# ACTION OF ENDOMORPHISM SEMIGROUPS ON DEFINABLE SETS

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Abstract. The aim of the paper is to construct, discuss and apply the Galois-type correspondence between subsemigroups of the endomorphism semigroup $End(A)$ of an algebra $A$ and sets of logical formulas. Such Galois-type correspondence forms a natural frame for studying algebras by means of actions of different subsemigroups of $End(A)$ on definable sets over $A$. We treat some applications of this Galois correspondence. The first one concerns logic geometry. Namely, it gives a uniform approach to geometries defined by various fragments of the initial language. The next prospective application deals with effective recognition of sets and effective computations with properties that can be defined by formulas from a fragment of the original language. In this way one can get an effective syntactical expression by semantic tools. Yet another advantage is a common approach to generalizations of the main model theoretic concepts to the sublanguages of the first order language and revealing new connections between well-known concepts. The fourth application concerns the generalization of the unification theory or more generally Term Rewriting Theory to the logic unification theory.

## Introduction.

The aim of the paper is to establish a sort of syntactical-semantical Galois correspondence between classes of First Order formulas and semigroups of endomorphisms of algebras. So, we suppose that a first order language $\mathbf{L}$ in the signature $\Omega$ and an algebra $A$ are given. The algebra $A$ (in fact, $\mathbf{L}$-algebra $A$) enjoys two natural semigroups: the semigroup $End(A)$ of endomorphisms and the semigroup $\mathbf{T}(A)$ of transformations of $A$. Throughout the paper we will deal (mostly) with $End(A)$ and its subsemigroups. However, all results can be formulated with respect to $\mathbf{T}(A)$ as well. Note, parenthetically, that a large part of our setting admits natural generalization to multi-sorted case (cf., [43]). A separate theory is related to the two-sorted case where semigroup $End(A)$ acts on algebra $A$ and we have the two-sorted algebra $(A, End(A))$ (cf., [28]). It is known that under some conditions this case can be reduced to the one-sorted one ([30], [28], [29]). We could also replace an algebra $A$ by an arbitrary structure $A$. This generalization is especially meaningful in applications.

The classical syntactical-semantical connection in Model Theory is given by correspondence between sets of formulas and the associated definable sets. In the series of papers (see, for example, [38], [42], [39]) this connection takes the form of a Galois correspondence between elements of syntactical algebra $\Phi(X)$ and elements of semantical algebra $Hal^X_\Theta(A)$ (see [43] for the detailed explanations and notations). In particular this means that every notion related to sets of formulas can be reformulated via definable sets and vice versa.

Our aim is to introduce another Galois correspondence between sets of formulas in **L** and subsemigroups of $End(A)$. These semigroups act naturally on the definable sets in the affine space.

Our motivation is as follows. Its algebraic-geometric part is related to classical Galois theory and ideas of Klein's Erlangen Program. On the other hand, the Model Theoretic motivation goes back to preservation theorems (see, for example, [18], [17], [23], and [13], [1], [49], [48] for finite models). This is a set of theorems that characterize classes of formulas by closeness of the corresponding elementary classes with respect to action of certain homomorphisms. In general, this is a way to make a bridge between syntax and semantics on classes of formulas. One of the basic facts of Model Theory states that any definable set of a model is closed under the action of the automorphism group of this model. So automorphisms preserve any class of formulas of the first order logic. The question is what are the natural subclasses of formulas which can be described as classes preserving the action of homomorphisms of a special sort. This idea of the classification of formulas by the maps which they preserve is explicitly presented in the well known encyclopedic book "Model Theory" by W. Hodges [18]. The easy part of the characterization of the popular classes of formulas can be summed up as follows. Let $\phi$ be a first order formula. Then

- Every formula $\phi$ is preserved under isomorphisms of algebras.
- Every formula $\phi$ without universal quantifiers is preserved under monomorphisms of algebras.
- Every positive formula $\phi$ is preserved under epimorphisms of algebras.
- Every positive formula $\phi$ without universal quantifiers is preserved under homomorphisms of algebras.

So, we obtain classes of formulas whose syntactical structure forces a special relationship between the corresponding algebras. The converse direction requires more subtle technique and leads to classical Los-Tarski's, Lyndon's, and homomorphism preservation theorems.

Our setting looks as follows. We define the correspondence **G** between languages $K$ (sets of the first order formulas in a given signature $\Omega$) and subsemigroups of $End(A)$. Namely we assign to $K$ the set $\mathbf{G}(K)$ of endomorphisms of $A$ such that any set $D$ defined in $A^n$ by a system $\Phi \subset K$ is closed under the action of $\mathbf{G}(K)$. Conversely, we assign to a set $E$ of endomorphisms of $A$ the set $\mathbf{G}(E)$ of the first order formulas such that any set $D$ defined by a system $\Phi \subset \mathbf{G}(E)$ is closed under the action of $E$. Sets $D$ defined by a system of formulas $\Phi$ are called type-definable, or *t-definable*, see Section 1.

The Galois correspondence **G** defines Galois closed subsemigroups of $End(A)$ and Galois closed classes of formulas. Mentioned above four subsemigroups of $End(A)$ and four classes of formulas are classical candidates for the Galois closed objects. It appears that these objects are not necessarily **G**-closed and there are many other Galois closed objects.

This correspondence gives rise to a general framework for generalizations of the well known concepts of Model Theory such as homogeneous structures, oligomorphic models, Ryll-Nardzewski property, the quantifier elimination property, etc. The correspondence $\mathbf{G}$ can be naturally applied to classification, investigation and description of the type-definable sets in models and algebras and for effective computations with the type-definable sets. In particular, it can be applied to the classification of $\mathbf{L}'$-definable sets over a structure $M$ up to the action of a semigroup of transformations of $M$, where $\mathbf{L}'$ is a subset of the first order language $\mathbf{L}$.

The Galois correspondence $\mathbf{G}$ can also find applications in the context of Logical Geometry and the Constraint Satisfaction Problems (CSPs). Theoretical aspects of CSP stimulated a beautiful Galois-type theory developed by M. Bodirsky (see, for example, [5], [9], [6], [10], [8], etc.). It extends to countably categorical structures the ideas laid in the works of Kaluzhnin-Krasner school ([20], [21], [44], [11], etc.).

A possible way to classify the type-definable sets is to cover them by traces of elements under the action of an endomorphism semigroup $S$. This idea is used with respect to algebraic sets in the frames of Unification Types Theory, a part of Term Rewriting Theory (see [2], [3]). Our Galois setting gives rise to natural generalizations of the ideas of Unification Types Theory. Namely, we suggest the concept of logic unification $S$-covers.

More precisely, given a semigroup of endomorphisms $S \subset End(A)$, consider four possibilities for an algebra (model) $A$: any type-definable set over $A$ defined by a system $\Phi \subset \mathbf{G}(S)$ can be covered either 1) by the trace of one point under the action of $S$, 2) by traces of a finite set of points under the action of $S$, 3) by traces of an infinite and minimal (relative to inclusion) set of points under the action of $S$, but not by traces of a finite set of points under the action of $S$ or, in opposite, 4) has no minimal cover by traces of any set of points under the action of $S$.

In the first case $A$, by definition, has the unitary logic unification $S$-cover. Similarly, in the second case $A$ has the finitary logic unification $S$-cover, if it has no unitary logic unification $S$-cover. In the third case $A$ has the infinitary logic unification $S$-cover. In the fourth case $A$ has the zero logic unification $S$-cover.

The paper is organized as follows. In the first section we introduce notation and main concepts used in the paper. In the second section we define a correspondence between sets of formulas of given signature and subsemigroups of the endomorphism semigroup $End(A)$ of an algebra $A$. We note that this correspondence is a Galois correspondence and consider first examples. In fact, one of the aims of this section is to present a lot of various examples of the Galois correspondence $\mathbf{G}$, of Galois closed subsemigroups of $End(A)$ and Galois closed classes of formulas. We also consider a generalization of this Galois correspondence.

Section 3 deals with notions of logical and algebraic homogeneity with respect to the action of a subsemigroup $S$ of the endomorphism semigroup $End(A)$. We apply these concepts to characterizing subsets and relations over an algebra $A$ closed under the action of $S$ and for characterizing logical and algebraic traces and orbits of points over $S$. We prove that $A$ is logically $S$-homogeneous if and only if any subset $M$ of $A^n$ closed under the action of $S$ is the union of $\mathbf{G}(S)$-t-definable sets. In Section 4 we suggest a classification of type-definable sets with respect to different sorts of their coverings by traces of elements under the action of a subsemigroup $S$ of $End(A)$. We look at this classification from the prospectives of the well known model theoretic concepts: oligomorphic, Ryll-Nardzewski and countable categoricity properties. In particular, we obtain an $S$-version of Ryll-Nardzewski, Engeler and Svenonius result and apply it to the investigation of unification types.

In the last fifth section we formulate some problems arising within the frames of the Galois correspondence under consideration.

We use [18], [26] as model theoretic sources. The paper [42] is the general reference for algebraic approach to the first order logic. Historical and bibliographical remarks can be also found in [18]. Among many works that have stimulated the results of this paper, note [15], [37], [27].

## 1. Conventions and notation

1.1. **Conventions.** We use the word "algebra" in the sense of universal algebra, i.e., an algebra is a set with certain algebraic operations on it. We use the same notation for an algebra and for the set of its elements. A class $C$ of algebras is called *elementary* or *axiomatizable* if it can be defined by a finite or an infinite set of first order sentences. So we assume that all variables in the first order formulas which determine an axiomatizable class lie inside the scope of a quantifier. An axiomatizable class is a variety if it can be defined by identities. Each variety of algebras $\Theta$ contains free algebras $F(X)$, where $X = \{x_1, \dots, x_n\}$ is the set of free generators of $F(X)$. Given an algebra $A$ in the variety $\Theta$, denote by $A^n$ the $n$-th Cartesian power of $A$. Then $A^n$ can be identified with $Hom(F(X), A)$ where $|X| = n$. Namely, a point $\mu \in Hom(F(X), A)$ defined by $\mu(x_i) = a_i$ corresponds to $\bar{a} = (a_1, \dots, a_n)$. We call $A^n$ (and $Hom(F(X), A)$) the *affine space* over the algebra $A$. Throughout the paper we freely use one of the implementations of the affine space without a special notice.

Let $\mathbf{L}$ be a language and $\phi \in \mathbf{L}$. We use characters $\alpha, \beta, \varphi, f$ for homomorphisms; $\phi$ for formulas; $\mu, \nu$ and $\bar{a} = (a_1, \dots, a_n)$ for points in affine spaces, $D, M...$ for definable sets, etc. The notation $\phi(x_1, \dots, x_n)$ means that the set $\{x_1, \dots, x_n\}$ includes the set of all free variables in formula $\phi$. Capital letters $A$, $B$, ... are used for algebras or, more generally, for $\mathbf{L}$ -structures. Let $\bar{a} = (a_1, \dots, a_n)$ be a point of the space $A^n$. If the relation $\phi(x_1, \dots, x_n)$ is fulfilled on the point $\bar{a} = (a_1, \dots, a_n)$ then we say that the point $\bar{a}$ satisfies the formula $\phi(x_1, \dots, x_n)$. An algebraic variant of the notion of satisfiability can be found, for example, in [42].

A set $M \subset A^n$ is called *definable* if there exists a first order formula $\phi(x_1, \dots, x_n)$ such that a point $\bar{a} \in A^n$ belongs to $M$ if and only if $\bar{a}$ satisfies $\phi(x_1, \dots, x_n)$.

A set $M \subset A^n$ is called *t-definable* if there exists a set $K$ of first order formulas such that a point $\bar{a} \in A^n$ belongs to $M$ if and only if $\bar{a}$ satisfies every formula $\phi(x_1, \ldots, x_n)$ from $K$.

1.2. **Notation.** We mostly use standard notations. For the sake of convenience we provide the reader with the short list of the frequently used terms.

**Notation 1.1.**

- $\mathbf{T}(A)$ *is the semigroup of all transformations of an algebra* $A$.
- $End(A)$ *is the semigroup of all endomorphisms of an algebra* $A$.
- $Aut(A)$ *is the group of all automorphisms of an algebra* $A$.
- $\mathbf{S}End(A)$ *is the semigroup of all surjective endomorphisms of an algebra* $A$.
- $\mathbf{I}End(A)$ *is the semigroup of all injective endomorphisms of an algebra* $A$.
- $Hom(A, B)$ *is the set of all homomorphisms of an algebra* $A$ *to an algebra* $B$.
- $Iso(A, B)$ *is the set of all isomorphisms of an algebra* $A$ *to an algebra* $B$.
- $\mathbf{S}Hom(A, B)$ *is the set of all surjective homomorphisms of an algebra* $A$ *to an algebra* $B$.
- $\mathbf{I}Hom(A, B)$ *is the set of all injective homomorphisms of an algebra* $A$ *to an algebra* $B$.

By transformations of an algebra we mean transformations of its underlying set. Recall once again that most of results proved in the paper are equally valid for $End(A)$ and $\mathbf{T}(A)$.

Given an algebra $A$, the elementary theory $Th(A)$ is the set of all closed formulas valid on $A$. We say that formulas $\phi$ and $\psi$ are logically equivalent modulo $Th(A)$ if the universal closure of $\phi \leftrightarrow \psi$ belongs to $Th(A)$.

**Convention 1.** *By default, formulas are considered up to the equivalence modulo* $Th(A)$.

We refer to [18] for the definitions of the sets of positive, existential, positive-existential, and universal formulas. We use for them the notations **Po**, **Ex**, **PoEx**, respectively. We do not use a special abbreviation for the set of universal formulas.

**Notation 1.2.**

(1) *Given a formula* $\phi(x_1, \ldots, x_n)$ *and a point* $\mu \in Hom(F(X), A)$ *denote by* $Val(\phi, \mu)$ *the truth value of the formula* $\phi$ *in the point* $\mu$. *If the value of* $\phi$ *in* $\mu$ *is "true" we say that the point* $\mu$ *satisfies* $\phi$. *This definition is equivalent to the one given in Subsection 1.1.*

(2) *We write* $Val(\phi, \mu) = T$ *if the value of* $\phi$ *in* $\mu$ *is "true". The "false" value is denoted by* $Val(\phi, \mu) = F$.

(3) *Let* $K$ *be a set of formulas in a language* $\mathbf{L}$. *We abbreviate these formulas as* $K$*-formulas.*

(4) *Let* $K$ *be a set of formulas in a language* $\mathbf{L}$, *and let* $\mu$ *be a point in the affine space. Then* $Ktp(\mu)$, *the* $K$*-type of* $\mu$, *denotes the set of all formulas of* $K$ *satisfied by* $\mu$. *That is,* $Ktp(\mu) = \{\phi \in K | Val(\phi, \mu) = T\}$. *Notation* $Ktp^0(\mu)$ *is used for the set of all atomic* $K$*-formulas satisfied by* $\mu$ .

## 2. The correspondence between subsemigroups of $End(A)$ and systems of elementary formulas.

For the reader's convenience we repeat here some facts from Introduction. We shall start with an excerpt from the book of Hodges.

**Definition 2.1** (cf., [18])**.** *Let $f : A \to B$ be a function on $\mathbf{L}$-structures, $\phi(x_1, \dots, x_n)$ be a formula in $\mathbf{L}$. A formula $\phi$ is preserved by $f$ if $Val(\phi, \mu) = T$, where $\mu \in A^n$ implies $Val(\phi, \nu) = T$, where $\nu = f(\mu) \in B^n$.*

A formula $\phi$ is preserved by a set of functions if it is preserved by each function from this set.

For example, a function preserving the atomic formulas of $\mathbf{L}$ is a homomorphism of $\mathbf{L}$-structures. In particular, if $M$ is the set definable by $\phi$ then $f$ preserves $\phi$ if $f(M) \subset M$.

**Theorem 1.** *(*[18]*). Let $\phi(x_1, \dots, x_n)$ be an $\mathbf{L}$-formula.*

(1) *If $f \in Iso(A, B)$ then $f$ preserves $\phi$.*
(2) *If $f \in \mathbf{I}Hom(A, B)$ and $\phi$ is a formula without quantifiers $\forall$ then $f$ preserves $\phi$.*
(3) *If $f \in \mathbf{S}Hom(A, B)$ and $\phi$ is positive then $f$ preserves $\phi$.*
(4) *If $f \in Hom(A, B)$ and $\phi$ is a positive formula without quantifiers $\forall$ then $f$ preserves $\phi$.*

As it was mentioned, the converse statements are classical Lyndon's, Los-Tarski's and homomorphism preservation theorems.

**Theorem 2** (see, for example, [18], [49])**.**

- *A formula $\phi$ is preserved under arbitrary monomorphisms of algebras if and only if it is equivalent to a formula without universal quantifiers (Los-Tarski theorem).*
- *A formula $\phi$ is preserved under arbitrary epimorphisms of algebras if and only if it is equivalent to a positive formula (Lyndon's positivity theorem).*
- *A formula $\phi$ is preserved under arbitrary homomorphisms of algebras if and only it if is equivalent to a positive formula without universal quantifiers (Homomorphism preserving theorem).*

**Definition 2.2.** *Let $K$ be a set of formulas in the language $\mathbf{L}$. We assign to $K$ the set $\mathbf{G}(K)$, consisting of all endomorphisms $\alpha$ of $A$ such that any formula $\phi \in K$ is preserved under the action of $\alpha \in \mathbf{G}(K)$.*

**Remark 2.3.** *Let $K$ be a set of formulas in the language language $\mathbf{L}$. The correspondence above can be equivalently presented as follows:*

(1) *$\alpha \in \mathbf{G}(K)$ if and only if any subset $D$ of $A^n$ defined by a system $\Phi \subset K$ is closed under the action of $\alpha$, i.e., $\alpha D \subset D$,*

*or as follows*

(2) $\mathbf{G}(K) = \{\alpha \in End(A) | (\forall \mu \in D)(\forall \phi \in K)(Val(\phi, \mu) = T \rightarrow Val(\phi, \alpha\mu) = T)\}.$

**Proposition 1.** *Let $K$ be a set of formulas of the language $\mathbf{L}$. Then*

(1) *$\mathbf{G}(K)$ is a subsemigroup of $End(A)$,*
(2) *$\mathbf{G}(K)$ contains $Aut(A)$.*

*Proof.* Statement (1) is straightforward. The 2-nd statement follows from Theorem 1. □

**Definition 2.4.** *Let $S$ be a subsemigroup of $End(A)$. We assign to $S$ the set $\mathbf{G}(S)$ of all $\mathbf{L}$-formulas preserved under the action of $S$.*

**Remark 2.5.** *Since the empty set and the set $A$ of all elements of an algebra $A$ are closed under the action of $End(A)$ we obtain that for any endomorphism semigroup $S \subset End(A)$ the set of formulas $\mathbf{G}(S)$ contains all false formulas (which define the emptyset) and $Th(A)$.*

Note that since any formula $\phi \in \mathbf{G}(S)$ is preserved under the action of $S$ the set over A defined by a system $\Phi \subset \mathbf{G}(S)$ is closed under the action of $S$.

**Proposition 2.** *Let $S$ be a subsemigroup of $End(A)$. Any set of formulas $\mathbf{G}(S)$ contains the set* **PoEx**.

*That is*

(1) *$\mathbf{G}(S)$ contains all atomic formulas of the first order language of $A$.*
(2) *$\mathbf{G}(S)$ is closed under conjunctions $\wedge$ and under disjunctions $\vee$.*
(3) *$\mathbf{G}(S)$ is closed under quantifiers $\exists x$.*

*Proof.* Let $\alpha \in S$. Let $\phi, \psi \in \mathbf{G}(S)$.

(1) The first statement follows from the definition of the endomorphism of an algebra $A$.
(2) Let $Val(\phi \wedge \psi, \mu) = T$. Then $Val(\phi, \mu) = T$ and $Val(\psi, \mu) = T$. Therefore $Val(\phi, \alpha\mu) = T$ and $Val(\psi, \alpha\mu) = T$ because $\phi, \psi \in \mathbf{G}(S)$. Consequently $Val(\phi \wedge \psi, \alpha\mu) = T$. Hence $\phi \wedge \psi \in \mathbf{G}(S)$.
Let $Val(\phi \vee \psi, \mu) = T$. Then $Val(\phi, \mu) = T$ or $Val(\psi, \mu) = T$. Thus $Val(\phi, \alpha\mu) = T$ or $Val(\psi, \alpha\mu) = T$ because $\phi, \psi \in \mathbf{G}(S)$. Consequently $Val(\phi \vee \psi, \alpha\mu) = T$. Hence $\phi \vee \psi \in \mathbf{G}(S)$.
(3) Let variable $x$ is free in $\phi$. Without loss of generality we suppose that $x = x_1$. Take a point $\mu$ such that $\mu(x_i) = a_i$, $a_i \in A$. Let $Val(\phi, \begin{pmatrix} x_1 & \dots & x_n \\ a_1 & \dots & a_n \end{pmatrix}) = T$. Then $Val(\phi, \begin{pmatrix} x_1 & \dots & x_n \\ \alpha(a_1) & \dots & \alpha(a_n) \end{pmatrix}) = T$, because $\phi \in \mathbf{G}(S)$. Let us choose $\alpha(a_1) \in A$ as a value of $x$. We obtain $Val((\exists x)(\phi), \alpha\mu) = T$. Hence $(\exists x)(\phi) \in \mathbf{G}(S)$.

□

Let us note that

(1) Any $t$-definable set over an $\mathbf{L}$-algebra $A$ is closed under the action of $Aut(A)$.
(2) Any $t$-definable set over an $\mathbf{L}$-algebra $A$ defined by atomic formulas of $\mathbf{L}$ is closed under the action of $End(A)$.
(3) Any $t$-definable set over an $\mathbf{L}$-algebra without operations and with "equality" to be a single relation is closed under the action of $\mathbf{T}(A)$.

**Theorem 3.** *The correspondence $S \to \mathbf{G}(S)$ and $K \to \mathbf{G}(K)$ (Definitions 2.2 and 2.4 above) between subsemigroups of the endomorphism semigroup $End(A)$ and subsets of first order formulas in $\mathbf{L}$ is the Galois-type correspondence.*

*Proof.* The check of necessary properties is straightforward (see, for example, definition of the Galois correspondence in [25]).

□

**Proposition 3.** *If a $t$-definable set $D$ over $A$ is closed under the action of $S \subset End(A)$ then $D$ can be defined by a system of formulas $\Phi \subset \mathbf{G}(S)$. Conversely, if a $t$-definable set $D$ over $A$ is not closed under the action of $S \subset End(A)$ then $D$ can not be defined by a system of formulas $\Phi \subset \mathbf{G}(S)$.*

This proposition is, in fact, a reformulation of the definition of $\mathbf{G}(S)$. However, it gives an effective semantical recognition method for the sets that can be defined by formulas of a special form.

A set of formulas $K$ in $\mathbf{L}$ is called Galois closed (or just $\mathbf{G}$-closed) if $\mathbf{GG}(K) = K$. Analogously, a subsemigroup $S$ in $End(A)$ is $\mathbf{G}$-Galois closed if $\mathbf{GG}(S) = S$. $\mathbf{G}(S)$ is a $\mathbf{G}$-closed set of formulas and $\mathbf{G}(K)$ is a $\mathbf{G}$-closed semigroup.

**Remark 2.6.** *Once we have a Galois correspondence, closed objects are the main ones in this theory. In particular, properties of an algebra $A$ can be studied via characterization of closed objects. In particular, let $S \subset T \subset \mathbf{GG}(S)$, $K \subset M \subset \mathbf{GG}(K)$ and $\mu \in A^n$. Then*

- $\mathbf{G}(S) = \mathbf{G}(T)$.
- $\mathbf{G}(S)tp(\mu) = \mathbf{G}(T)tp(\mu)$.
- $\mathbf{G}(K) = \mathbf{G}(M)$.

$\mathbf{G}$-closure operators define natural derived structures on an algebra $A$. Note that the intersection of $\mathbf{G}$-closed subsemigroups of $End(A)$ and correspondingly the union of G-closed sets of formulas are G-closed. Therefore the $\mathbf{G}$-closed objects form 2 lattices. The second lattice operation in the lattice of $\mathbf{G}$-closed subsemigroups is the closure of the union of the semigroups.

We say that subsemigroups $S$ and $T$ of $End(A)$ are $\mathbf{G}$-equivalent if $\mathbf{GG}(S) = \mathbf{GG}(T)$ and denote this relation by $S \equiv_{\mathbf{G}} T$. Similarly subsets $K$ and $M$ of $\mathbf{L}$ are $\mathbf{G}$-equivalent if $\mathbf{GG}(K) =$

$\mathbf{GG}(M)$. They are denoted $K \equiv_{\mathbf{G}} M$. These equivalence relations are completely described by the corresponding $\mathbf{G}$-closed objects.

**Example 1.** *The following proposition gives examples of* $\mathbf{G}$*-closed semigroups.*

**Proposition 4.**

(1) *Denote by* $\mathbf{E}lEnd$ *the semigroup of elementary embeddings of* $A$ *into itself.* $\mathbf{G}(\mathbf{L}) = \mathbf{E}lEnd$ *and* $\mathbf{G}(\mathbf{E}lEnd) = \mathbf{L}$.
(2) $\mathbf{G}(\mathbf{Ex}) = \mathbf{I}End(A)$.
(3) *The semigroups* $End(A)$, $\mathbf{I}End(A)$ *are Galois closed.*
(4) *The semigroup* $Aut(A)$ *is Galois closed whenever* $A$ *is finite or co-Hopfian.*

*Proof.*

(1) Elementary embedding preserves any formula of $\mathbf{L}$. Conversely, let $\alpha \in End(A)$ preserve any formula of $\mathbf{L}$. It is injective because it preserves formula $x \neq y$ and it is elementary embedding because it preserves any formula of $\mathbf{L}$. Note that any automorphism is elementary embedding but not any elementary embedding of $A$ into itself is automorphism.

    The equality $\mathbf{G}(\mathbf{E}lEnd)) = \mathbf{L}$ follows from the fact that $\mathbf{G}(\mathbf{E}lEnd) = \mathbf{GG}(\mathbf{L}) = \mathbf{L}$.
(2) $\mathbf{I}End(A) \subset \mathbf{G}(\mathbf{Ex})$ by Theorem 1. Conversely, any endomorphism $\alpha \in \mathbf{G}(\mathbf{Ex})$ preserves formula $x \neq y$ and, therefore, $\alpha \in \mathbf{I}End(A)$.
(3) $End(A)$ is closed due to our convention to deal in this paper only with subsemigroups of $End(A)$.

    Suppose that $\mathbf{GG}(\mathbf{I}End(A)) = S$ and $\alpha \in S$. Let $D \subset A^2$ be the set defined by a formula $x \neq y$. This formula belongs to $\mathbf{G}(\mathbf{I}End(A))$ because any $\beta \in \mathbf{I}End(A)$ preserves it. Consequently $\alpha$ preserves formula $x \neq y$. Therefore $\alpha \in \mathbf{I}End(A)$. Thus $S = \mathbf{GG}(\mathbf{I}End(A)) \subset \mathbf{I}End(A)$. Conversely, $\mathbf{I}End(A) \subset \mathbf{GG}(\mathbf{I}End(A))$ by Theorem 3.
(4) If algebra $A$ is finite then $Aut(A) = \mathbf{S}End(A) = \mathbf{I}End(A)$. Hence $Aut(A)$ is $\mathbf{G}$-closed. The same is true for co-Hopfian algebras where $\mathbf{I}End(A) = Aut(A)$.

□

In this example we use Convention 1. Let us call semigroups of endomorphisms and the corresponding sets of formulas from Theorem 2 the *classical* semigroups of endomorphisms and sets of formulas or just classical objects. Example 1 gives us the classical $\mathbf{G}$-Galois closed semigroups and $\mathbf{G}$-Galois closed sets of formulas. Examples below show that classical objects are not necessarily $\mathbf{G}$-closed and there are many non classical $\mathbf{G}$-closed sets and $\mathbf{G}$-closed semigroups. For example $\mathbf{G}(\mathbf{L}) = \mathbf{E}lEnd$ does not necessarily coincide with $Aut(A)$, i.e. there exist endomorphisms, which are not automorphisms, but preserve all sets definable over $A$.

If for specific algebras some classical endomorphism semigroups coincide, then by Proposition 3 formulas of a wider class are equivalent to formulas of a smaller class. This is the case for a vast class of algebras, see further examples.

**Example 2.**

- *Let $\Gamma = (Z_p)^{\mathbb{N}}$, i.e., $\Gamma$ is infinite direct product of cyclic groups of prime order. Then a monomorphism $\varphi : \Gamma \to \Gamma$ which shifts the $i$-th component to the $(i+1)$-th one preserves all definable sets. So, $\mathbf{G}(\mathbf{L}) \neq Aut(\Gamma)$. This example shows that $Aut(A)$ is not necessarily $\mathbf{G}$-closed even for countably categorical algebras (cf.,* [6]*).*
- *Let $F_2$, $F_3$ be two- and three-generator free groups, respectively. Take $\Gamma = F_2 \times F_3^{\mathbb{N}}$. By* [46] *the natural embedding $f : F_2 \to F_3$ is elementary. Then monomorphism $\varphi : \Gamma \to \Gamma$ which sends $F_2$ to $f(F_2)$ and shifts all other copies of $F_3$ as above preserves all definable sets. Hence, $\mathbf{G}(\mathbf{L}) \neq Aut(\Gamma)$. The similar examples can be constructed on the base of hyperbolic groups* [32], [33].
- *Let $\Gamma = (\mathbb{Q}, <)$ be the ordered set of rational numbers. It also can be considered as a semigroup with operation min. It is well known that $(\mathbb{Q}, <)$ possesses quantifier elimination. So, any embedding $\Gamma \to \Gamma$ is elementary. Therefore, the injective endomorphism $x \to x^3$ belongs to $\mathbf{G}(\mathbf{L})$. Thus, $\Gamma$ is not co-Hopfian and $Aut(\Gamma)$ is not $\mathbf{G}$-closed.*
- *Consider the abelian group $A = (\mathbb{Q}, +)$, which is model complete and co-Hopfian. It means that $\mathbf{G}(\mathbf{L}) = Aut(A)$, and $Aut(A)$ is $\mathbf{G}$-closed.*

**Remark 2.7.** *Recall that algebra $A$ is model complete if every embedding of $A$ in $A$ is elementary. Suppose that $Th(A)$ is model complete. Then $Aut(A)$ is $\mathbf{G}$-closed if and only if $A$ is co-Hopfian. Note that apart from abelian groups like $(\mathbb{Q}, +)$, see also* [12]*, there are lots of non-abelian co-Hopfian groups. Among them are some torsion-free nilpotent groups* [4]*, some hyperbolic groups* [47]*, $SL_n(\mathbb{Z})$, $n \geq 3$* [16]*, etc. Moreover, if $A$ is countably categorical and model complete then every formula is equivalent to $\mathbf{Ex}$-formula, see* [18] *and* [5].

**Example 3.**

(1) *If $A$ is a congruence free algebra then it has the unique endomorphism, namely the identical automorphism. Therefore all first order formulas in the language of $A$ form the unique $\mathbf{G}$-Galois closed system of formulas and correspondingly $Aut(A) = End(A)$ is the unique closed subsemigroup of $End(A)$. Hence, any definable over $A$ set can be defined by positive existential formulas. Thus, one can get an effective syntactical expression by semantic tools.*

(2) *Let $\Phi$ be a set of the first order formulas with $n$ free variables which define the empty set over algebra $A$. Then $\mathbf{G}(\Phi) = End(A)$ (or even $\mathbf{G}(\Phi) = \mathbf{T}(A)$ in the setting with*

*the transformation semigroup* $\mathbf{T}(A)$*) and the Galois closed set of formulas* $\mathbf{GG}(\Phi)$ *equals* **PoEx**.

The notion of Galois closed semigroup admits the following generalization.

**Definition 2.8.** *Let* $\mathbf{L}(f)$ *be the extension of language* $\mathbf{L}$ *of* $A$ *by the unary function symbol* $f$ *and* $\Phi$ *be a set of formulas in* $\mathbf{L}(f)$*. We say that a subsemigroup* $S$ *of* $End(A)$ *is defined in* $End(A)$ *by a set of formulas* $\Phi$*, i.e.,* $S$ *is* $\Phi$-definable *if it consists of all elements of* $End(A)$ *which satisfy* $\Phi$.

*Let* $B$ *be an algebra with the same signature as* $A$*.* $\Phi$ *is defined as above. We say that a subset* $S$ *of* $Hom(A,B)$ *is defined in* $Hom(A,B)$ *by a set of formulas* $\Phi$*, i.e.,* $S$ *is* $\Phi$-definable *if it consists of all elements of* $Hom(A,B)$ *which satisfy* $\Phi$.

**Example 4.** *Let* $S$ *be a subsemigroup of* $End(A)$.

1. $End(A)$ *is* $(\forall x)(\forall y)(f(xy) = f(x)f(y))$*-definable.*
2. $\mathbf{I}End(A)$ *is* $(\forall x)(\forall y)(f(xy) = f(x)f(y)\&((x \neq y) \to (f(x) \neq f(y))))$*-definable.*
3. $\mathbf{S}End(A)$ *is* $(\forall x)(\forall y)(f(xy) = f(x)f(y)\&(\forall z)(\exists t)(f(t) = z)))$*-definable.*

**Example 5.** *Let* $A$ *be an algebra such that the semigroup* $Const = End_{\leq 1}(A) = \{f_c | c \in A, (\forall a \in A)f(a) = c\}$ *lies in* $End(A)$*. Note that semigroup of transformations* $Const$ *is not, in general, a semigroup of endomorphisms. However,* $Const \subset End(A)$ *for some algebras. For example this is the case for* $A = LZ$ *a left zero semigroup (we remind that* $xy = x$ *for all* $x, y \in LZ$*). Define* $C_n \subset A^n$ *by* $C_n = \{(a, ..., a) | a \in A\}$.

*Define* $\mathbf{E}lEnd_C(A) = \{\alpha \in End(A) | \alpha(D) \subset D \cup C_n\}$ *for any definable set* $D$ *and call it the semigroup of* $C$*-elementary embeddings.*

*A non empty definable over* $A$ *set* $D \subset A^n$ *is* $\mathbf{G}(Const)$*-definable if and only if* $C_n \subset D$*. Indeed, for any* $\alpha \in Const$ *we obtain* $\alpha(D) \subset C_n \subset D$*. Conversely, let* $D \neq \emptyset$ *be closed under the action of* $Const$*,* $(a_1, ..., a_n) \in D$ *and* $(a, ..., a) \in C_n$*. Then* $f_a(a_1, ..., a_n) = (a, ..., a) \in D$.

**Proposition 5.** *Let* $Const = End_{\leq 1}(A) = \{f_c | c \in A, (\forall a \in A)f_c(a) = c\} \subset End(A)$*. Let* $\Phi \subset \mathbf{L}$ *be the set of all formulas* $\phi(x_1, ..., x_n)$ *of* $\mathbf{L}$ *such that* $\phi(x_1, ..., x_n)$ *implies the sentence* $(\forall x)\phi(x, ..., x)$*. Then* $\mathbf{G}(Const) = \Phi$ *and* $\mathbf{GG}(Const)$ *consists of the set of all* $\alpha \in \mathbf{E}nd(A)$ *such that* $\alpha(D) \subset D \cup C_n$ *for any definable set* $D$ *over* $A$*, i.e. of all* $C$*-elementary embeddings. Thus* $\mathbf{GG}(Const) = \mathbf{E}lEnd_C(A)$.

*Proof.* Let $\phi(x_1, ..., x_n)$ define the set $D \subset A^n$. Let $\phi(x_1, ..., x_n) \in \mathbf{G}(Const)$. Then $C_n = \{\alpha(D) | \alpha \in Const\} \subset D$. It means that if $Val(\phi(x_1, ..., x_n), \mu) = T$ then $Val(\phi(x_1, ..., x_n), \alpha\mu) = T$ for any $\alpha \in Const, \mu \in A^n$. Therefore $Val((\forall x)\phi(x, ..., x)) = T$ and, thus, $\phi(x_1, ..., x_n)$ implies the sentence $(\forall x)\phi(x, ..., x)$, i.e., $\phi(x_1, ..., x_n) \in \Phi$. So, $\mathbf{G}(Const) \subset \Phi$. Conversely, let $\phi(x_1, ..., x_n)$ implies the sentence $(\forall x)\phi(x, ..., x)$. Let $\alpha \in Const$. Let $\mu = (a_1, ..., a_n) \in D$ and $(\alpha(a_1), ..., \alpha(a_n)) = (a, ..., a)$. By the assumption, $Val((\forall x)\phi(x, ..., x)) = T$. Therefore

$Val(\phi(a,...,a)) = T$, $(a,...,a) \in D$ and $D$ is closed under the action of $Const$. Hence $\Phi \subset \mathbf{G}(Const)$. Thus $\Phi = \mathbf{G}(Const)$.

Let $\alpha \in \mathbf{E}nd(A) \setminus \mathbf{E}lEnd_C(A)$. Suppose that $\alpha(D) \not\subset D \cup C_n$, where $D$ is defined by a formula $\phi(x_1,...,x_n)$. The set $D \cup C_n$ is definable by the set of two formulas $\{\phi(x_1,...,x_n)\}$ and $\{\phi(x_1,...,x_n) \to (\forall x)\phi(x,...,x) | \phi \in \Psi\}$. It is closed under the action of $Const$ but it is not closed under the action of $\alpha$. Therefore $\alpha$ does not belong to the $\mathbf{G}$-closure of $Const$. Conversely all $C$-elementary embeddings belong to $\mathbf{GG}(Conat)$. Thus $\mathbf{GG}(Const)$ consists of the set of all $\alpha \in \mathbf{E}nd(A)$ such that $\alpha(D) \subset D \cup C_n$ for any definable set $D$ over $A$, i.e., $\mathbf{GG}(Const) = \mathbf{E}lEnd_C(A)$. Note that $\mathbf{E}lEnd_C(A)$ is a semigroup. □

**Remark 2.9.** *$\mathbf{G}(Const)$ consists of formulas $\phi(x_1,...,x_n)$ such that $\phi(x_1,...,x_n) \to (\forall x)\phi(x,...,x)$ is a true formula with respect to $A$ or, equivalently, of formulas $\phi(x_1,...,x_n)$ such that $\phi(x_1,...,x_n)$ implies $\phi(x_1,...,x_n) \to (\forall x)\phi(x,...,x)$.*

*Note that $\mathbf{G}(Const)$ and $\mathbf{GG}(Const)$ are non classical $\mathbf{G}$-closed objects. Similar description can be obtained for $\mathbf{G}(\mathbf{T}_{\leq k}(A))$ and $\mathbf{GG}(\mathbf{T}_{\leq k}(A))$, where $\mathbf{T}_{\leq k}(A)$ is the semigroup of rank $\leq k$ transformations of such an algebra $A$ that $\mathbf{T}_{\leq k}(A) \subset End(A)$. For example, it is the case for $A = LZ$ a left zero semigroup.*

By reasoning similar to the previous proof one can obtain non classical $\mathbf{G}$-Galois closed sets for models.

**Example 6.** *Let $A$ be an algebra and let $z \in A$ be an element such that $t(z,...,z) = z$ for any signature operation $t$ of $A$. For example, $z$ is the identity element in a group or $z$ is the zero element in a ring. Then the function $0(x) = z$ for all $x \in A$ belongs to $End(A)$ and forms a one-element subsemigroup $\mathbf{O}$ of $End(A)$.*

*Define $\mathbf{E}lEnd_0(A) = \{\alpha \in End(A) | \alpha(D) \subset D \cup \{(z,...,z)\}$ for any definable set $D$ and call it the semigroup of the $0$-elementary embeddings.*

*A non empty definable over $A$ set $D \subset A^n$ is $\mathbf{G}(\mathbf{O})$-definable if and only if $(z,...,z) \in D$. Indeed, if $(a_1,...,a_n) \in D$ and $D$ is closed under the action of $\mathbf{O}$ then $(z,...z) = (0(a_1),...,0(a_n)) \in D$. Conversely, if $(z,...z) = (0(a_1),...,0(a_n)) \in D$ then $D$ is closed under the action of $\mathbf{O}$.*

**Proposition 6.** *An algebra $A$, element $z \in A$ and the one element subsemigroup $\mathbf{O}$ of $End(A)$ were defined above. Let $\{z\}$ be a definable set (in particular, $z$ is a constant). Then $\mathbf{G}(\mathbf{O})$ consists of all formulas $\phi(x_1,...,x_n)$ such that $\phi(x_1,...,x_n)$ implies formula $(\exists y)(\psi(y) \wedge \phi(y,...,y))$, where the formula $\psi$ defines the set $\{z\}$. $\mathbf{GG}(\mathbf{O})$ consists of all $\alpha \in \mathbf{E}nd(A)$ such that $\alpha(D) \subset D \cup \{(z,...,z)\}$ for any definable set $D$ over $A$, i.e., of all $0$-elementary embeddings.*

*Proof.* Let $D$ be a non empty set definable by a formula $\phi(x_1,...,x_n)$. Recall that $D$ is $\mathbf{G}(\mathbf{O})$-definable if it is closed under the action of $\mathbf{O}$. We have proved that this is the case if and only if

$(z,...,z) \in D$. Let us prove that $D$ is $\mathbf{G}(\mathbf{O})$-definable if and only if $D$ is definable by the set of formulas $\phi(x_1,...,x_n)$ such that $\phi(x_1,...,x_n)$ implies $(\exists y)(\psi(y) \wedge \phi(y,...,y))$, where formula $\psi$ defines set $\{z\}$. Let $D$ be $\mathbf{G}(\mathbf{O})$-definable, $D$ is defined by a a formula $\phi$. Then $(z,...,z) \in D$. Therefore $Val(\phi(z,...,z)) = T$ and $\phi(x_1,...,x_n)$ implies $(\exists y)(\psi(y) \wedge \phi(y,...,y))$, where formula $\psi$ defines set $\{z\}$. Conversely, suppose that $D$ is defined by a formula $\phi$ which implies $(\exists y)(\psi(y) \wedge \phi(y,...,y))$, where formula $\psi$ defines the set $\{z\}$. Let $\mu = (a_1,...,a_n) \in D$. Then $(z,...,z) \in D$ because $Val((\exists y)(\psi(y) \wedge \phi(y,...,y)), \mu) = T$. Thus $D$ is closed under the action of $\mathbf{O}$. Therefore $\mathbf{G}(\mathbf{O})$ consists of all formulas $\phi(x_1,...,x_n)$ such that $\phi(x_1,...,x_n)$ implies$(\exists y)(\psi(y) \wedge \phi(y,...,y))$.

Let $\alpha \in \mathbf{E}nd(A) \backslash \mathbf{E}lEnd(A)$. Suppose that $\alpha(D) \not\subset D \cup \{(z,...,z)\}$, where $D$ is defined by a formula $\phi(x_1,...,x_n)$. We proved above that the set $D \cup \{(z,...,z)\}$ is definable by two formulas $\phi(x_1,...,x_n)$ and $\phi(x_1,...,x_n) \rightarrow (\exists y)(\psi(y) \wedge \phi(y,...,y))$ and it is closed under the action of $\mathbf{O}$ but it is not closed under the action of $\alpha$. Therefore $\alpha$ does not belong to the $\mathbf{G}$-closure of $\mathbf{O}$. Hence $\mathbf{GG}(\mathbf{O})$ consists of of all $\alpha \in \mathbf{E}nd(A)$ such that $\alpha(D) \subset D \cup \{(z,...,z)\}$ for any definable set $D$ over $A$, i.e. of all 0-elementary embeddings. Thus $\mathbf{GG}(\mathbf{O}) = \mathbf{E}lEnd_0(A)$.

□

This example shows that one can obtain non classical $\mathbf{G}$-closed sets for algebras with zero or identity.

Consider the following natural generalization of the notion of Galois closed semigroup. Define the correspondence $\overline{\mathbf{G}}$ as follows. Given a subsemigroup $S \subset End(A)$, define $\overline{\mathbf{G}}(S)$ as the set of all formulas of an extension of the first order language $\mathbf{L}$ such that any formula $\varphi \in \overline{\mathbf{G}}(S)$ is preserved under the action of $S$. If we denote this extension of $\mathbf{L}$ by $\overline{\mathbf{L}}$ then one can take $\overline{\mathbf{G}}(S) = \mathbf{G}_{\overline{L}}(S)$. A semigroup $S$ of $End(A)$ is $\overline{\mathbf{G}}$-closed if $\overline{\mathbf{GG}}(S) = S$.

**Example 7.** *Let $M$ be a subset of $A$. Consider semigroup $S = Fix_M$ of all endomorphisms of $A$ which fix each element of a subset $M$.*

**Proposition 7.** *Semigroup $S = Fix_M$ is $\overline{\mathbf{G}}$-closed.*

*Proof.* Let $D$ be a subset defined by **PoEx** formulas with parameters from $M$. Then $D$ is closed under the action of $S$ because each **PoEx** formula is preserved by the whole $End(A)$ and elements of $M$ are fixed by any $\beta \in S$. Let $\alpha \in End(A)$ and $\alpha \notin S$. There is a point $c \in M$ such that $\alpha(c) \neq c$. Let $\phi$ be a **PoEx** formula with the parameter $c$. Then $\phi$ is preserved by $S$ but is not preserved by $\alpha$. Therefore $\alpha$ does not belong to the closure $\overline{\mathbf{GG}}(S)$ of $S$. Thus $S = Fix_M$ is $\overline{\mathbf{G}}$-closed. □

**Remark 2.10.** *Note that $\overline{\mathbf{G}(Fix_M)}$ consists of* **PoEx** *formulas with parameters from $M$. Thus we can consider the class $\overline{\mathbf{G}(Fix_M)}$ in order to enrich the language* **L** *by a set of constants.*

Let $\mathbf{I}$ be the identity automorphism. We know that $G(\mathbf{I}) = \mathbf{L}$ and $\mathbf{GG}(\{\mathbf{I}\}) = \mathbf{G}(\mathbf{L})$. On the other hand we can look at $\{\mathbf{I}\}$ as a $\overline{\mathbf{G}}$-closed object. Then $\overline{\mathbf{G}}(\mathbf{I}) = \mathbf{L}_A$ where the language $\mathbf{L}_A = \mathbf{L} \cup \{a | a \in A\}$ is obtained of $\mathbf{L}$ by adding all elements of $A$ as constants.

**Remark 2.11.** *If each element of $M$ is $\mathbf{G}(Fix_M)$-definable, then we do not need to enrich the language by a set of constants and $S = Fix_M$ is $\mathbf{G}$-closed.*

The following example is similar to Example 7.

**Example 8.** *Let $S = Stab_M$ be the semigroup of all endomorphisms (automorphisms) of $A$ which fix a subset $M$ of $A$, i.e., for any $f \in S$ we have $f(M) \subset M$. Then $S = Stab_M$ is $\overline{\mathbf{G}}$-closed.*

We shall finish with the example based on the variety of groups.

**Example 9.** *Let $\Gamma$ be a non-abelian group with the nontrivial center $Z(\Gamma) \neq 1$. Subgroup $Z(\Gamma)$ is definable by the formula $\phi$ of the form $\forall x(xy = yx)$. Denote $\mathbf{G}(\phi) = S$. Clearly $Aut(\Gamma) < S$. Moreover $\mathbf{S}End(\Gamma) < S$, since the center is a strictly characteristic subgroup, that is an epimorphism-invariant subgroup. It is clear that $\mathbf{S}End(\Gamma) \neq S$ because endomorphism $s_0$, defined by $s_0(g) = 1$ for every $g \in \Gamma$, belongs to $S$. It is well known that center $Z(\Gamma)$ is not a fully invariant subgroup (see, for example,* [45]*), that is $Z(\Gamma)$ is not endomorphism invariant. So, $\mathbf{S}End(\Gamma) < S < End(\Gamma)$. $S$ also cannot be equal to $\mathbf{I}End(\Gamma)$ because $S$ contains $\mathbf{S}End(\Gamma)$. See below an explicit example*

*There are standard examples showing that $Z(\Gamma)$ is not necessarily injective-invariant subgroup. One of the examples is as follows. Take $\Gamma$ to be a simple group and let $H$ be isomorphic to an abelian subgroup in $\Gamma$. Let $\Gamma_1$ be the infinite direct product of $H$ and of countable number copies of $\Gamma$. Then $Z(\Gamma_1) = H \times (1) \times (1)....$ An injective endomorphism of $\Gamma_1$ which sends $H$ to its copy in $\Gamma$ and shifts the i-copy of $\Gamma$ to the $(i+1)$-copy, does not preserve the center of $\Gamma_1$.*

*Hence, $S \neq \mathbf{I}End(\Gamma)$ and $S$ is a $\mathbf{G}$-closed non classical subgroup. Take $\mathbf{G}(S) = \Phi$. Then by definition, $\Phi$ is a $\mathbf{G}$-closed non classical set of formulas.*

## 3. $\mathbf{G}(S)$-$t$-definable sets in $S$-homogeneous algebras.

Let $A$ be an algebra, $S$ a subsemigroup of $End(A)$ and $I$ the identity endomorphism. If $I \notin S$ then $S^1 = S \cup I$ is a submonoid of $End(A)$. In this section we assume that $S = S^1$ whenever this assumption is required. In Section 2 we defined the class $\mathbf{G}(S)$ of the first order formulas which corresponds to $S$. Recall that the $\mathbf{G}(S)$-type of a point $\mu \in A^n$ is the class of all $\mathbf{G}(S)$-formulas which are satisfied by $\mu$ and we denote this class by $\mathbf{G}(S)tp(\mu)$.

**Definition 3.1.** *Let $S = S(A)$ be a subsemigroup of $End(A)$. Let $\mu \in A^n$ be a point in the affine space.*

(1) *The algebraic $S$-trace of $\mu$ is the set $S\mu = \{\nu = \alpha\mu | \alpha \in S\}$.*

(2) *The algebraic* $S$*-orbit of* $\mu$ *is the set* $S_{AO}(\mu) = \{\nu | S\mu = S\nu\}$.

(3) *The logical* $S$*-trace of* $\mu$ *is the set* $S_{LT}(\mu) = \{\nu | \mathbf{G}(S)tp(\mu) \subset \mathbf{G}(S)tp(\nu)\}$.

(4) *The logical* $S$*-orbit of* $\mu$ *is the set* $S_{LO}(\mu) = \{\nu | \mathbf{G}(S)tp(\mu) = \mathbf{G}(S)tp(\nu)\}$.

Let $S, T$ be subsemigroups of $End(A)$ such that $S \subset T \subset \mathbf{GG}(S)$. Then it follows from Remark 2.6 that the logical $S$-trace ($S$-orbit) of $\mu$ coincides with the logical $T$-trace ($T$-orbit) of $\mu$.

Now we turn to homogeneous algebras. These structures are extensively studied in the literature (see, for example, [24], [15]).

In view of the next definition observe that the condition $\mathbf{G}(S)tp(\mu) \subset \mathbf{G}(S)tp(\nu)$ means that the point $\nu$ satisfies any system of $\mathbf{G}(S)$-formulas satisfied by $\mu$.

**Definition 3.2.** *Let* $S = S(A)$ *be a subsemigroup of* $End(A)$. *We call an algebra logically* $S$*-homogeneous if for any* $\mu \in A^n$ *the logical* $S$*-trace of* $\mu$ *is contained in the algebraic* $S$*-trace of* $\mu$*, i.e.,* $(\forall \mu \in A^n)(S_{LT}(\mu) \subset S\mu)$. *In other words an algebra is logically* $S$*-homogeneous if and only if for any* $\mu, \nu \in A^n$ *such that* $\mathbf{G}(S)tp(\mu) \subset \mathbf{G}(S)tp(\nu)$ *there exists* $\alpha \in S$ *such that* $\nu = \alpha\mu$.

In particular*:*

*An algebra* $A$ *is logically End-homogeneous if for any* $\mu, \nu \in A^n$ *such that the* **PoEx***-type of* $\nu$ *contains the* **PoEx***-type of* $\mu$ *(*$\nu$ *satisfies any system of* **PoEx***-formulas satisfied by* $\mu$*) there exists* $\alpha \in End(A)$ *such that* $\nu = \alpha\mu$.

*An algebra* $A$ *is logically Aut-homogeneous or just logically homogeneous if for any iso-typed* $\mu, \nu \in A^n$ *(*$\mu$ *and* $\nu$ *satisfy the same formulas) there exists* $\alpha \in Aut(A)$ *such that* $\nu = \alpha\mu$.

**Definition 3.3.** *Let* $S$ *be a subsemigroup of* $End(A)$ *defined by a set of formulas* $\Phi$ *(see Definition 2.8). We call an algebra* $A$ *algebraically* $S$*-homogeneous if, for any finitely generated subalgebra* $B$ *of* $A$*, any homomorphism* $\varphi \in Hom(B, A)$ *which satisfies* $\Phi$ *is the restriction of some* $\varphi' \in S$.

In particular*:*

*An algebra* $A$ *is algebraically End-homogeneous if for any finitely generated subalgebra* $B$ *of* $A$ *any homomorphism* $\varphi \in Hom(B, A)$ *is the restriction of some* $\varphi' \in End(A)$.

*An algebra* $A$ *is algebraically Aut-homogeneous or just algebraically homogeneous if for any finitely generated subalgebra* $B$ *of* $A$ *any isomorphism* $\varphi \in \mathbf{I}Hom(B, A)$ *is the restriction of some* $\varphi' \in Aut(A)$.

**Remark 3.4.** *A few words about terminology. In the classical "Model theory" by C. C. Chang and H. J. Keisler (*[14]*) authors use the name w-homogeneous structures for structures we called logically Aut-homogeneous. In the fundamental "Model theory" by W. Hodges (*[18]*) the author uses the name ultrahomogeneous structures for structures we call algebraically Aut-homogeneous. On the other hand it seems that in most of the papers the term homogeneous structures is used for structures W. Hodges call ultrahomogeneous (see for example* [24] *and the bibliography therein).*

*W. Hodges also mentions this fact. The notion of homogeneous structures was developed in the papers by B. Jonson, M. Morley and R. Vaught (see, for example, the review* [19] *on the paper "Homogeneous universal models" by M. Morley and R. Vaught).*

For the reader's convenience we collect several immediate consequences of the definitions in the following remark.

**Remark 3.5.** *Let $A$ be an algebra. Let $S$ be a subsemigroup of $End(A)$.*

1. *Consider a point $\mu \in A^n$ then $S\mu \subset S_{LT}(\mu)$.*
2. *$A$ is a logically $S$-homogeneous algebra if and only if $S\mu = S_{LT}(\mu)$ for any point $\mu \in A^n$.*
3. *The logical $S$-orbit of a point $\mu \in A^n$ contains the algebraic $S$-orbit of $\mu$, i.e. $S_{AO}(\mu) \subset S_{LO}(\mu)$.*

**Definition 3.6.** *Let $\mu \in A^n$. Let $S$ be a subsemigroup of $End(A)$.*

1. *$< \mu >$ is the intersection of all $t$-definable sets which contain $\mu$. We call $< \mu >$ the $t$-definable set generated by $\mu$.*
2. *$< \mu >_S$ is the intersection of all $\mathbf{G}(S)$-$t$-definable sets which contain $\mu$. We call $< \mu >_S$ the $\mathbf{G}(S)$-$t$-definable set generated by $\mu$.*

By definition, $< \mu >_S$ is the minimal $\mathbf{G}(S)$-$t$-definable set, containing $\mu$. Note that $< \mu >=< \mu >_{Aut(A)}$ and that different minimal $\mathbf{G}(S)$-$t$-definable sets can intersect.

**Example 10.**

1. *The left zero semigroup $LZ$ is the semigroup of the constant maps on $X$, i.e., $(\forall a, b \in LZ)(ab = a)$. It is easy to see that any transformation of $LZ$ is its endomorphism. Therefore $End(LZ) = \mathbf{T}(LZ)$ and $Aut(LZ) = Sym(LZ)$. $LZ$ is both algebraically and logically End-homogeneous, Aut-homogeneous and $S$-homogeneous for any subsemigroup $S$ of $End(LZ)$. Logical $S$-homogeneity follows from Corollary 4.6.*
2. *The free semigroup $A = X^+$ is not algebraically End-homogeneous. Indeed, let $x \in X$. Then $<< x^2 >> \cong << x^3 >>$ (here $<< a >>$ denotes the monogenic subsemigroup of $X^+$ generated by $a$). Let $\varphi \in End(X^+)$, $\varphi(x) = x_{i_1}...x_{i_s}$. Suppose that $\varphi(x^2) = x_{i_1}...x_{i_s}x_{i_1}...x_{i_s} = x^3$. This equality is impossible because we have in the left side even and in the right side odd number of free factors. On the other hand $A = X^+$ is logically End-homogeneous (*[50]*).*
3. *The free group $A = FG(X)$ for any set $X$ of free generators is logically Aut-homogeneous (*[34], [31]*). Therefore the variety $G$ of groups is perfect (following B. I. Plotkin (*[41]*), we call a variety $V$ perfect if each free in $V$ algebra is logically Aut-homogeneous). The free group is not algebraically Aut-homogeneous; to see this the reader can use reasoning*

*similar to that of the previous example or just notice that the free non abelian group $A$ is not algebraically Aut-homogeneous since it has a non trivial characteristic subgroup $A'$.*

**Theorem 4.** *Let $S$ be a submonoid of $End(A)$. The following statements are equivalent.*

(1) *$A$ is logically $S$-homogeneous, i.e., $S\mu = S_{LT}(\mu)$ (see Remark 3.5).*
(2) *$S\mu =< \mu >_S$ for any $\mu \in A^n$.*
(3) *$S\mu$ is a $\mathbf{G}(S)$-$t$-definable set for any $\mu \in A^n$.*
(4) *Any subset $M$ of $A^n$ closed under the action of $S$ is a union of $\mathbf{G}(S)$-$t$-definable sets.*

*Proof.*

1⇒ 2. Suppose that $A$ is logically $S$-homogeneous. Let $\mu \in A^n$. It follows from the definition of $\mathbf{G}(S)$ that $S\mu \subset D$ for any $\mathbf{G}(S)$-$t$-definable set $D$ which contains $\mu$. Therefore $S\mu \subset < \mu >_S$. Let $\nu \in< \mu >_S$. It means that $\nu$ satisfies any system of $\mathbf{G}(S)$-formulas satisfied by $\mu$, i.e., the $\mathbf{G}(S)$-type of $\nu$ contains the $\mathbf{G}(S)$-type of $\mu$. $A$ is logically $S$-homogeneous, therefore there exists $\alpha \in S$ such that $\nu = \alpha\mu \in S\mu$. Thus, $S\mu =< \mu >_S$.

2 ⇒ 3. $S\mu =< \mu >_S$ is a $\mathbf{G}(S)$-$t$-definable set by the definition of $< \mu >_S$.

3 ⇒ 4. Let $M$ be a subset of $A^n$ closed under the action of $S$, i.e., $SM \subset M$. Since $S$ is a monoid, for any $\mu \in M$ we have $\mu \in S\mu$ and by the assumption $S\mu$ is a $\mathbf{G}(S)$-$t$-definable set. Hence, $M$ is the union of $\mathbf{G}(S)$-$t$-definable sets.

4 ⇒ 1. Suppose that every subset $M$ of $A^n$ closed under the action of $S$ is a union of $\mathbf{G}(S)$-$t$-definable sets. Let $\mu, \nu \in A^n$. Suppose that the $\mathbf{G}(S)$-type of $\nu$ contains the $\mathbf{G}(S)$-type of $\mu$, i.e., $\nu \in< \mu >_S$. We have $\mu \in S\mu \subset< \mu >_S$, and $< \mu >_S$ is the minimal $\mathbf{G}(S)$-$t$-definable set which contains $\mu$. Therefore, $S\mu =< \mu >_S$ and $\nu \in S\mu$. Thus, $A$ is logically $S$-homogeneous.

□

**Corollary 3.7.** *Let $S, T$ be subsemigroups of $End(A)$ such that $S \subset T \subset \mathbf{GG}(S)$. Then logical $S$-homogeneity implies logical $T$-homogeneity.*

Indeed, let $S\mu$ be a $\mathbf{G}(S)$-$t$-definable set. Then it is $\mathbf{G}(T)$-$t$-definable (Remark 2.6). Then $T\mu = S\mu$ because $S\mu$ is closed under the action of $T$.

**Corollary 3.8.** *Let $A$ be an algebra. The following statements are equivalent.*

(1) *$A$ is logically Aut-homogeneous (i.e., logically homogeneous).*
(2) *$Aut(A)\mu =< \mu >$ for any $\mu \in A^n$.*
(3) *Any subset $M$ of $A^n$ closed under the action of $Aut(A)$ is a union of $t$-definable sets.*

**Corollary 3.9.** *Let $A$ be an algebra. The following statements are equivalent.*

(1) *$A$ is logically End-homogeneous.*

(2) $End(A)\mu =< \mu >_{End(A)}$ *for any* $\mu \in A^n$.

(3) *Any subset* $M$ *of* $A^n$ *closed under the action of* $End(A)$ *is a union of* **PoEx**-*t-definable sets.*

**Corollary 3.10.** *Let* $A$ *be a logically* $S$*-homogeneous algebra and* $\mu \in A^n$. *Let* $S$ *be a submonoid of* $End(A)$. *Then* $S_{LO}(\mu) = S_{AO}(\mu)$, *i.e., the logical* $S$*-orbit of a point* $\mu$ *coincides with its algebraic* $S$*-orbit in* $A$.

*Proof.* $A$ is a logically $S$-homogeneous algebra. Therefore it follows from Theorem 4 that $< \nu >_S = S\nu$ for any $\nu \in A^n$. $S_{LO}(\mu) = \{\nu | \mathbf{G}(S)tp(\mu) = \mathbf{G}(S)tp(\nu)\} = \{\nu | < \nu >_S =< \mu >_S\} = \{\nu | S\nu = S\mu\} = S_{AO}(\mu)$. □

**Corollary 3.11.** *Let* $A$ *be a logically* $S$*-homogeneous algebra. Let* $S$ *be a submonoid of* $End(A)$ *such that different* $S$*-traces do not intersect each other. Then* $S_{LO}(\mu) = S\mu =< \mu >_S$ *for any* $\mu \in A^n$, *i.e., the logical* $S$*-orbit of a point* $\mu$ *coincides with the* $S$*-trace of* $\mu$ *and coincides with the minimal* $\mathbf{G}(S)$*-definable set which contains* $\mu$ *in any logically* $S$*-homogeneous algebra.*

*Proof.* $A$ is a logically $S$-homogeneous algebra. Therefore it follows from Theorem 4 that $< \nu >_S = S\nu$ for any $\nu \in A^n$. $S_{LO}(\mu) = \{\nu | \mathbf{G}(S)tp(\mu) = \mathbf{G}(S)tp(\nu)\} \subset < \mu >_S$. Conversely, let $\nu \in < \mu >_S$. Then $S\nu \subset < \mu >_S = S\mu$. Consequently, $S\nu = S\mu$ because different $S$-traces do not intersect. Hence, $\nu \in S_{AO}(\mu) = S_{LO}(\mu)$. Thus, $S_{LO}(\mu) = S\mu =< \mu >_S$. □

## 4. Logical unification types.

4.1. **The $S$-oligomorphic structures and the Ryll-Nardzewski $\mathbf{G}(S)$-property.** Let $A$ be an algebra, $S$ an arbitrary subsemigroup of $End(A)$. Let $n \in N$. We recall that the $n$-$\mathbf{G}(S)$-type of a point $\mu \in A^n$ is the class of all $\mathbf{G}(S)$-formulas with $n$ free variables which are satisfied by $\mu$.

Below we define versions of the well-known model theoretic concepts with respect to Galois correspondence $\mathbf{G}$.

**Definition 4.1.** *Let* $S$ *be a semigroup of endomorphisms of an algebra* $A$.

(1) *We call a* $\mathbf{G}(S)$*-type* $T$ *principal, if there exists a formula* $\psi \in T$ *such that* $\psi$ *logically implies any* $\phi \in T$ *(modulo* $Th(A)$*).*

(2) *We call an algebra* $A$ $\mathbf{G}(S)$*-atomic if the* $\mathbf{G}(S)$*-type of any point of* $A^n$ *is principal.*

(3) *We call an algebra* $A$ $S$*-oligomorphic if there exists only finitely many algebraic* $S$*-orbits under the action of* $S$ *on* $A^n$ *for any* $n \in N$.

(4) *We say that an algebra* $A$ *possesses the Ryll-Nardzewski* $\mathbf{G}(S)$*-Property if for any* $n \in N$ *there exist only finitely many logically non equivalent formulas with* $n$ *free variables in* $\mathbf{G}(S)$ *modulo* $Th(A)$. *This means that there is a finite subset* $M$ *in the set* $\mathbf{G}(S)_n$ *of all formulas of* $\mathbf{G}(S)$ *with* $n$ *free variables such that any* $\phi \in \mathbf{G}(S)_n$ *is logically equivalent to some* $\psi \in M$ *modulo* $Th(A)$.

**Remark 4.2.** *Let $S$ be a semigroup of endomorphisms of an algebra $A$. The following statements are equivalent.*

(1) *There exist only finitely many algebraic $S$-orbits on $A^n$.*
(2) *There exist only finitely many $S$-traces on $A^n$.*

**Remark 4.3.** *Let $S$ be a semigroup of endomorphisms of an algebra $A$.*

(1) *It is well known that for $S = Aut(A)$ and any $\mu \in A^n$ the algebraic $S$-trace of $\mu$ coincides with the algebraic $S$-orbit of $\mu$.*
(2) *If $\nu \in S\mu$ then $S\nu \subset S\mu$ for any $\mu, \nu \in A^n$.*
(3) *An algebraic $S$-orbit is the intersection of all algebraic $S$-traces which contain this $S$-orbit.*
*Indeed, let $D$ be an $S$-orbit and $a, b \in D$. Let $x = \alpha a = \beta b$.*
*Then $x \in Sx = S\alpha a \subset Sa$ and $x \in Sx = S\beta b \subset Sb$. Therefore, $D \subset Sa \cap Sb$. Conversely, let $x, y \in \cap_{a \in D} Sa$. Then $x = \alpha y, y = \beta x$ and $Sx = S\alpha y \subset Sy$. $Sy = S\beta x \subset Sx$. Thus $x, y$ are in the same algebraic $S$-orbit.*
(4) *Points with different $\mathbf{G}(S)$-types belong to different algebraic $S$-orbits.*
(5) *Points in different algebraic $S$-orbits have different $\mathbf{G}(S)$-types in logically $S$-homogeneous structures.*

The next theorem is a $\mathbf{G}(S)$-version of a result by Ryll-Nardzewski, Engeler and Svenonius.

**Theorem 5.** *Let $A$ be an algebra. Let $S$ be a subsemigroup of $End(A)$. The following statements are equivalent.*

(1) *$A$ possesses the Ryll-Nardzewski $\mathbf{G}(S)$-property.*
(2) *$A$ realizes only finitely many $n$-$\mathbf{G}(S)$-types for each $n \in N$.*
(3) *$A$ is $\mathbf{G}(S)$-atomic.*
(4) *If $A$ is countable, then $A$ is $S$-oligomorphic.*

*Proof.*

(1) $\Leftrightarrow$ (2). Let any formula in $\mathbf{G}(S)$ with $n$ free variables be logically equivalent modulo $Th(A)$ to one of the formulas $\varphi_1, ..., \varphi_s$. Then there are not more than $2^s$ different $n$-$\mathbf{G}(S)$-types.

Conversely, let $T_1, ..., T_s$ be all $\mathbf{G}(S)$-types of points of $A^n$. We note that any two formulas of $\mathbf{G}(S)$ with exactly $n$ free variables which simultaneously belong or do not belong to any $n$-$\mathbf{G}(S)$-type are logically equivalent modulo $Th(A)$. We choose $s$ pairwise non-equivalent formulas: $\varphi_1, ..., \varphi_s; \varphi_i \in T_i$ and construct $2^s$ formulas $\psi_{\alpha_1 ... \alpha_s} = \varphi_1^{\alpha_1} \wedge ... \wedge \varphi_s^{\alpha_s}$, where $\alpha_i \in \{0, 1\}$, $\varphi_i^0 = \neg\varphi_i$, $\varphi_i^1 = \varphi_i$. Then any formula $\phi$ with $n$ free variables in $\mathbf{G}(S)$ is logically equivalent to one of the formulas $\psi_{\alpha_1 ... \alpha_s}$. Namely, if $\phi$ belongs to types $T_{i_1}, ..., T_{i_k}$ and only to these types then $\phi$ is logically equivalent to $\varphi_1^{\alpha_1} \wedge ... \wedge \varphi_s^{\alpha_s}$, where

$\alpha_i = 1 \Leftrightarrow i \in \{i_1, ..., i_k\}$. Indeed, these two formulas simultaneously belong or do not belong to any $n$-$\mathbf{G}(S)$-type.

(2) $\Leftrightarrow$ (3). Suppose that $A$ satisfies Condition (2). Then, as we have proved above, $A$ satisfies Condition (1). Hence, any formula with $n$ free variables in $\mathbf{G}(S)$ is logically equivalent modulo $Th(A)$ to one of the formulas of the finite set $M = \{\varphi_1, ..., \varphi_s\}$. Let $\{\varphi_{i_1}, ..., \varphi_{i_k}\} = M \cap \mathbf{G}(S)tp_A(a)$. Then the formula $\chi \equiv (\varphi_{i_1} \wedge ... \wedge \varphi_{i_k})$ implies every formula of $\mathbf{G}(S)tp_A(a)$. Thus, $\mathbf{G}(S)tp_A(a)$ is principle and algebra $A$ is $\mathbf{G}(S)$-atomic.

Conversely, suppose that $A$ is $\mathbf{G}(S)$-atomic. Let $\{T_i | i \in I\}$ be the set of all $\mathbf{G}(S)$-types of points of $A$. Since $A$ is $\mathbf{G}(S)$-atomic, all these $\mathbf{G}(S)$-types are principle. Hence, we have a list $\{\chi_i | i \in I\}$ of formulas such that $\chi_i$ generates $T_i$ for any $i \in I$. The set $Th(A) \cup \{\neg\chi_i | i \in I\}$ is inconsistent. Therefore by the compactness theorem there is a finite subset $J = \{j_1, ..., j_k\} \subset I$ such that $Th(A) \cup \{\neg\chi_j | j \in J\}$ is inconsistent. Therefore $Th(A) \models (\forall x)(\chi_{j_1} \vee ... \vee \chi_{j_k})$. Thus $T_{j_1}, ..., T_{j_k}$ is the list of all $n$-$\mathbf{G}(S)$-types of points of $A$. Hence there exist only finitely many $n$-$\mathbf{G}(S)$-types.

(4) $\Rightarrow$ (2). Suppose that $A$ is $S$-oligomorphic. Then the action of $S$ on $A$ has only finitely many orbits. If $a, b \in A^n$ belong to the same orbit, then there exist $f, g \in S$ such that $f(a) = b$ and $g(b) = a$. Therefore $\mathbf{G}(S)tp_A(a) = \mathbf{G}(S)tp_A(b)$. Hence $A$ realizes only finitely many $n$-$\mathbf{G}(S)$-types for each $n \in N$.

(3) $\Rightarrow$ (4). This item is the only one where we use Fraisse type (back-and-forth) arguments and hence we add new restriction: the countability of $A$. By Fraisse type arguments (see details in the proof of Proposition 8) we obtain that for an atomic countable algebra $A$ and any two points $\mu, \nu \in A^n$ which satisfy $\mathbf{G}(S)tp(\mu) \subset \mathbf{G}(S)tp(\nu)$ there exists $\alpha \in S$ such that $\alpha(\mu) = \nu$. Hence $A$ is logically $S$-homogeneous. Therefore by Remark 4.3, item 5, $A$ is $S$-oligomorphic.

□

In view of Remark 2.6 we have

**Corollary 4.4.** *Let $S, T$ be subsemigroups of $End(A)$ such that $S \subset T \subset \mathbf{GG}(S)$. Then an algebra $A$ satisfies $\mathbf{G}(S)$-properties of Theorem 5 if and only if $A$ satisfies $\mathbf{G}(T)$-properties of Theorem 5.*

We remind that the notion of a $\Phi$-definable subset of $Hom(A, B)$ is defined in the extension of the language $\mathbf{L}$ (Definition 2.8). In the next Proposition we consider formulas in this extension of $\mathbf{L}$.

**Proposition 8.** *Let $S$ be a $\Phi$-definable subset of $Hom(A, B)$. If $A$ and $B$ are countable $\mathbf{G}(S)$-atomic algebras and $Th(A) \subset Th(B)$ then there exists $\alpha \in S$ such that $\alpha(A) \subset B$. If $\mu \in A^n$ and $\nu \in B^n$ satisfy $\mathbf{G}(S)tp_A(\mu) \subset \mathbf{G}(S)tp_B(\nu)$ then there exists $\alpha \in S$ such that $\alpha(A) \subset B$ and $\alpha(\mu) = \nu$.*

*Proof.* We use the classical Fraisse-type arguments. First we prove that for any $c \in A$ there exists $d \in B$ such that $\mathbf{G}(S)tp_A(c) \subset \mathbf{G}(S)tp_B(d)$. Let $c \in A$ and $\phi \in \mathbf{G}(S)tp_A(c)$. Then $(\exists y)(\phi) \in Th(A) \subset Th(B)$. Therefore there exists $d \in B$ such that $\phi \in \mathbf{G}(S)tp_B(d)$. $A$ is a $\mathbf{G}(S)$-atomic algebra. We choose $\phi$ to be a formula which generates $\mathbf{G}(S)tp_A(c)$. Thus $\mathbf{G}(S)tp_A(c) \subset \mathbf{G}(S)tp_B(d)$.

Let $\mu \in A^n, \nu \in B^n$ and $\mathbf{G}(S)tp_A(\mu) \subset \mathbf{G}(S)tp_B(\nu)$. Let us prove that for any $c \in A$ there exists $d \in B$ such that $\mathbf{G}(S)tp_A(\mu, c) \subset \mathbf{G}(S)tp_B(\nu, d)$. Let $c \in A$ and $\phi \in \mathbf{G}(S)tp_A(\mu, c)$. Then $(\exists y)(\phi) \in \mathbf{G}(S)tp_A(\mu)$. Thus $(\exists y)\phi \in \mathbf{G}(S)tp_B(\nu)$. Therefore there exists $d \in B$ such that $\phi \in \mathbf{G}(S)tp_B(\nu, d)$. $A$ is a $\mathbf{G}(S)$-atomic algebra. We choose $\phi$ to be a formula which generates $\mathbf{G}(S)tp_A(\mu, c)$. Thus $\mathbf{G}(S)tp_A(\mu, c) \subset \mathbf{G}(S)tp_B(\nu, d)$.

Let $\mu \in A^n, \nu \in B^n$ satisfy $\mathbf{G}(S)tp_A(\mu) \subset \mathbf{G}(S)tp_B(\nu)$. We construct $\alpha : A \to B$ recursively. We define $\alpha(\mu) = \nu$. Suppose that $\alpha(a_{n+1}) = b_{n+1}, ..., \alpha(a_{n+k}) = b_{n+k}$ are already defined. Let $a_{n+k+1}, ...$ be the list of all elements of $A$ not included in $\{\mu, a_{n+1}, ..., a_{n+k}\}$. We proved already that there exists $b_{n+k+1}$ such that $\mathbf{G}(S)tp_A(\mu, a_{n+1}, ..., a_{n+k+1}) \subset \mathbf{G}(S)tp_B(\nu, b_{n+1}, ..., b_{n+k+1})$. Define $\alpha(a_{n+k+1}) = b_{n+k+1}$. Thus we obtain a map $\alpha$ from $A$ to $B$. Since $tp_A^0(a) \subset \mathbf{G}(S)tp_A(a)$, $\alpha$ is a homomorphism. Besides that, $\alpha$ satisfies $\Phi$. Therefore $\alpha \in S$. □

In particular we obtain the following corollary.

**Corollary 4.5.** *Let $A$ be a countable $\mathbf{G}(S)$-atomic algebra. Let $\mu, \nu \in A^n$ be such that $\mathbf{G}(S)tp_A(\mu) \subset \mathbf{G}(S)tp_A(\nu)$. Let $S$ be a $\Phi$-definable subsemigroup of $End(A)$. Then there exists $\alpha \in S$ such that $\alpha(\mu) = \nu$.*

Theorem 5 and Corollary 4.5 imply Corollary 4.6. It gives the correspondence between the logical $S$-homogeneity of a countable algebra $A$ and the properties from Theorem 5.

**Corollary 4.6.** *Countable $S$-oligomorphic algebra $A$ is logically $S$-homogeneous.*

4.2. **Action of subsemigroups of $End(A)$ on $t$-definable subsets of $A$. Logical $S$-unification types.** Unification Types theory is a part of Term Rewriting theory [2], [3]. We extend its setting from equations to logical formulas. In particular, it can be applied to Constraint Satisfaction Problems.

Let $A$ be an algebra. Let $S$ be a subsemigroup of $End(A)$. Let $D \subset A^n$ be a $\mathbf{G}(S)$-$t$-definable set (i.e., $D$ is defined by a set $\Phi \subset \mathbf{G}(S)$). Algebraic $S$-trace $S\mu$ is a subset of $D$ for any point $\mu \in D$. There are four possibilities for covering of $\mathbf{G}(S)$-$t$-definable sets by algebraic $S$-traces:

(1) Any $\mathbf{G}(S)$-$t$-definable set over $A$ can be covered by algebraic trace $S\mu$ of one point $\mu$. In this case $A$ has *unitary logical $S$-unification type.*

(2) Any $\mathbf{G}(S)$-$t$-definable set over $A$ can be covered by the union $S\mu_1 \cup ... \cup S\mu_n$ of algebraic traces of a finite set of points $\mu_1, ..., \mu_n$. In this case $A$ has *finitary logical $S$-unification type*.

(3) For any $\mathbf{G}(S)$-$t$-definable set $D$ over A there is a minimal (with respect to inclusion) set $M$ of points such that $D$ can be covered by the union of algebraic traces $S\mu$, $\mu \in M$ and $A$ has no finitary $S$-unification type. In this case $A$ has infinitary logical $S$-unification type.

(4) There is a $\mathbf{G}(S)$-$t$-definable set $D$ over $A$ such that the minimal (with respect to inclusion) set $M$ of points such that $D$ can be covered by the union of the algebraic traces $S\mu$, $\mu \in M$ does not exist. In this case $A$ has zero logical $S$-unification type.

**Remark 4.7.** *Let $D$ be a $\mathbf{G}(S)$-$t$-definable over $A$ set and $\mu_1, \mu_2 \in D$. We say that $\mu_1$ is less than $\mu_2$ and use the notation $\mu_1 <_S \mu_2$ if $\mu_2 \in S\mu_1$ and $\mu_1 \notin S\mu_2$. A point $\nu \in A^n$ is called $S$-minimal if it is minimal relative to the partial order $<_S$. We say that two points are $S$-equivalent if they belong to the same algebraic $S$-orbit.*

*Consider the following condition which we call the condition $C(A)$: for any $\mathbf{G}(S)$-$t$-definable set $D$ over $A$ there exists minimal (with respect to inclusion) set $M$ of points in $D$ such that $D$ can be covered by the $S$-traces of points of $M$, i.e., $D = \cup_{\mu \in M} S\mu$. In other words, for any $\mu \in D$ either there exists $\nu \in D$ such that $\nu$ is $S$-minimal and $\nu <_S \mu$ or $\mu$ is $S$-minimal.*

(1) *$A$ has* unitary logical $S$-unification type *if and only if any $\mathbf{G}(S)$-$t$-definable set $D$ over $A$ contains the minimum element.*

(2) *$A$ has* finitary logical $S$-unification type *if and only if it satisfies the condition $C(A)$ and for any $\mathbf{G}(S)$-$t$-definable set $D$ over $A$ the set of non $S$-equivalent $<_S$-minimal elements is finite.*

(3) *$A$ has* infinitary logical $S$-unification type *if and only if it satisfies the condition $C(A)$ and there exists a $\mathbf{G}(S)$-$t$-definable set $D$ over $A$ with the infinite set of non $S$-equivalent $<_S$-minimal elements.*

(4) *$A$ has* zero logical $S$-unification type *if it does not satisfy the minimal cover condition $C(A)$.*

We define the linear order on the set of logical $S$-unification types. The unitary logical $S$-unification type is greater than the finitary logical $S$-unification type which is in turn greater than the infinitary logical $S$-unification type which is greater than the zero logical $S$-unification type. In view of Theorem 3 we have the following corollaries.

**Corollary 4.8.** *Let $A$ be an algebra. Let $S, T$ be subsemigroups of $End(A)$ and $S \subset T$. Then the logical $S$-unification type of $A$ is less than or equal to the logical $T$-unification type of $A$.*

**Corollary 4.9.** *Let $A$ be an algebra. Let $S$ be a subsemigroup of $End(A)$. The logical End-unification type of $A$ is less than or equal to the unification type of $A$. For example if $A$ has* *zero unification type then $A$ has zero logical unification type with respect to the positive existential theory.*

**Proposition 9.**

(1) *If $A$ is $S$-oligomorphic then $A$ has finitary logical $S$-unification type.*
(2) *If $A$ has finitary logical $S$-unification type and different $S$-traces do not intersect then $A$ is $S$-oligomorphic.*

*Proof.*

(1) If $A$ is $S$-oligomorphic, then there exist only finitely many $S$-orbits under the action of $S$ on $A$. Therefore, $A$ has the finitary logical $S$-unification type.
(2) Let $A$ has the finitary logical $S$-unification type and different $S$-traces do not intersect each other. Any $\mathbf{G}(S)$-$t$-definable set over $A$ can be covered by the union of traces $S\mu_1 \cup ... \cup S\mu_n$ of a finite set of points $\mu_1, ..., \mu_n$. In particular the set $A^n$ can be covered by the union of traces $S\mu_1 \cup ... \cup S\mu_n$. Any other trace should intersect one of these traces. But it is impossible by our assumption. Therefore $A$ is $S$-oligomorphic.

□

**Corollary 4.10.** *$A$ is Aut-oligomorphic if and only if $A$ has finitary logical Aut-unification type.*

**Corollary 4.11.** *Let $A$ be an algebra and $S = Aut(A)$ or $S = \mathbf{I}End(A)$. The following conditions are equivalent.*

(1) *$A$ has the finitary logical $S$-unification type.*
(2) *$A$ is $S$-oligomorphic.*
(3) *$A$ satisfies any of equivalent conditions of Theorem 5.*

**Proposition 10.** *Let $S$ be a submonoid of $End(A)$. Any logically $S$-homogeneous algebra $A$ has the non zero logical $\mathbf{G}(S)$-unification type.*

*Proof.* Any $\mathbf{G}(S)$-$t$-definable set has a minimal cover by algebraic orbits of the form $\mu_S$ and $S\mu = < \mu >_S$ for any $\mu \in A^n$ because $A$ is logically $S$-homogeneous. □

**Corollary 4.12.** *Any logically Aut-homogeneous algebra $A$ has the non zero logical unification type. So, if $A$ has the zero logical unification type then it is not logically Aut-homogeneous algebra.*

## 5. Some Problems.

Let $A$ be an algebra, $\mathbf{L}$ a first-order language, $S$ a semigroup of endomorphisms of $A$, i.e., $S \subset End(A)$.

**Question 1.** *Describe the Galois closed objects for the Galois correspondence* $\mathbf{G}$ *(Definitions 2.2 and 2.4), i.e., for an algebra $A$ of the given class of algebras (e.g. groups, semigroups, associative algebras) find*

- *subsemigroups $S$ of $End(A)$ such that* $\mathbf{GG}(S) = S$,
- *subsemigroups $S$ of $End(A)$ such that* $\mathbf{G}_{\overline{L}}\mathbf{G}_{\overline{L}}(S) = S$ *holds in some extension* $\overline{L}$ *of* $\mathbf{L}$,
- *subsets $K$ of* $\mathbf{L}$ *such that* $\mathbf{GG}(K) = K$.
- *Describe the lattice of $G$-closed subsemigroups of $End(A)$.*

**Question 2.** *In view of Examples 1 and 2 and Remark 2.7 we have the following question. What are the algebras $A$ such that*

- *classical semigroups of endomorphisms and sets of formulas are* $\mathbf{G}$*-closed (cf. Proposition 4),*
- *classical semigroups of endomorphisms and sets of formulas are the only* $\mathbf{G}$*-closed objects,*
- *in particular, what are the algebras such that any their elementary embedding into itself is an automorphism.*

The following question is closely connected with the investigation of the minimal $\mathbf{G}(S)$-$t$-definable sets, namely, minimal elements in the set of $\mathbf{G}(S)$-$t$-definable sets over an algebra $A$ with respect to the "subset" relation.

**Question 3.**

- *Given class $M$ of endomorphism subsemigroups $S$ (injective, surjective, stabilizors etc.), describe algebras $A$ which have the non zero logical $S$-unification type.*

Following [40], [39] we say that an algebra $A$ is logically $S$-noetherian if any subset of formulas of $\mathbf{G}(S)$ is logically equivalent to a formula modulo $Th(A)$, i.e., any $\mathbf{G}(S)$-$t$-definable set over $A$ is $\mathbf{G}(S)$-definable.

**Question 4.** *Let $S$ and $S'$ be subsemigroups of $End(A)$ and $S' \subset S$.*

- *Study relations between logical $S$-noetherianity and logical $S'$-noetherianity. For example, study the cases when $S$-noetherianity and $S'$-noetherianity do not coincide for some $A$.*
- *What are the pairs $(A, S)$ such that $A$ is logically $S$-noetherian. Vary $A$ and $S$.*

Partial transformations (see for example the book of Ljapin-Evseev [22]) and clones (e.g., [7],[36]) are extensively studied. In particular, they have many applications concerning problems discussed in this paper.

**Question 5.** *Generalize the* $\mathbf{G}$*-Galois correspondence to the Galois correspondence*

- *between clones of transformations of algebras and sets of formulas,*
- *between partial transformations (endomorphisms) of algebras and sets of formulas.*

Many natural questions can be collected under the roof of the following one: when some properties of an algebra $A$ are inherited by an algebra $B$; in particular, what are the conditions that provide the existence of $\varphi \in S \subset Hom(A, B)$ such that $\varphi(A) = B$, i.e., $B$ is an $S$-morphic image of $A$.

In view of Proposition 8, Corollary 4.5 and Problems 14–19 from [43] we have

**Question 6.**
- *Find conditions on $\mathbf{G}(S)tp(A)$ and $\mathbf{G}(S)tp(B)$ sufficient for $B$ to be an $S$-morphic image of $A$. Consider relatively free algebras, vector spaces, algebraic groups, abelian groups etc.*

**Question 7.** *It is clear that $\mathbf{G}(S)$-t-definable sets in $A^n$ give rise to a Zarisky-type topology. So, we obtain a series of $\mathbf{G}(S)$-topologies.*

- *Study $\mathbf{G}(S)$-topologies in a uniform way with respect to compactness, unification theory, countable categoricity, etc.*

Let $\Theta$ be a variety of algebras and $A = F(X)$, the free in $\Theta$ algebra over the set $X = \{x_1, \ldots, x_t\}$ of free generators. The next problem is related to a well-known question whether the rank of a free algebra is elementary definable.

**Question 8.** *Given $A = F(X)$, define the subsemigroup $T_k(A)$ of $\mathbf{E}nd(A)$ by $\alpha \in T_k(A)$ if the set $\alpha(X)$ consists of at most $k$ elements. So, $T_k(A) = \{\alpha \in \mathbf{E}nd(A) | |\alpha(X)| \leq k\}$. Recall that subsemigroups $S_1$ and $S_2$ of $End(A)$ are $\mathbf{G}$-equivalent if $\mathbf{GG}(S_1) = \mathbf{GG}(S_2)$.*

- *Describe $\mathbf{GG}(T_k(A))$. In particular, what are algebras $A$ such that $\mathbf{GG}(T_k(A)) = T_k(A) \cup \mathbf{E}lEnd(A)$.*
- *Describe algebras $A$ such that $T_k(A)$ and $T_s(A)$ are $\mathbf{G}$-equivalent for all $k, s \in N$ or for all $k, s \geq m$ for some $m \in N$.*

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