s of the last century. Some practical discussions have led to the problem of constructing an algebraic model of databases and knowledge bases. Step by step, research related to this topic has spawned the book [13]. Later, probably influenced by ideas and discussions with V.Remeslennikov and I.Rips and their colleagues, I used this book to develop a unified approach to algebraic geometry and algebraic logic.

Let me say that the way to a universal approach allows at least two possibilities. First of all, you can go to the "extension" of the theory in the direction of general model-theoretical ideas. On the other hand, one can sacrifice a model-theoretic generality of research, having instead a kind of universal algebraization of classical geometric ideas. Both approaches are equally significant.

This article is based on the ideas of the second approach. We fix an arbitrary variety of algebras Θ and an algebra $H \in \Theta$. Let us attach to H two invariants. The first invariant is the category $AG_\Theta(H)$ of all algebraic sets over a given H. Its objects are the sets of algebraic sets $AG^X_\Theta(H)$, corresponding to a given set of variables X. It is known that if the algebras H_1 and H_2 are geometrically equivalent [12], then their geometries are the same, that is, the categories $AG_\Theta(H_1)$ and $AG_\Theta(H_2)$ are isomorphic (see [15] and the references therein). All algebraic sets over A are in one-to-one correspondence with their X-coordinate algebras. Each coordinate algebra belongs to the same variety Θ. Thus, the semantic condition for the uniformity of geometries over H_1 and H_2 can be raised to the syntactic level of isomorphism of the categories of the corresponding coordinate algebras.

The second invariant of H is the category $LG_\Theta(H)$ of all definable sets over H. Its objects are the lattices of definable sets $LG^X_\Theta(H)$ corresponding to the given set X. When we look for the coincidence of geometries of definable sets, it turns out that the correct condition is a condition of the logical equivalence of algebras. However, the simple transfer of the concept of coordinate algebra to definable sets fails. This paper is devoted to the study of syntactic-semantic correspondences in universal algebraic and logical geometries.

General references for the paper are [2], [4], [6], [7], [8].

2. On syntax and semantics

By **syntax** we will mean a language intended to describe a certain subject area. In syntax we ask questions, express hypotheses, and formulate the results. In syntax we also build chains of formal consequences.

For our aims we use first-order languages or their fragments. Each language is based on some finite set of variables that serve as the alphabet, and a number of rules that allow us to build words based on this alphabet. In general, its signature includes Boolean operations, quantifiers, constants, and also functional symbols and predicate symbols. The latter ones are included in atomic formulas and, in fact, determine the face of a particular language. Atomic formulas will be called words. Words together with logical operations between them will be called formulas.

By **semantics** we understand the world of models, or in other words, the subject area of our knowledge. This world exists by itself, and develops according to its laws.

Many mathematical (and not only mathematical) questions are reduced to a very general correspondence:

$$\begin{array}{ccc} SYNTAX & \longrightarrow & SYNTAX \\ \downarrow & & \downarrow \\ SEMANTICS & \longrightarrow & SEMANTICS. \end{array}$$

We call this correspondence **syntactic-semantic square**.

- *The goal of universal algebraic geometry is to make the transitions from syntax to semantics as algebraic as possible and to relate tightly the arising structures of algebra, logic, geometry and model theory.*

3. General approach

In the syntactic-semantic square, the upper level corresponds to syntax. The upper vertices of the square correspond to syntactic objects, that is, to sets of formulas of a first-order language. The upper arrows correspond to some transitions from one set of formulas to another. These transitions can be realized through logical inference, or just through maps between sets of formulas.

The lower level is the level of semantics. The lower vertices should be semantic objects, such as data tables, sets of points in affine or vector spaces, matrices. Transitions between objects are realized by special mappings that take into account the syntactic structure of the upper floor.

The following connections play an important role:

$$\text{SYNTAX} \rightleftarrows \text{LANGUAGE} \qquad \text{SEMANTICS} \rightleftarrows \text{MODEL},$$

$$\text{SYNTAX} \rightleftarrows \text{ALGEBRA} \qquad \text{SEMANTICS} \rightleftarrows \text{GEOMETRY}.$$

The main condition in the general approach of universal algebraic geometry is the existence of a set-theoretic Galois correspondence.

A Galois correspondence is any pair of functions $\varphi : \mathbb{A} \to \mathbb{B}$ and $\psi : \mathbb{B} \to \mathbb{A}$ between partially ordered sets $\mathbb{A}$ and $\mathbb{B}$, satisfying:

1. if $a \leqslant a'$, then $\varphi(a) \geqslant \varphi(a')$,
2. if $b \leqslant b'$, then $\psi(b) \geqslant \psi(b')$,
3. $\psi(\varphi(a)) \geqslant a$,
4. $\varphi(\psi(b)) \geqslant b$.

Let us denote the functions φ and ψ in the Galois correspondence with one symbol $'$. Each Galois correspondence leads to Galois-closed objects $a'' = a$. Finally, it is easy to see that in the Galois correspondence there is a one-to-one correspondence between closed objects in $\mathbb{A}$ and $\mathbb{B}$.

We now formulate the main conditions for syntactic-semantic transitions. We denote by $T_1, T_2, \dots, T_k$... syntactic objects and by $A, B, C, ...$ semantic objects.

- The sets of objects is partially ordered.

- There is a Galois correspondence between syntactic and semantic objects.
- Galois correspondence has a functorial property, in the sense that the diagram of a syntactic-semantic square

$$\begin{tikzcd} T_1 \arrow[r] \arrow[d, "\prime"'] & T_2 \arrow[r] \arrow[d, "\prime"] & T_1 \arrow[d, "\prime"] \\ A_1 \arrow[r] & A_2 \arrow[r] & A_1, \end{tikzcd}$$

is commutative with a suitable choice of directions of arrows.

4. Universal algebraic geometry

4.1. **General view on universal algebraic geometry.** First of all we note that the subject area of classical algebraic geometry consists of subsets in the n-dimensional complex affine space which are defined by systems of polynomial equations. The task of universal algebraic geometry is to get away from these objects and extend the ideas of classical algebraic geometry to arbitrary varieties of algebras. By a variety of algebras we mean a class of algebraic structures defined by a set of identities. These can be semigroups, monoids, groups, Lie algebras, databases, automata, etc.

Fixing the variety Θ, we thereby determine the subject area, to be studied. In fact, algebras from Θ constitute the semantics of universal algebraic geometry. The variety Θ is also associated with a special syntax that takes into account the set of identities of this variety. Universal algebraic geometry studies syntax and semantic transitions defined over an arbitrary variety Θ. From this point of view, classical algebraic geometry is geometry for the special case of the variety $Com - K$ of commutative associative algebras with a unit over a fixed field.

Thus, one of the goals is to study the sets of solutions of equations over given algebra $H \in \Theta$. Such a study in the case of equations over free groups led to the geometry of the free group and served as the main tool for solving the Tarski problem.

However, in what follows we focus our attention on another goal of universal algebraic geometry, namely on the study of **geometric invariants of algebras** from Θ.

Note that we distinguish two parts of universal algebraic geometry. The first one is **equational geometry**. It means that algebraic sets are defined by systems of equations in free algebras from Θ. This section is devoted to it. The second part is **logical geometry**. This is the topic of the next section.

4.2. **System of notions.** All further reasoning is based on the variety of algebras Θ. Accordingly, all the basic concepts of classical algebraic geometry must be modernized for arbitrary Θ.

From the algebraic point of view, the language of classical geometry is the polynomial algebra $K[X] = K[x_1, \dots, x_n]$. Indeed, this algebra is the free finitely generated algebra in the variety of commutative associative algebras over the field K. Free algebra is a syntactic object of any variety. Therefore, the role of $K[x_1, \dots, x_n]$ in the general case is played by the free in Θ algebra $W(X)$, with the free system of generators $X = \{x_1, \dots, x_n\}$. Equations are written in the algebra $W(X)$. In this case, the role of equations of the form $f(x_1, \dots, x_n) = 0$, where f is a polynomial, is played by equations of the form $w \equiv w'$, where $w, w' \in W(X)$.

Now we need to determine the place for solutions of equations. In classical algebraic geometry, these were affine spaces over the ground field or over its extensions. In the general case, Cartesian powers of H^n are taken as affine spaces, where H is a certain algebra in Θ. An important feature is the representation of this space in the form of systems of homomorphisms $Hom(W(X), H)$). The resulting system of notions is presented in the following comparative table.

Classical AG	**Universal AG**
Variety	
$Com - K$	Θ
Free Algebra	
$K[X]$, $\lvert X\rvert = n$	$W(X)$, $\lvert X\rvert = n$
Elements of free algebra	
$f(x_1, \dots, x_n) \in K[X]$	$w(x_1, \dots, x_n) \in W(X)$
Equations	
$f(x_1, \dots, x_n) \equiv 0$	$w \equiv w'$
Ground field	Algebra in Θ
K	H
Affine space	
$K^n \cong Hom(K[X], K)$	$H^n \cong Hom(W(X), H)$
Points	
$\mu = (a_1, \dots, a_n)$	$\mu = (a_1, \dots, a_n)$
$\mu \in Hom(K[X], K)$	$\mu \in Hom(W(X), H)$
Solutions	
$f(a_1, \dots, a_n) = 0$	$w(a_1, \dots, a_n) = w'(a_1, \dots, a_n)$
or	
$\mu(f) = 0$	$\mu(w) = \mu(w')$
μ is a solution of f	μ is a solution of $w_i \equiv w_j$
$\Leftrightarrow f \in Ker(\mu)$	$\Leftrightarrow (w_i, w_j) \in Ker(\mu)$
Galois correspondence	
ideal T	congruence T
$\Updownarrow$	$\Updownarrow$
algebraic set A	algebraic set A
Galois-closed objects	
Radical ideal $I(A)$	Closed congruence A'_H
algebraic set $V(A)$	algebraic set T'_H
Topology	
Zariski topology	Zariski topology
Coordinate algebra	
Coordinate ring	Coordinate algebra
$K[X]/I(A)$	$W(X)/A'_H$
Category of algebraic sets	
$AG(K)$	$AG_\Theta(H)$
Morphisms	
Polynomial (regular) maps	

A few comments on this table. Since the point μ of an affine space is considered as a homomorphism from $Hom(W(X), H)$, it has a kernel $Ker(\mu)$. By definition, the kernel is exactly the set of all equations for which the point μ is a solution. This is how the Galois correspondence between points of an affine space and sets of equations arises. It is immediately transferred to a correspondence between subsets of an affine space and sets of equations.

Unfortunately, we no longer have a comfortable situation when closed syntactic objects are radical ideals. They are replaced by closed congruences. The description of closed congruences over a specific algebra H is the Hilbert's Nullstellensatz for a given $H \in \Theta$. The next Theorem 1 is true for an arbitrary variety of algebras.

Theorem 1.
- *The category of H-closed congruences is dually isomorphic to the category of algebraic sets*
- *The category of H-coordinate algebras is dually isomorphic to the category of algebraic sets.*

This duality theorem leads to the idea of considering these categories as algebraic-geometric (aka syntactic-semantic) invariants of algebras from the variety Θ. The geometric insight yields the following definition:

Definition 1 ([10], [12]). *Algebras H_1 and H_2 are called geometrically similar, if the corresponding categories $AG_\Theta(H_1)$ and $AG_\Theta(H_2)$ of algebraic sets are isomorphic.*

By virtue of the Galois correspondence, algebras are geometrically similar if and only if the categories of closed congruences and, respectively, coordinate algebras are isomorphic. This leads to the key notion of **geometrically equivalent algebras**.

Definition 2 ([10], [12]). *Algebras H_1 and H_2 are called geometrically equivalent if for every set of equations T and every set of variables X the corresponding H_1- and H_2-closed congruences coincide.*

It turns out that geometric equivalence admits a very clear syntactic-semantic description.

Theorem 2 ([10], [15]). *Two algebras are geometrically equivalent if and only if they have the same infinitary quasi-identities, or, equivalently, they generate the same infinitary quasi-varieties.*

We also note that classical algebraic geometry is Noetherian, and, according to Hilbert's theorem, every radical ideal is finitely generated. This is not true for an arbitrary variety of algebras, but it is true for some varieties, in particular for groups. For such varieties the coincidence of usual quasi-identities is enough for geometric equivalence.

To conclude this brief review of the foundations of universal algebraic geometry, we give another look at the concept of geometrically similar algebras. Suppose we have a specific algebra and want to deform it somehow, while preserving the geometry of the resulting algebra unchanged. The question is how many of such deformed algebras exist and how to describe them all. It turns out that the syntactic category Θ^0 of free in Θ algebras is responsible for such deformations. If this category does not have automorphisms other than internal ones, then there is no deformation and all the freedom of geometry is reduced to renaming variables. If there are external automorphisms, then there are also derived from H algebras that preserve the geometry.

5. Logical geometry

5.1. **General view of logical geometry.** Logical geometry appeared in the paper [9] (2004). In this paper, universal algebraic geometry is extended to the geometry of first-order logic over an arbitrary variety of algebras, that is, to logical geometry. It means, that algebraic sets are defined by arbitrary first-order formulas. In the case of logical geometry, algebraic sets are called definable (elementary) sets, and arbitrary first-order formulas replace equations.

In other words, the syntax and semantics of logical geometry coincide with the syntax and semantics of first-order logic. Therefore, the whole theory and all transitions are closely related to various concepts of logic and model theory.

We note two features of the approach to logical geometry, which play an essential role in what follows.

1. The system of concepts of algebraic logic is used as a working tool. In principle, everything can be translated into the usual language of model theory. However, the use of algebraic logic makes the main ideas more explicit and consistent. In particular, the apparatus of Halmos algebras is used for algebraization of syntax. Halmos algebras are algebras that correspond to first-order logic in the same way as Boolean algebras correspond to the propositional calculus.

2. Multi-sorted theory. There are many reasons for considering multi-sorted syntactic constructions. Some of them are associated with potential applications of algebraic logic and logical geometry in computer science, while others are purely algebraic in nature.

Finally, the main feature of logical geometry is that it assumes the presence of a Galois transition and, as a consequence, the realization of a syntactic-semantic square for the general case of a first-order language. This correspondence generalizes the constructions from algebra and geometry considered earlier.

As always, the basis of all further arguments lies in the variety of algebras Θ. Accordingly, all the basic concepts of logical geometry refer to some fixed Θ. Let us list them.

• Θ, variety of algebras which determines logic and geometry.

• Θ^0, syntactic category of all free algebras $W(X)$. Algebras $W(X)$ determine the syntax of elementary formulas, this is the place were the equations $w = w'$ live. Morphisms in Θ^0 are homomorphisms of free algebras.

• $\Theta(H)$, semantic category of affine spaces. The points μ of affine spaces are homomorphisms from $Hom(W(X), H)$.

• $\widetilde{\Phi} = (\Phi(X), X \in \Gamma)$, syntactic multi-sorted algebra of formulas. It is a Halmos algebra, i.e., an algebraization of the first order language. Its signature consists of boolean operations, quantifiers, and constants presented by atomic formulas $M = (M_X, X \in \Gamma)$, where M_X is the set of formulas $w \equiv w'$, $w, w' \in W(X)$. Algebras of the form $\Phi(X)$ is the place where the formulas live, where the logical deduction is built, i.e., the place where the rules of the given syntax play the game.

• Algebra $\widetilde{\Phi} = (\Phi(X), X \in \Gamma)$ can be treated as the category Hal^0_Θ, i.e., the category of formulas $\Phi(X)$ with morphisms $s_* : \Phi(X) \to \Phi(Y)$.

• $Hal_\Theta(H)$, semantic multi-sorted Halmos algebra of the form $Hal^X_\Theta(H)$. The algebra $Hal^X_\Theta(H)$ is the algebra of all subsets of the space $Hom(W(X), H)$. The operations of Halmos algebra in it are realized as intersection, union, addition, and cylindrical operations corresponding to quantifiers. Constants corresponding to the elements of $M = (M_X, X \in \Gamma)$ are also defined. This is the place where the solutions of systems of formulas, i.e., sets of points that satisfy the formulas of $\Phi(X)$, live. $Hal_\Theta(H)$ is also treated as a category with special polynomial morphisms associated with morphisms $s_* : \Phi(X) \to \Phi(Y)$.

Universal AG	**Logical geometry**
Variety	
Θ	Θ
Syntactic algebra	
$W(X)$, $\lvert X\rvert = n$	$\widetilde{\Phi} = (\Phi(X), X \in \Gamma)$
Elements of the syntactic algebra	
Words – $w(x_1, \ldots, x_n) \in W(X)$	Formulas – $\varphi(x_1, \ldots, x_n) \in \Phi(X)$
Equations	
$w \equiv w'$	$\varphi(x_1, \ldots, x_n) \in \Phi(X)$
Algebra in Θ	Algebra in Θ
H	H
Affine space	
$H^n \cong Hom(W(X), H)$	$Hal_X(H) = Bool(Hom(W(X), H))$
Points	
$\mu = (a_1, \ldots, a_n)$	$\mu = (a_1, \ldots, a_n)$
$\mu \in Hom(W(X), H)$	$\mu \in Hom(W(X), H)$
Solutions	
μ is a solution of $w_i \equiv w_j$	μ satisfies formula $\varphi \in \Phi(X)$
$\Leftrightarrow (w_i \equiv w_j) \in Ker(\mu)$	$\Leftrightarrow \varphi \in LKer(\mu)$
Galois correspondence	
congruence T	filter T
$\Updownarrow$	$\Updownarrow$ Value
algebraic set A	definable set A
Galois-closed objects	
closed congruence A'_H	closed filter A^L_H
algebraic set T'_H	definable set T^L_H
Topology	
Zariski Topology	Zariski Topology
Coordinate algebra	
Coordinate algebra	Coordinate algebra
$W(X)/A'_H$	$\Phi(X)/A^L_H$
Category of algebraic/definable sets	
$AG_\Theta(H)$	$LG_\Theta(H)$
Morphisms	
Polynomial (regular) maps	

• Hal_Θ is the variety of multi-sorted Halmos algebras. Algebras $\widetilde{\Phi} = (\Phi(X), X \in \Gamma)$ and $Hal_\Theta(H) = (Hal^X_\Theta(H),\ X \in \Gamma)$ belong to this variety.

• $Val_H : \widetilde{\Phi} \to Hal_\Theta(H)$. For every $X \in \Gamma$ there is a homomorphism $Val^X_H : \Phi(X) \to Hal^X_\Theta(H)$. Homomorphism Val^X_H calculates values of the formulas from $\Phi(X)$ in algebras $Hal^X_\Theta(H)$.

• $LKer(\mu)$ is the logical kernel of the point μ. It is defined due to existence of the homomorphism $Val^X_H = Val_X(H)$. $LKer(\mu)$ is the set of formulas φ, such that μ

belongs to the set of values of every $\varphi \in LKer(\mu)$. We will say that the point μ **satisfies** every $\varphi \in LKer(\mu)$.

Definition 3. *A subset A in $Hom_\Theta(W(X), H)$ is called definable if there exists the set T in $\Phi(X)$, such that A is the set of points μ satisfied by every formula φ from T.*

Now we can define a Galois correspondence and complete the construction of a syntactic-semantic square in logical geometry.

• L-Galois correspondence between sets of formulas T in $\Phi(X)$ and sets of points A in $Hom_\Theta(W(X), H)$ in terms of $LKer$ and Val_H^X. We have

$$T_H^L = A = \{\mu \in Hom(W(X), H) \mid T \subset LKer(\mu)\} = \bigcap_{u \in T} Val_H^X(u).$$

$$A_H^L = T = \bigcap_{\mu \in A} LKer(\mu) = \{u \in \Phi(X) \mid A \subset Val_H^X(u)\}.$$

Closed objects in L-Galois correspondence are definable sets and closed filters. This results in a required transition between syntax and semantics. L-Galois correspondence can be also treated as a passage between algebra and geometry, and between logic and models. It implies the existence of a commutative diagram.

$$\begin{tikzcd} \Phi_\Theta(Y) \arrow[r, "s_*"] \arrow[d, "Val_H^Y"'] & \Phi_\Theta(X) \arrow[d, "Val_H^X"] \\ Hal_\Theta^Y(H) \arrow[r, "s_*^H"] & Hal_\Theta^X(H). \end{tikzcd}$$

5.2. **Isotypeness and logical equivalence.** The key concept of the model theory is the concept of a type and, in particular, of the type of a point. The type of a point is the set of all formulas that are satisfied in a given point. In geometrical terminology, this is exactly its logical kernel. The intersection of all logical kernels gives a set of formulas that are valid in any point of the affine space over a fixed algebra H, i.e., its elementary theory.

Having syntactically semantic Galois correspondence and the whole system of concepts, we can reason at the level of semantics, that is, geometrically. Then the corresponding syntactic concepts arise automatically and naturally. Recall the initial question of the universal algebraic geometry.

• *Let two algebras H_1 and H_2 be given. When algebraic geometries associated with these algebras coincide? More precisely: when is the category of algebraic sets over H_1 equivalent to the category of algebraic sets over H_2?*

This question is purely geometric, it naturally arises because of the desire to understand the geometric characteristics of algebras. Raising it to the algebraic (syntactic) level we arrive at the concept of geometrically equivalent algebras and the answer: two algebras are geometrically equivalent if and only if they generate the same infinitary quasivariety.

• *Let us ask exactly the same question as above: when do the geometries of definable sets over H_1 and H_2 coincide? This means: when the category of definable sets over H_1 is isomorphic to the category of definable sets over H_2?*

Since there is an L-Galois correspondence, we again raise this question to the level of syntax. We immediately come to the concept of isotypic algebras and the question of isotipicity. Let us give formal definitions.

Definition 4 ([11], [14])**.** *Two algebras H_1 and H_2 are called isotypic if for any X and for any point $\mu : W(X) \to H_1$ there exists a point $\nu : W(X) \to H_2$ such that the types of μ and ν are the same, and for every point $\nu : W(X) \to H_2$ there exists a point $\mu : W(X) \to H_1$ such that their types are the same.*

Definition 5 ([11], [14])**.** *Two algebras H_1 and H_2 are called logically equivalent if for any X and for any set of formulas T in $\Phi(X)$ their H_1 and H_2 Galois closures coincide:*

$$T^{LL}_{H_1} = T^{LL}_{H_2.}$$

It turns out that:

Theorem 3 ([16])**.** *Algebras H_1 and H_2 are logically equivalent if and only if they are isotypic.*

Now one can argue exactly as it was done in Definition 1.

Definition 6 ([11], [15])**.** *Two algebras H_1 and H_2 are called logically similar if the categories of definable sets LG_{H_1} and LG_{H_2} are isomorphic.*

Isotypeness implies logical similarity, the opposite is not always the case. It is important that both geometric equivalence and isotypeness arose equally geometrically. The concept of elementary equivalence stands between them almost in the middle: isotypeness implies elementary equivalence, elementary equivalence implies geometric equivalence (in the case of Noetherian equations). Since all constructions are valid for arbitrary universal algebras, then the detailed description of the situation for specific varieties requires a separate study.

In addition, in the language of Galois correspondence, other concepts of the model theory acquire geometrical sounding: homogeneity, saturation, categoricity, Ryll-Nardzewski properties, and so on.

We shall note that universal algebraic geometry is an actively developing area. Recently, the book [3] where the topic is viewed from a slightly different angle, was published. Syntax and semantics are studied deeply in [1]. Among the urgent open problems, we note

- construction and description of coordinate algebras in the case of logical geometry,
- definition of rational morphisms of algebraic sets,
- study of the problem of dimension in the case of logical geometry.

In conclusion, it should be noted that another important task is the study of objects of universal logical geometry for various interesting concrete varieties of algebras and the determination of the exact form of syntactic-semantic transitions for these categories. Among such varieties we indicate a variety of near-rings associated with tropical geometry, a variety of Lie algebras, a variety of quasigroups (cf., [5]).

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