Logically-geometrical similarity for algebras and models with the same identities

October 18, 2015

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The article is dedicated to the memory of the outstanding Belorussian scientist, academician D.A.Suprunenko who made invaluable contribution to the development of mathematics in Belarus. The elder of the authors is proud to be his friend for many years.

Abstract

The paper is related to the field which we call Universal Algebraic Geometry (UAG). All algebras under consideration belong to a variety of algebras Θ. For an arbitrary Θ we construct a system of notions which lead to a bunch of new problems. As a rule, their solutions depend on the choice of a specific Θ. This can be variety of groups Grp, variety of associative or Lie algebras, etc. In particular, it can be the classical variety $Com-P$ of commutative and associative algebras with unit over a field.

For example, the paper concerns with the following general problem. For every algebra $H \in \Theta$ one can define the category of algebraic sets over H. Given H_1 and H_2 in Θ the question is what are the relations between these algebras that provide an isomorphism of the corresponding categories of algebraic sets. Similar problem stands with respect to situation when algebras are replaced by models and the categories of algebraic sets are replaced by the categories of definable sets. The results on the stated problem are applicable to knowledge theory and, in particular, to knowledge bases.

This paper was written in the time of the 70th anniversary of the victory over fascism in Second World War 1939-1945. We would like to dedicate this paper to this valuable date. The elder of the authors participated in this war as a machine-gun platoon commander and was seriously wounded in the fighting during the liberation of his native Belarus.

Mathematics Subject Classification 2010: 08-02, 08C05, 03G15. UDK: 512.573, 512.579

1 Preliminary notions

Let Θ be a variety of algebras, let H be an algebra in Θ and $X = \{x_1, \dots, x_n\}$ be a finite set of variables. Consider points $\mu : X \to H$, μ is a mapping. From one hand, these points can be viewed as tuples $\bar{a} = (a_1, \dots, a_n)$, $a_i = \mu(x_i)$. From the other hand, these points can be regarded as homomorphisms

$$\mu : W(X) \to H$$

where $W = W(X)$ is the free in Θ algebra over X. Algebraic nature of points in Θ is given by algebras H, $W(X)$ and homomorphisms $\mu : W(X) \to H$. We arrive to the affine space of points $Hom(W(X), H)$.

Take further an infinite set of variables X^0 and let Γ be a set of finite subsets X in X^0. Define two categories Θ^0 and $\Theta^0(H)$. The first one is the category of all free in Θ algebras $W(X)$, $X \in \Gamma$. Morphisms in $\Theta^0(H)$ are homomorphisms $s : W(Y) \to W(X)$. The second one is the category of all affine spaces $Hom(W(X), H)$ with morphisms

$$\widetilde{s} : Hom(W(X), H) \to Hom(W(Y), H)$$

for each $s : W(Y) \to W(X)$. Here for each point $\mu : W(X) \to H$ we define $\nu = \widetilde{s}(\mu) : W(Y) \to H$ by the rule $\nu = \mu s$, that is, $\nu(w) = \mu(s(w))$, where $w \in W(Y)$. The transitions $W(X) \to Hom(W(X), H)$ and $s \to \widetilde{s}$ determine a contravariant functor $\Theta^0 \to \Theta^0(H)$. It is checked that this functor determines duality of categories if and only if $Var(H) = \Theta$, see [8], [16].

Note further that for each point $\mu : W(X) \to H$ we have its classical kernel $Ker(\mu)$. It is a system of all equality relations $w \equiv w'$, $w, w' \in W(X)$, such that $w^\mu = w'^\mu$. Here and throughout the paper $w^\mu = \mu(w)$. By definition, a point μ satisfies an equality $w \equiv w'$ if and only if $w \equiv w'$ lies in $Ker(\mu)$. Let M_X be the set of all equalities $w \equiv w'$, $w, w' \in W(X)$. Denote by $\Phi_0(X)$ the free boolean algebra over M_X.

Consider the category $\widetilde{\Phi}_0$ of all free over M_X boolean algebras $\Phi_0(X)$ where X runs Γ. Observe that this category depends on the choice of Θ. Morphisms of this category are homomorphisms of boolean algebras $s_* : \Phi_0(Y) \to \Phi_0(X)$ satisfying the condition $s_*(w \equiv w') = (sw \equiv sw')$. So we assume that all s_* are correlated with equalities.

For each space of points $Hom(W(X), H)$ take the boolean power algebra of all subsets of $Hom(W(X), H)$. Assign to every formula $w \equiv w'$ in $\Phi_0(X)$ the subset A which consists of all points $\mu \in Hom(W(X), H)$, satisfying $w \equiv w'$. Denote this set by $[w \equiv w']_H$ and denote by $Bool_\Theta^X(H)$ the boolean power algebra with the specified elements $[w \equiv w']_H$. We call these elements equalities of $Bool_\Theta^X(H)$. The mapping $w \equiv w' \to [w \equiv w']_H$ determines the boolean homomorphism

$$Val_H^X : \Phi_0(X) \to Bool_\Theta^X(H).$$

We say that a point $\mu : W(X) \to H$ satisfies the formula $u \in \Phi_0(X)$ if and only if $\mu \in Val_H^X(u)$. This definition fits well to the standard model theoretic definition.

Let us define the category $Bool_\Theta^0(H)$. Its objects are algebras $Bool_\Theta^X(H)$. Given $B \in Bool_\Theta^Y(H)$, define $s_*(B) = A \in Bool_\Theta^X(H)$ by the rule: $\mu \in A$ if and only if $\widetilde{s}(\mu) = \nu = \mu s \in B$. One can check that s_* is a homomorphism of boolean algebras. Note also that every homomorphism s_* transforms equalities of $Bool_\Theta^Y(H)$ to equalities of $Bool_\Theta^X(H)$, that is, $s_*[w \equiv w']_H = [sw \equiv sw']_H$. The correspondence between objects in $\widetilde{\Phi}_0$ and objects in $Bool_\Theta^0(H)$ gives rise to a functor from $\widetilde{\Phi}_0$ to $Bool_\Theta^0(H)$.

Passing to the general case, we use models of the form $F = (H, \Psi, f)$. Here H is an algebra in Θ, Ψ is a set of symbols of relations φ of an arbitrary arity m (we write $\varphi_m = \varphi(w_1, \ldots, w_m)$, $w_i \in W(X)$) and f is a function which interprets each φ_m in H. For every φ_m consider the set of all m-tuples from H^m which satisfy the relation φ_m in the interpretation f. A point $\mu : W(X) \to H$ satisfies φ_m if the tuple $(w_1^\mu, \ldots, w_m^\mu)$ lies in $f(\varphi_m)$. In the sequel we consider relations, i.e., formulas of the form $\varphi(w_1, \ldots, w_m)$ along with equalities.

Warning. For the sake of convenience, in the notation of objects related to a model $F = (H, \Psi, f)$ we use the letter H instead of pointing out the whole model F.

Define automorphisms σ of the model F as automorphisms of H which keep every $f(\varphi)$ invariant under σ. It is clear that if a tuple μ satisfies the formula $\varphi(w_1, \ldots, w_m)$ and σ is an automorphism of the model, then the point $\sigma(\mu)$ satisfies this formula as well. Moreover,

Theorem 1.1 ([14], cf.,[5], [7]) *If σ is an automorphism of the model $F = (H, \Psi, f)$, then every definable set A in the category $LG_\Theta(H)$ is invariant under the action of automorphism σ.*

Further on we will expand the algebra of formulas $\Phi_0(X)$ taking into account formulas of the form $\varphi(w_1, \ldots, w_m)$.

2 Logical and geometrical terminology

2.1 A system of ongoing notions

We shall define some new categories. First of all, this is the category of algebras of logical formulas $\widetilde{\Phi}_\Theta$, whose objects are denoted by $\Phi(X)$. Morphisms $s_* : \Phi(Y) \to \Phi(X)$ in $\widetilde{\Phi}_\Theta$ correspond to homomorphisms $s : W(Y) \to W(X)$. They are correlated with the signature of algebras $\Phi(X)$ (see 2.2). Algebraic part of $\widetilde{\Phi}_\Theta$ will be presented by a functor $\Theta^0 \to \widetilde{\Phi}_\Theta$.

Another important category is the category $Hal_\Theta(H)$ of extended boolean algebras $Hal^X_\Theta(H)$, $H \in \Theta$, with morphisms $s_* : Hal^Y_\Theta(H) \to Hal^X_\Theta(H)$. The situation of $Hal^X_\Theta(H)$ differs from the one of $Bool^X_\Theta(H)$ by adding operations of existential quantifiers to $Bool^X_\Theta(H)$. So, morphism s_* should be correlated with quantifier operations.

Both categories are treated also as multi-sorted algebras with the set of sorts Γ based on an infinite set of variables X^0. Objects of these categories are domains of the corresponding algebras while morphisms give rise to operations in these algebras.

In our setting the categories $Hal_\Theta(H)$, $H \in \Theta$ precedes the category $\widetilde{\Phi}_\Theta$ and, in some sense, determines it. We will define the variety Hal_Θ as a variety determined by the identities of algebras $Hal_\Theta(H)$. The identities of Hal_Θ arise naturally from the properties of $Hal_\Theta(H)$. Then, the algebra (and the category) $\widetilde{\Phi}_\Theta = (\Phi(X), X \in \Gamma)$ is the free in Hal_Θ algebra. The set of the atomic formulas $M_F = (M_X, X \in \Gamma)$ is a system of free generators of this algebra. All this allows us to define $\Phi(X)$ as domains of the algebra $\widetilde{\Phi}_\Theta$.

These complications are necessary in order to pass from the propositional calculus to first order logic. In particular, algebras $\widetilde{\Phi}_0$ and $\widetilde{\Phi}_0(X)$ are defined in the very simple and natural way. The definition of $\widetilde{\Phi}_\Theta = (\Phi(X), X \in \Gamma)$ is more complicated.

2.2 Extended boolean algebras

Along with the algebra of formulas $\Phi_0(X)$ we will consider an algebra $\Phi(X)$ enriched by quantifiers which are added to signature of operations. Recall (see [12], [15], [16]) that an extended boolean algebra is defined for each finite set of variables X. Its signature L_X consists of three parts:

1. Boolean operations $\vee, \wedge, \neg$.
2. Atomic formulas. These are equalities $w \equiv w'$ and formulas of the form $\varphi(w_1, \ldots, w_m)$ where all w_i are elements in the free in Θ algebra $W(X)$.
3. Existential quantifiers $\exists x$ for $x \in X$.

Note that an existential quantifier $\exists$ of the Boolean algebra B is an unary operation $\exists : B \to B$ with the properties

1. $\exists 0 = 0$.
2. $a \leq \exists a$.
3. $\exists(a \wedge \exists b) = \exists a \wedge \exists b$.

We have also $\exists(a \vee b) = \exists a \vee \exists b$.

Universal quantifier $\forall : B \to B$ is defined dually and its properties are

1. $\forall 1 = 1$.
2. $a \geq \forall a$.
3. $\forall(a \vee \forall b) = \forall a \vee \forall b$.

The equalities $\forall(a \wedge b) = \forall a \wedge \forall b$ and $\neg(\forall a) = \exists(\neg a)$ are always true. We also require $\exists x \exists y = \exists y \exists x$ for x, y in X. In the formulas above 0 and 1 are zero and unit in B and a, b are elements in B.

Define the extended boolean algebra $Hal_\Theta^X(H)$. First of all, it is the power algebra of all sets A in $Hom(W(X), H)$. Atomic formulas $[u]_F$ in $Hal_\Theta^X(H)$ are defined to be values of $u \in M_X$ as is in Section 1. Quantifiers $\exists x$ on $Hal_\Theta^X(H)$ are defined as follows: for $x \in X$ and $A \in Hom(W(X), H)$ we define $B = \exists x A$ by the rule: a point μ lies in B if there is ν in A, such that $\mu(x') = \nu(x')$ for $x' \neq x$, $x' \in X$. We obtained an extended boolean algebra $Hal_\Theta^X(H)$.

We will define algebras $\Phi(X)$ in such a way that $\Phi(X)$ and $Hal_\Theta^X(H)$ have the same signature. Besides, we need that for any $X \in \Gamma$ and any $H \in \Theta$ we have a homomorphism $Val_F^X : \Phi(X) \to Hal_\Theta^X(H)$ which takes every formula $u \in \Phi(X)$ into its value $Val_F^X(u) = [u]_F$. This value is, indeed, the set of points $\mu : W(X) \to H$, satisfying the formula u.

Since Val_F^X is a homomorphism, we have:

$$Val_F^X(\exists x u) = \exists x Val_F^X(u).$$

We will be interested in two categories. As earlier, we proceed from an infinite set X^0 and let Γ be a set of all finite subsets X in X^0. We had already defined the categories Θ^0 and $\Theta^0(H)$. Define now the categories $\widetilde{\Phi}_\Theta$ and $Hal_\Theta(H)$.

Define the Halmos category $Hal_\Theta(H)$ of all $Hal_\Theta^X(H)$. Let us discuss its morphisms

$$s_* : Hal_\Theta^Y(H) \to Hal_\Theta^X(H)$$

corresponding to $s : W(Y) \to W(X)$ in more detail. We define s_* in the way as it was done in Section 1, that is, given $B \in Hal_\Theta^Y(H)$, define $s_*(B) = A \in Hal_\Theta^X(H)$ by the rule: $\mu \in A$ if and only if $\widetilde{s}(\mu) = \nu = \mu s \in B$.

These s_* are homomorphisms of the corresponding boolean algebras (see Section 1). Further, on the elements $[\varphi(w_1, \dots, w_m)]_H$ morphisms s_* act by the rule: $s_*[\varphi(w_1, \dots, w_m)]_H = [\varphi(sw_1, \dots, sw_m)]_H$, $\varphi \in \Psi$. Thus, s_* takes atomic formulas to atomic formulas, preserving symbols of relations, but not necessarily preserving relations themselves.

Let us discuss the interaction of s_* with quantifiers $\exists y$. Take the set $\exists yB$, $B \in Hal^Y_\Theta(H)$. Denote $x = s(y)$, $s_*(B) = A$. We shall treat the equality $s_*(\exists yB) = \exists x s_*(B) = \exists xA$ which holds not for every s.

Definition 2.1 *A morphism $s : W(Y) \to W(X)$ is called y-admissible, if*

1. $s(y)$ is a variable x.

2. Element x does not belong to the support of every $w' = s(y')$, where $y' \neq y$.

This definition means also that if $y' \neq y$ and $w' = s(y')$ then $x' \neq x$, $x = s(y)$ for every x' in the support of $w' = s(y')$.

Proposition 2.2 (cf. [16]) *If a morphism $s : W(Y) \to W(X)$ is y-admissible, then for every $\mu : W(X) \to H$ from $\exists xA$ the point $\widetilde{s}(\mu)$ lies in $\exists yB$. This means that*

$$\exists xA \subseteq s_*(\exists yB).$$

Proof. Let a point $\mu : W(X) \to H$ belongs to $\exists xA$. By definition of $\exists x$, there exists a point $\nu \in W(X) \to H$, such that $\nu \in A$ and $\mu(x') = \nu(x')$ for every $x' \neq x$, $x' \in X$. Points $\widetilde{s}(\mu) = \mu s$ and $\widetilde{s}(\nu) = \nu s$ belong to $Hom(W(Y), H)$. Besides, $\widetilde{s}(\nu)$ lies in B, since $\nu \in A = s_*B$. Apply $\widetilde{s}(\mu)$ and $\widetilde{s}(\nu)$ to $y' \neq y$. We have

$$\widetilde{s}(\mu)(y') = (\mu s)(y') = \mu(s(y')) = \mu(w')$$

and

$$\widetilde{s}(\nu)(y') = (\nu s)(y') = \nu(s(y')) = \nu(w').$$

By the condition, μ and ν coincide for every $x' \neq x$. According to the condition on s, element w' is generated by the elements x' of such kind. Since $\mu(x') = \nu(x')$ for every x' in the support of w', then $\mu(w') = \nu(w')$. Hence, $\mu s(y') = \nu s(y')$, $y' \neq y$. By definition of a quantifier $\exists y$ this means that the point μs belongs to $\exists y(B)$.

□

Moreover,

Proposition 2.3 (cf. [16]) *If a morphism $s : W(Y) \to W(X)$ is y-admissible, then for every $\mu : W(X) \to H$ such that $\widetilde{s}(\mu) \in \exists yB$, the point μ lies in $s_*(\exists yB) = \exists xA$. This means that*

$$\exists xA \supseteq s_*(\exists yB).$$

Proof. Let $\mu : W(X) \to H$ and $\widetilde{s}(\mu)$ belong to $\exists yB$, that is μ lies in $s_*(\exists yB)$. We want to prove that $\mu \in \exists xA$, that is there exists $\nu \in A$ such that $\nu(x') = \mu(x')$ for all $x' \neq x$, $x = s(y)$.

Since $\widetilde{s}(\mu) = \mu s \in \exists yB$, there exists $\xi : W(X) \to H$ such that $\xi \in B$ and $\xi(y') = (\mu s)(y')$, where $y' \neq y$. Let us show that if $\xi : W(X) \to H$ and $\xi \in B$, then there exists $\nu : W(X) \to H$ such that $(\mu s)(y') = (\nu s)(y')$ for every $y' \neq y$, i.e., $\xi = \nu s$. If $y' \in B$ then we have a commutative diagram

$$\begin{tikzcd} W(Y) \arrow[r, "s"] \arrow[dr, "\xi"'] & W(X) \arrow[d, "\nu"] \\ & H \end{tikzcd}$$

Denote $s(y') = w' \in W(X)$ for every $y' \neq y$. These w' induce the homomorphism $\nu : W((X) \to H$, such that $\nu(w') = \mu(w')$. Define $\mu(x') = \nu(x')$ for all x' which lie in the support of the element w'. This definition is correct since ν is the same for all w'. However, there can exist x' which does not lie in the support w'. Define $\mu(x') = \nu(x')$ in this case as well.

It remains to define $\nu(x)$, where $s(y) = x$. Geometrically, the point μ lies on the cylinder over the set A. So to define ν we need to take the "projection" of μ on A along the x.

So we constructed a point $\nu : W(X) \to H$, such that $\mu(x') = \nu(x')$, for every $x' \neq x$. The latter means that μ belongs to $\exists xA$, where $A = s_*B$.

$\square$

Summing up Propositions 2.2 and 2.3 above we obtain that for every $s : W(Y) \to W(X)$ and the corresponding $s_* : Hal_\Theta^Y(H) \to Hal_\Theta^X(H)$ the equality

$$s_*(\exists yB) = \exists xA$$

where $x = s(y)$, $A = s_*(B)$, $B \in Hal_\Theta^Y(H)$, takes place if and only if s is y-admissible.

From propositions above and Propositions 2.7 and 2.8 from [16] follow the rules for morphisms s_* in the category $Hal_\Theta(H)$.

Proposition 2.4 ([16]) *1. All morphisms s_* are homomorphisms of boolean algebras $Bool_\Theta^X(H)$, $X \in \Gamma$.*

2. Let $\varphi(w_1, \ldots, w_m) \in \Psi$. Then $s_*[\varphi(w_1, \ldots, w_m)]_H = [\varphi(sw_1, \ldots, sw_m)]_H$.

3. Let s_1 and s_2 be morphisms $W(Y) \to W(X)$ and let $s_1(y') = s_2(y')$ for all $y' \in Y$, $y' \neq y$. Then the equality

$$s_{1*}\exists y(B) = s_{2*}\exists y(B),$$

where $B \subset Hom(W(X), H)$, holds in $Hal_\Theta(H)$.

4. *Let $s : W(Y) \to W(X)$ be a morphism. Take $y \in Y$ and let $s(y) = x$. Let s be y-admissible. Then the equality*

$$s_* \exists y(B) = \exists s(y) s_*(B) = \exists x A,$$

where $B \subset Hom(W(Y), H)$, holds in $Hal_\Theta(H)$.

So, each $s : W(Y) \to W(X)$ induces a morphism $s_* : Hal_\Theta^Y(H) \to Hal_\Theta^X(H)$ in the category $Hal_\Theta(H)$. Every s_* satisfies conditions from Proposition 2.4. The morphisms s_* are not homomorphisms of the extended boolean algebras $Hal_\Theta^X(H)$ in contrast to morphisms in the category $\widetilde{\Phi}_0$.

Now we shall present another look at the Halmos categories. Define the subcategory Θ^1 of Θ^0. These categories have the same objects which are free finitely generated algebras $W(X)$. Morphisms of Θ^1 are admissible morphisms in Θ^0.

Proposition 2.5 *Θ^1 is a subcategory in Θ^0.*

Proof. Check that if $s_1 : W(Z) \to W(Y)$ is z-admissible morphism and $s_2 : W(Y) \to W(X)$ is y-admissible morphism, then $s_2 s_1 : W(Z) \to W(X)$ is z-admissible where $s_1(z) = y$, $s_2(y) = x$.

Take $z' \in Z$, $z' \neq z$ and let $s_1(z') = w$, $w \in W(Y)$. Then $w = w(y'_1, \ldots, y'_n)$ where $y'_i \neq y$ for all $i = 1, \ldots, n$. Apply an y-admissible s_2. We have $(s_2 s_1)(z') = s_2(w) = w(s_2(y'_1), \ldots, s_2(y'_n))$, that is $(s_2 s_1)(z') = w'(w'_1, \ldots, w'_n)$ where $w'_i \in W(X)$. Moreover, x does not belong to the support of w'. This means that $s_2 s_1$ is z-admissible morphism.

Basing on the category Θ_1 one can define the category of *correct* Halmos algebras $Hal_{\Theta_1}(H)$ exactly in the way it was done beforehand for the category $Hal_\Theta(H)$. Since we restricted ourselves to morphisms s in Θ_1, that is, to admissible morphisms, all morphisms s_* in $Hal_{\Theta_1}(H)$ become homomorphisms of extended boolean algebras. In plain words the category $Hal_{\Theta_1}(H)$ has less morphisms than $Hal_\Theta(H)$.

In what follows we stay on the positions of the logic defined by the category $Hal_\Theta(H)$ because in many cases there is no reason to assume an arbitrary morphism to be admissible. However, we impose this assumption if we deal with formulas with existential quantifiers.

The introduced category $Hal_\Theta(H)$ gives rise to a multi-sorted Halmos algebra $Hal_\Theta(H)$ (cf., [4]) whose domains and operations are objects and morphisms of the category, respectively.

Define further the variety of multi-sorted algebras Hal_Θ. The construction is as follows. We use the conditions described by Proposition 2.4 as axioms. The complete list of axioms can be found, for example, in [16], [1]. These axioms are either identities or conditional identities. The latter means that these conditions contain operations s which act on algebras from Θ^0 (see axioms 2–4 of Proposition 2.4).

Define Hal_Θ to be the class of algebras, generated by algebras $Hal_\Theta(H)$, H runs Θ, that is, the class of algebras which possess the same identities and conditional identities as algebras $Hal_\Theta(H)$. The class of algebras of such kind we call LG-variety (logically-geometric variety).

Straightforward calculations show that

Proposition 2.6 *The class of algebras Hal_Θ is closed with respect to operators of taking subalgebras, Cartesian products and homomorphic images.*

Hence, in view of Birkhoff's theorem [11] the LG-variety Hal_Θ is a variety in the sense of universal algebra, i.e., Hal_Θ is determined by the identities from 2.4, see also [16], [1]. Another set of identities which is more transparent can be found in [9], [10], [3]. In particular they show that "conditional" identities can be replaced by simpler ones. This fact requires introducing additional morphisms.

Let $\mathfrak{L}$ be the absolutely free algebra (algebra of multi-sorted terms) with respect to multi-sorted signature $L_\Theta = \{L_X, s_*, M_\Theta\}$, where the set of atomic formulas $M_\Theta = (M_X, X \in \Gamma)$ is the generating set of $\mathfrak{L}$. Here M_X is the set of all equalities $w \equiv w'$, $w, w' \in W(X)$. Given algebra H, assign to each formula $u \in M_X$ its value $Val_H^X(u)$ in the algebra $Hal_\Theta^X(H)$. This correspondence gives rise to the homomorphism $\mathfrak{L} \to Hal_\Theta(H)$. Let $Ker(H)$ be the kernel of this homomorphism. Define the algebra $\widetilde{\Phi}_\Theta = (\Phi(X),\ X \in \Gamma)$ as the quotient algebra of $\mathfrak{L}$ modulo $\bigcap_{H\in\Theta} Ker(H)$. Then the value homomorphism is defined for every algebra $Hal_\Theta(H)$, where $H \in \Theta$. In particular we have a commutative diagram for every $s : W(Y) \to W(X)$.

$$\begin{tikzcd}
\Phi(Y) \arrow[r, "s_*"] \arrow[d, "Val_F^Y"'] & \Phi(X) \arrow[d, "Val_F^X"] \\
Hal_\Theta^Y(F) \arrow[r, "s_*"] & Hal_\Theta^X(F)
\end{tikzcd}$$

Regarding s_*, we call these morphisms quasi-homomorphisms meaning that their behavior is ruled by axioms of $Hal_\Theta(H)$. Denote $Val_F^Y(v) = B$, $Val_F^X(u) = A$, where $u = s_*v$, $v \in \Phi(Y)$, $u \in \Phi(X)$. Then, in particular,

$$\begin{tikzcd}
\exists yv \arrow[r, "s_*"] \arrow[d, "Val_F^Y"'] & \exists xu \arrow[d, "Val_F^X"] \\
\exists yB \arrow[r, "s_*"] & \exists xA
\end{tikzcd}$$

Consider now the Galois correspondence. Note first of all that for any point $\mu : W(X) \to H$ we have its logical kernel $LKer(\mu)$ consisting of formulas $u \in \Phi(X)$ for which the points μ satisfy u. This kernel is a boolean ultrafilter in $\Phi(X)$. The Galois correspondence is a correspondence between the sets T of formulas in $\Phi(X)$ and the sets A of points in the space

$Hom(W(X),H)$. We define $T_F^L = A$ by the rule: a point μ is contained in A if $T \subset LKer(\mu)$. In other words, μ satisfies every $u \in T$. Then, in terms of the value homomorphism we can write:

$$T_F^L = A = \bigcap_{u \in T} Val_F^X(u).$$

Here we say that the set A is a definable set which is defined by the set of formulas T.

Let now $A \subset Hom(W(X),H)$ be given. We set:

$$A_F^L = T = \bigcap_{\mu \in A} LKer(\mu).$$

According to this definition a formula u is contained in the set T if and only if $A \subset Val_F^X(u)$.

The set $T = A_H^L$ is an F-closed boolean filter in $\Phi(X)$. We always have T_F^{LL} and A_F^{LL}. If $T = A_F^L$, then $T_F^{LL} = T$. If $A = T_F^L$, then $A_F^{LL} = A$.

3 Similarity of algebras and models

3.1 Category of definable sets

Denote the category of definable sets by $LG_\Theta(H)$. Let us define this category for the given model $F = (H, \Psi, f)$. Its objects are the sets $LG_\Theta^X(H)$. Each set consists of definable sets A of the space $Hom(W(X),H)$. Let A_1 and A_2 be definable sets in $Hom(W(Y),H)$, and let $A_1 = {T_1^L}_F$ and $A_2 = {T_2^L}_F$. Proceed from the fact that T_1 and T_2 are filters in $\Phi(X)$ and make some remarks on Galois transitions.
Take two filters T_1 and T_2 in $\Phi(X)$. Then

$$(T_1 \cup T_2)_F^L = {T_1^L}_F \cap {T_2^L}_F$$

follows directly from the definition of the operator L. Proceed further from $T_1 \cap T_2$. We have

$$(T_1 \cap T_2)^L \supset T_1^L \cap T_2^L.$$

Take now definable sets A_1 and A_2 in $Hom(W(X),H)$. By the definition,

$$(A_1 \cup A_2)_F^L = {A_1^L}_F \cap {A_2^L}_F$$

and

$$(A_1 \cap A_2)_F^L \supset {A_1^L}_F \cup {A_2^L}_F.$$

Theorem 3.1 *Let A_1 and A_2 be definable sets of points in $Hom(W(X),H)$. Then $A_1 \cup A_2$ and $A_1 \cap A_2$ are definable sets as well.*

Proof.

We have also $(T_1 \cup T_2)^L = T_1^L \cap T_2^L = A_1^{LL} \cap A_2^{LL} = A_1 \cap A_2$. So, $A_1 \cap A_2$ is a definable set.

Prove that $A_1 \cup A_2$ is definable. Take $T_1 = A_1^L$ and $T_2 = A_2^L$. Take the set T of all formulas $u \vee v$, $u \in T_1$, $v \in T_2$. Since $u \in T_1$, then $u \vee v$ also lies in T_1. The same for $v \in T_2$. Therefore, $T \subset T_1 \cap T_2$, $T^L \supset (T_1 \cap T_2)^L$.

We have $T_1 \cap T_2 \subset T_1$ and $A_1 = T_1^L \subset (T_1 \cap T_2)^L$. So, $A_1 \cup A_2 \subset (T_1 \cap T_2)^L \subset T^L$.

Take a point $\mu : W(X) \to H$ satisfying all formulas $u \vee v \in T$ and let μ doesn't belong to A_1. Then μ doesn't satisfy some $u \in T_1$. Hence, μ satisfies every $v \in T_2$, that is $\mu \in A_2$. So, if μ satisfies every $u \vee v \in T$, then μ is contained in $A_1 \cup A_2 = T_F^L$.

We checked that $T^L = A_1 \cup A_2$, thus, $A_1 \cup A_2 = (T_1 \cap T_2)^L$. The theorem is proved.

Corollary 3.2 *The system $LG_\Theta^X(H)$ is a lattice.*

Thus, objects of the category of all definable sets $LG_\Theta(H)$ are lattices $LG_\Theta^X(H)$. The corresponding morphisms

$$s_* : LG_\Theta^Y(H) \to LG_\Theta^X(H)$$

are morphisms of lattices, since they preserve boolean operations.

Let us make a remark on the functor

$$Cl_H : \widetilde{\Phi}_\Theta \to Lat.$$

Denote $Cl_H(\Phi(X)) = LF_\Theta^X(H)$. It is the lattice of all F-closed filters T in $\Phi(X)$ for the model F. Here $T = A_F^L$ for $A \subset Hom(W(Y), H)$. The lattice $LF_\Theta^X(H)$ is anti-isomorphic to the lattice $LG_\Theta^X(H)$. The transition $A \to A^L = T$ transposes union with intersection: $(A_1 \cup A_2)^L = A_1^L \cap A_2^L$ and $(T_1 \cup T_2)^L = T_1^L \cap T_2^L$. We have also: $s_* : \Phi(Y) \to \Phi(X)$ implies $s_* : LF_\Theta^Y(H) \to LF_\Theta^X(H)$. Here s_* is correlated with the lattice operations. We have also the diagram

$$\begin{tikzcd}
\Phi(Y) \arrow[r, "s_*"] \arrow[d, "Cl_F^Y"'] & \Phi(X) \arrow[d, "Cl_F^X"] \\
LF_\Theta(Y) \arrow[r, "s_*"] & LF_\Theta(X)
\end{tikzcd}$$

The next important step on the way to the main theorem is to recall the definition of isomorphism of two functors (cf., [6]).

Definition 3.3 *Let φ_1, φ_2 be two functors from a category C_1 to C_2. We say that an isomorphism of functors $S : \varphi_1 \to \varphi_2$ is defined if for any morphism $\nu : A \to B$ in C_1 the following commutative diagram takes place*

$$\begin{tikzcd}
\varphi_1(A) \arrow[r, "S_A"] \arrow[d, "\varphi_1(\nu)"'] & \varphi_2(A) \arrow[d, "\varphi_2(\nu)"] \\
\varphi_1(B) \arrow[r, "S_B"] & \varphi_2(B).
\end{tikzcd}$$

Here S_A is the A-component of S, that is, a function which makes a bijective correspondence between $\varphi_1(A)$ and $\varphi_2(A)$. The same is valid for S_B.

We have also

$$\varphi_2(\nu) = s^B \varphi_1(\nu)(s^A)^{-1}.$$

A particular case of this definition is the notion of inner automorphism of categories. An automorphism φ of the category C is called inner (see [13]) if φ is isomorphic to the identity functor 1_C. This provides the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "s_A"] \arrow[d, "\nu"'] & \varphi(A) \arrow[d, "\varphi(\nu)"] \\
B \arrow[r, "s_B"] & \varphi(B),
\end{tikzcd}$$

that is, $\varphi(\nu) = s_B \nu s_A^{-1}$.

In the next diagram

$$\begin{tikzcd}
\widetilde{\Phi}_\Theta \arrow[rr, "\varphi"] \arrow[dr, "Cl_{H_1}"'] & & \widetilde{\Phi}_\Theta \arrow[dl, "Cl_{H_2}"] \\
& Lat_\Theta &
\end{tikzcd}$$

φ is an automorphism of the category $\widetilde{\Phi}_\Theta$ meaning the substitution of variables in the algebra $\widetilde{\Phi}_\Theta$. Commutativity of the diagram means that there is an isomorphism of functors

$$\alpha\varphi : Cl_{H_1} \to Cl_{H_2}\varphi.$$

Here $F_1 = (H_1, \Psi, f_1)$, $F_2 = (H_2, \Psi, H_2)$.

Definition 3.4 ([16],[2]) *We call the models F_1 and F_2 automorphically equivalent, if this diagram is commutative.*

Recall (see [2]) that the models F_1 and F_2 are similar if the categories $LG_\Theta(H_1)$ and $LG_\Theta(H_2)$ are isomorphic.

Theorem 3.5 *Let $Var(H_1) = Var(H_2) = \Theta$. If the models $F_1 = (H_1, \Psi, f_1)$ and $F_2 = (H_2, \Psi, H_2)$ are automorphically equivalent, then they are similar.*

Proof. Let the models (H_1, Ψ, f_1) and (H_2, Ψ, H_2) be automorphically equivalent. Then there is an isomorphism of functors

$$\alpha\varphi : Cl_{H_1} \to Cl_{H_2}\varphi.$$

This means that there is a commutative diagram

$$\begin{array}{ccc} Cl_{H_1}(\Phi(Y)) & \xrightarrow{(\alpha\varphi)_{\Phi(Y)}} & Cl_{H_2}\varphi(\Phi(Y)) \\ {\scriptstyle Cl_{H_1}(s_*)}\downarrow & & \downarrow{\scriptstyle Cl_{H_2}\varphi(s_*)} \\ Cl_{H_1}(\Phi(X)) & \xrightarrow{(\alpha\varphi)_{\Phi(X)}} & Cl_{H_2}\varphi(\Phi(X)). \end{array}$$

Here, $(\alpha\varphi)_{\Phi(Y)}$ and $(\alpha\varphi)_{\Phi(X)}$ are bijections. Denote $\eta_* = \varphi(s_*)$.

Homomorphism $s : W(Y) \to W(X)$ defines uniquely the vertical arrows of the diagram. Our aim is to choose the horizontal arrows in such a way that there will be an isomorphism of $LG_\Theta(H_1)$ and $LG_\Theta(H_2)$.

Since $Var(H_1) = Var(H_2) = \Theta$, categories $LG_\Theta(H_1)$ and $LG_\Theta(H_2)$ are isomorphic if and only if categories $LF_\Theta(H_1)$ and $LF_\Theta(H_2)$ are isomorphic. Hence, it is enough to establish an isomorphism of the categories of the lattices of the closed filters.

We have T in the lattice $Cl_{H_1}(\Phi(Y))$ and we need to construct T^* in the lattice $Cl_{H_2}\varphi(\Phi(Y))$. So, the main problem is to assign T^* to the distinguished T.

First of all, we will define the semigroup $End_*(\Phi(X))$. For each $W(X)$ consider the semigroup $End(W(X))$. Take $s : W(X) \to W(X)$. The morphism $s_* : \Phi(X) \to \Phi(X)$ corresponds to s. Hence, for every $u \in \Phi(X)$ we have an element s_*u in $\Phi(X)$. All s_* constitute a semigroup which we denote $End_*(\Phi(X))$.

Remark 3.6 *It is worth to mention that $\Phi(X)$ is not a purely algebraic structure. So, we should assume correlation with quantifiers and atomic formulas. This correlation is given by cited above axioms of Halmos algebras related with the morphisms s_*. Moreover, in view of this remark, one can treat $End_*(\Phi(X))$ as the semigroup of logical endomorphisms of $\Phi(X)$.*

Let an H_1-closed filter T in $\Phi(Y)$ be given. By definition, T is a boolean filter. Correspondingly, $\Phi(Y)/T$ is a boolean algebra. Consider the homomorphism of boolean algebras

$$\mu_T : \Phi(Y) \to \Phi(Y)/T.$$

Take elements s_*^1, s_*^2 in $End_*(\Phi(Y))$. Define the relation ρ on $End_*(\Phi(Y))$ by the rule: $s_*^1 \rho s_*^2$ if and only if $(\mu_T s_*^1) = (\mu_T s_*^2)$, that is, $(\mu_T s_*^1)(u) = (\mu_T s_*^2)(u)$ for every $u \in \Phi(Y)$. This means that $s_*^1 u$ and $s_*^2 u$ coincide in $\Phi(Y)/T$ for every $u \in \Phi(Y)$.

Note that arbitrary formulas t_1 and t_2 from $\Phi(Y)$ coincide in $\Phi(Y)/T$ if and only if the formula $(t_1 \to t_2) \wedge (t_2 \to t_1)$ belongs to the filter T (Lindenbaum-Taski approach).

Apply this note to our situation, substituting t_1 and t_2 by $s_*^1 u$ and $s_*^2 u$, respectively. We have $s_*^1 \rho s_*^2$ if and only if $(s_*^1 u \to s_*^2 u) \wedge (s_*^2 u \to s_*^1 u)$ belongs to T.

This allows us to define the relation $\widetilde{\rho}$ on $\Phi(Y)$ as follows: $(s_*^1 u)\widetilde{\rho}(s_*^2 u)$ if and only if $(s_*^1 u \to s_*^2 u) \wedge (s_*^2 u \to s_*^1 u)$ belongs to T. Let $T = A_{H_1}^L$, where A lies in $Hom(W(Y), H_1)$. Using the L-Galois correspondence, the latter condition defining $\widetilde{\rho}$ can be reformulated now as: $(s_*^1 u)\widetilde{\rho}(s_*^2 u)$ if and only if the formula $(s_*^1 u \to s_*^2 u) \wedge (s_*^2 u \to s_*^1 u)$ where $u \in \Phi(Y)$ is satisfied on A.

So, $T \to \rho(T) = \rho \to \widetilde{\rho}$. Identifying $\widetilde{\rho}$ with the set of formulas $(s_*^1 u \to s_*^2 u) \wedge (s_*^2 u \to s_*^1 u)$, we have,

$$\tau(\widetilde{\rho}(T)) = T,$$

where $\tau(\widetilde{\rho}(T)) = (\widetilde{\rho}(T))^{LL}$.

Define relation ρ^* on $End_*\varphi(\Phi(Y)) = End_*(\Phi(Y'))$ according to action of the functor φ on morphisms. Namely, let $\varphi(s_*^1) = \eta_*^1$ and $\varphi(s_*^2) = \eta_*^2$. Define $\rho^* = \varphi(\rho)$, that is, $\eta_*^1 \rho^* \eta_*^2$ if and only if $s_*^1 \rho s_*^2$.

Take in $\Phi(Y')$ set all formulas of the form $(\eta_*^1 v \to \eta_*^2 v) \wedge (\eta_*^2 v \to \eta_*^1 v)$, where $v \in \Phi(Y')$ and $\eta_*^1 \rho^* \eta_*^2$. Denote this set by $\widetilde{\rho}^*$. Denote by $A_* = (\widetilde{\rho}^*)_{H_2}^L$ the corresponding definable set in $Hom(W(Y'), H_2)$. Then the filter T^* is defined as $T^* = (A_*)_{H_2}^L = (\widetilde{\rho}^*)_{H_2}^{LL}$.

Observe that one can define the relation $\widetilde{\rho}^*$ on $\Phi(Y')$ in the way similar to previously done with respect to $\widetilde{\rho}$ on $\Phi(Y)$. Define this relation by the rule: $(\eta_*^1 v)\widetilde{\rho}^*(\eta_*^2 v)$, where $v \in \Phi(Y')$ if and only if the formula $(\eta_*^1 v \to \eta_*^2 v) \wedge (\eta_*^2 v \to \eta_*^1 v)$ is satisfied on A_*. This relation characterizes T^* in the unique way.

Apply this setting to the diagram which defines isomorphism of functors Cl_{H_1} and $Cl_{H_2}\varphi$. Take $T \in Cl_{H_1}(\Phi(Y))$ and denote $(\alpha\varphi)_{\Phi(Y)}(T)$ by T^*. Then, T^* lies is $Cl_{(H_2)}\varphi(\Phi(Y))$. We have $T = \tau_{\Phi(Y)}(\rho(T))$. Hence, $T^* = \tau_{\varphi(\Phi(Y))}(\varphi(\rho(T)))$. This means that filter T defines T^* uniquely. Because of the diagram the correspondence $T \to T^*$ gives rise to the isomorphism of categories $LF_\Theta(H_1)$ and $LF_\Theta(H_2)$.

Acknowledgements. E. Aladova and E.Plotkin were supported by the Israel Science Foundation grant No. 1207/12, by the Minerva foundation through the Emmy Noether Research Institute. The support of these institutions is gratefully appreciated.

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