

BOUNDED GENERATION OF STEINBERG GROUPS OVER DEDEKIND RINGS OF ARITHMETIC TYPE

BORIS KUNYAVSKIĬ, ANDREI LAVRENOV, EUGENE PLOTKIN,
AND NIKOLAI VAVILOV

ABSTRACT. The main result of the present paper is bounded elementary generation of the Steinberg groups $\mathrm{St}(\Phi, R)$ for simply laced root systems Φ of rank ≥ 2 and arbitrary Dedekind rings of arithmetic type. Also, we prove bounded generation of $\mathrm{St}(\Phi, \mathbb{F}_q[t, t^{-1}])$ for all root systems Φ , and bounded generation of $\mathrm{St}(\Phi, \mathbb{F}_q[t])$ for all root systems $\Phi \neq A_1$.

The proofs are based on a theorem on bounded elementary generation for the corresponding Chevalley groups, where we provide uniform bounds.

In memory of our dear friend and teacher Kolya Vavilov

Remark *This paper was completed in summer 2023 and was, in fact, the last work of Nikolai Vavilov. Nowadays, the style and scope of this paper could be slightly different. But in the present circumstances, the authors find it neither reasonable nor moral to alter the original style of N. A. Vavilov.*

INTRODUCTION

In the present paper, we consider simply-connected Chevalley groups $G = G_{\mathrm{sc}}(\Phi, R)$, and the corresponding Steinberg groups $\mathrm{St}(\Phi, R)$ over Dedekind rings of arithmetic type. G is generated by the elementary root unipotents $x_\alpha(r)$, $\alpha \in \Phi$, $r \in R$, and we are interested in the classical problem of estimating the width of G with respect to the generators $x_\alpha(r)$. The width is defined as the smallest possible m such as every element of G is representable as a product of m generators $x_\alpha(r)$. If there is no such m , we say that the width is infinite. If the width is finite, we say that G is **boundedly elementarily generated**.

The problem of bounded generation has attracted considerable attention over the last 40 years or so. We refer the reader to [KPV] containing a survey of this long activity as well as some applications to Kac–Moody groups and model theory.

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To make a long story short, given a reduced irreducible root system Φ of rank ≥ 2 , a Dedekind ring of arithmetic type R , and a Chevalley group $G = G_{\text{sc}}(\Phi, R)$, until now there were many cases where it was known that G is boundedly elementarily generated, and two kinds of general upper estimates for the elementary width of G were available:

- explicit estimates depending on Φ and the fraction field of R ;
- estimates depending on Φ alone in the case $\Phi = A_l$.

Combining the methods of [KPV] and [Tr22], we are now able to come up with a complete solution in the general case:

Theorem A ([KPV2]). *Let Φ be a reduced irreducible root system of rank l .*

- (1) *If $l \geq 2$, then there exists a constant $L = L(\Phi)$, depending on Φ alone, such that for any Dedekind ring of arithmetic type R , any element in $G_{\text{sc}}(\Phi, R)$ is a product of at most L elementary root unipotents.*
- (2) *If $l = 1$, then there exists a constant $L = L(\Phi)$ such that for any Dedekind ring of arithmetic type R **with infinite multiplicative group R^*** , any element in $G_{\text{sc}}(\Phi, R)$ is a product of at most L elementary root unipotents.*

An important — and unexpected! — aspect of this result is the existence of *explicit uniform* bounds in the function case. In the number case the bounds are also uniform, but if we wish to cover all R and not just those with infinite multiplicative group R^* , they are not explicit. Note that for symplectic groups the result is new even in the number case.

We will sketch the proof of Theorem A in Section 2. However, since the goal of the present work is bounded generation for Steinberg groups, we intentionally do not provide any computational arguments, especially taking into account that in the most tricky case $\Phi = C_2$ they are long and tedious. Moreover, no care of explicit numerical bounds is taken here. Observe in particular that the case of $\text{SL}(3, R)$ for function rings, solved in [Tr21], is redone in [KPV2] with better bounds. We refer to [KPV2] for all details and discussions of the proof of Theorem A, taking uniform bounded generation for Chevalley groups for granted. Note, however, that for the sake of completeness we collect in Theorem E all cases of Theorem A needed for the treatment of Steinberg groups and provide a full proof.

At this point it is natural to ask whether Theorem A or maybe its weaker forms can be generalised to Steinberg groups. This question was explicitly raised by Alexei Myasnikov at the conference GAGTA 2022. The reason was that bounded generation of Steinberg groups would have important model-theoretic applications. It plays a crucial role in problems such as first order rigidity, elementary equivalence of groups, Diophantine theory, and the like. So, having Theorem A for Chevalley groups as a base, one can think about similar properties for their coverings.

More precisely, behind recent successes in model theory of classical and Chevalley groups stands the notion of bi-interpretability. In fact, one needs to find a

logical formula that defines the ground ring together with its ring operations inside the group in terms of the group operation, and vice versa. The groups in question should be interpretable in the ring, see [BMP], [AvMe], [SeTe], [BuGv]. Since bounded generation gives a finite expression of any element via the group generators, it is clear that it gives a desired formula for interpretation of ground rings in groups in question provided the group generators are definable. When do we have bounded generation of Chevalley groups? As a rule, we need various low dimensional stability conditions on rings for that [St78]. In particular, Dedekind rings of arithmetic type serve as a major example of such conditions, see [Tav], [KPV]. Going beyond these rings is a very difficult problem. The following open question goes back to Hyman Bass: *Is it true that for any n there exists a ring R of $\dim \operatorname{Max} R = n$ such that the Chevalley group $G_{\operatorname{sc}}(\Phi, R)$ is boundedly generated?*

Again, we are interested in the bounded generation in terms of the set

$$X = \{x_\alpha(r) \mid \alpha \in \Phi, r \in R\}$$

of elementary generators (which we continue to denote by the same letter).

However, this case turned out to be much more challenging. Apart from the bounded generation of the Chevalley groups themselves, it depends on the deep results on the finiteness of the (linear) K_2 -functor, and on bunch of other difficult results of K-theory, such as stability theorem for K_2 , centrality of K_2 , etc.

It is not clear how one could get *uniform* bounds in this case. Even with the bounds that depend on R so far we could only prove it for the case when the root system Φ is *simply-laced*, i.e., $\Phi = A_l, D_l, E_l$.

Theorem B. *Let Φ be a reduced irreducible simply laced root system of rank ≥ 2 , and let R be a Dedekind ring of arithmetic type. If $\Phi = A_2$, assume additionally that R^* is infinite. Then $\operatorname{St}(\Phi, R)$ is boundedly elementarily generated.*

The idea is to derive this result from Theorem A. It suffices to establish that the kernel $K_2(\Phi, R)$ of the projection $\operatorname{St}(\Phi, R) \rightarrow G_{\operatorname{sc}}(\Phi, R)$ is finite and central, and thus bounded elementary generation of $G_{\operatorname{sc}}(\Phi, R)$ implies that of $\operatorname{St}(\Phi, R)$. Here are the main sources on which we rely in this proof.

- The *stable* linear $K_2(R)$ is finite, for the function case this is proven by Hyman Bass and John Tate [BaTa] and for the number case by Howard Garland [Gar]. (These finiteness results were generalised to higher K-theory by Daniel Quillen and Günter Harder, see the survey by Chuck Weibel [W]).
- However, we need similar results for the unstable K_2 -functors $K_2(\Phi, R)$. For the *linear* case $\operatorname{SL}(n, R)$ there is a definitive stability theorem by Andrei Suslin and Marat Tulenbaev [SuTu]. However, injective stability for Dedekind rings only starts with $n \geq 4$, so that for $\operatorname{SL}(3, R)$ one has to refer to Wilberd van der Kallen [Ka81] instead, which accounts for the extra-condition in this case.
- For other embeddings there are no stability theorems in the form we need them and starting where we want them to start. For instance, in the even orthogonal

case the theorem of Ivan Panin [Pan] starts with $\mathrm{Spin}(10, R)$, whereas we would like to cover also $\mathrm{Spin}(8, R)$. In any case, there are no similar results for the exceptional embeddings.

Thus, we have to prove a comparison theorem relating $K_2(\Phi, R)$ to $K_2(A_3, R)$. This is obtained as a corollary of partial stability results for Dedekind rings developed by Hideya Matsumoto [Mat] and *surjective* stability of K_2 for some embeddings, established by Michael Stein [St78] and the third author [Pl91, Pl98].

We also remark that the centrality of K_2 for all Chevalley groups over arbitrary rings is accomplished by the second author, Sergey Sinchuk and Egor Voronetsky, also in collaboration [Lav, Sin, LS17, LS20, Vor, LSV24].

- An essential obstacle in the symplectic case is that $K_2(C_l, R)$ is the Milnor–Witt K_2^{MW} , rather than the usual Milnor K_2^{M} , as for all other cases (compare [Sus] for an explicit connection between $K_2\mathrm{Sp}(R)$ and $K_2(R)$). As is well known, it may fail to be finite, which means that our approach does not work at all in this case. This does not mean that the result itself fails, but the proof would require an entirely different idea.

But even for non-symplectic multiply laced systems, where our approach could theoretically work, we were unable to overcome occurring technical difficulties related to the K_2 -stability and comparison theorems. At least, as yet.

In contrast to Theorem B, in our second generalisation of Theorem A, we put the restrictions on the ring rather than the root system.

Namely, using specific calculations of $K_2(\Phi, \mathbb{F}_q[t])$ and $K_2(\Phi, \mathbb{F}_q[t, t^{-1}])$ by Eiichi Abe, Jun Morita, Jürgen Hurrelbrink and Ulf Rehmann [AbMo, Hur, MoRe, Reh] we were able to establish similar results over $\mathbb{F}_q[t]$ and $\mathbb{F}_q[t, t^{-1}]$ also for the multiply laced systems, even the symplectic ones.

Theorem C. *Let Φ be a reduced irreducible root system, and $R = \mathbb{F}_q[t, t^{-1}]$ or $R = \mathbb{F}_q[t]$. In the latter case assume additionally that $\Phi \neq A_1$. Then $\mathrm{St}(\Phi, R)$ is boundedly elementarily generated.*

The paper is organised as follows. In § 1 we recall notation and collect some preliminary results. In § 2 we give a sketch of the proof of Theorem A. In § 3 we recall some basic facts concerning $K_2(\Phi, R)$ and in § 4 collect the necessary facts concerning finiteness of K_2 in the arithmetic case. In § 5 we prove comparison theorems for $K_2(\Phi, R)$ in the case of simply laced Φ , and thus prove Theorem B. In § 6 we recall computation of K_2 of polynomial rings, which imply Theorem C. Finally, in § 7 we mention some further generalisations.

1. NOTATION AND PRELIMINARIES

In this section we briefly recall the notation that will be used throughout the paper and some background facts. For more details on Chevalley groups over rings see [Va91] or [VaPl], where one can find many further references.

1.1. Chevalley groups. Let Φ be a reduced root system and $W = W(\Phi)$ be its Weyl group. In our main results, Φ will be assumed irreducible, though in some proofs one has to use subsystems that are not. As usual, we choose an order on Φ and let Φ^+ , Φ^- and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be the corresponding sets of positive, negative and fundamental roots, respectively. Further, we consider a lattice \mathcal{P} intermediate between the root lattice $\mathcal{Q}(\Phi)$ and the weight lattice $\mathcal{P}(\Phi)$. Finally, let R be a commutative ring with 1, with the multiplicative group R^* .

These data determine the Chevalley group $G = G_{\mathcal{P}}(\Phi, R)$, of type (Φ, \mathcal{P}) over R . It is usually constructed as the group of R -points of the Chevalley–Demazure group scheme $G_{\mathcal{P}}(\Phi, -)$ of type (Φ, \mathcal{P}) . In the case $\mathcal{P} = \mathcal{P}(\Phi)$ the group G is called simply connected and is denoted by $G_{\text{sc}}(\Phi, R)$. In another extreme case $\mathcal{P} = \mathcal{Q}(\Phi)$ the group G is called adjoint and is denoted by $G_{\text{ad}}(\Phi, R)$.

Many results do not depend on the lattice \mathcal{P} and hold for all groups of a given type Φ . In all such cases, or when \mathcal{P} is determined by the context, we omit any reference to \mathcal{P} in the notation and denote by $G(\Phi, R)$ *any* Chevalley group of type Φ over R . However in some cases specific bounds may depend on \mathcal{P} . Usually, we work with a simply connected group, but in some cases it is convenient to work with the adjoint group, which is then reflected in the notation.

In what follows, we also fix a split maximal torus $T = T(\Phi, R)$ in $G = G(\Phi, R)$ and identify Φ with $\Phi(G, T)$. This choice uniquely determines the unipotent root subgroups, X_{α} , $\alpha \in \Phi$, in G , elementary with respect to T . As usual, we fix maps $x_{\alpha}: R \mapsto X_{\alpha}$, so that $X_{\alpha} = \{x_{\alpha}(r) \mid r \in R\}$, and require that these parametrisations are interrelated by the Chevalley commutator formula with integer coefficients, see [C], [S]. The above unipotent elements $x_{\alpha}(r)$, where $\alpha \in \Phi$, $r \in R$, elementary with respect to $T(\Phi, R)$, are also called [elementary] unipotent root elements or, for short, simply root unipotents.

Further,

$$E(\Phi, R) = \langle x_{\alpha}(r) \mid \alpha \in \Phi, r \in R \rangle$$

denotes the *absolute* elementary subgroup of $G(\Phi, R)$, spanned by all elementary root unipotents, or, what is the same, by all [elementary] root subgroups X_{α} , $\alpha \in \Phi$. For $\epsilon \in \{+, -\}$ denote

$$U^{\epsilon}(\Phi, R) = \langle x_{\alpha}(r) \mid \alpha \in \Phi^{\epsilon}, r \in R \rangle \leq E(\Phi, R).$$

1.2. Steinberg groups. Denote by $\text{St}(\Phi, -)$ the *Steinberg group* functor corresponding to Φ . For Φ that does not have irreducible components $\cong A_1$, and a commutative ring R the Steinberg group $\text{St}(\Phi, R)$ is a group defined by the set of generators

$$\{x_{\alpha}(r) \mid \alpha \in \Phi, r \in R\}$$

subject to the *Steinberg relations*

- Additivity

$$x_{\alpha}(r)x_{\alpha}(s) = x_{\alpha}(r+s) \quad \text{for } \alpha \in \Phi, r, s \in R,$$

- Chevalley commutator formula

$$(1) \quad [x_\alpha(r), x_\beta(s)] = \prod_{\substack{i,j \in \mathbb{N} \setminus 0 \\ i\alpha + j\beta \in \Phi}} x_{i\alpha + j\beta}(N_{\alpha\beta ij} r^i s^j) \quad \text{for } \alpha, \beta \in \Phi, \beta \neq -\alpha, r, s \in R,$$

where, as usual, $[g, h] = ghg^{-1}h^{-1}$ denotes the left normed commutator, whereas $N_{\alpha\beta ij} \in \mathbb{Z}$ are the structure constants of the Chevalley group $G_{sc}(\Phi, R)$.

The choice of the structure constants $N_{\alpha\beta ij} \in \mathbb{Z}$ and the order of factors in (1) are not unique, and we fix any possible choice, see [VaPl, Va08] for many more details and further references. It is not a problem to specify signs for classical cases, see [B]. On the other hand in [Va01] one can find specific choice of the structure constants $N_{\alpha\beta}$ for E_6 , E_7 and E_8 , corresponding to a positive Chevalley base (in this case automatically $i = j = 1$, so that $N_{\alpha\beta 11} = N_{\alpha\beta}$ are just the structure constants of the corresponding Lie algebra). All structure constants $N_{\alpha\beta ij}$ for F_4 and G_2 are tabulated in [VaPl].

- For A_1 one needs another relation

$$w_\alpha(u)x_\alpha(r)w_\alpha(u)^{-1} = x_{-\alpha}(-u^{-2}r) \quad \text{for } \alpha \in \Phi, u \in R^*, r \in R,$$

where

$$(2) \quad w_\alpha(u) = x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u).$$

Remark. If Φ does not have irreducible components $\cong A_1$ this extra relation follows from additivity and the Chevalley commutator formula.

1.3. Arithmetic case. Let F be a global field and S be a finite non-empty set of places of F containing all archimedean places when F is a number field. Following [BMS] we will say that

$$R = \{x \in F \mid v(x) \geq 0 \ \forall v \notin S\}$$

is the *Dedekind ring of arithmetic type* defined by the set S . Obviously, R is indeed a Dedekind domain, and one can canonically identify the maximal ideals of R with the places outside S .

The following result is proven by Matsumoto in [Mat, Théorème 12.7].

Theorem 1.1. *Let R be a Dedekind ring of arithmetic type and Φ a reduced irreducible root system of rank at least 2. Then*

$$E_{sc}(\Phi, R) = G_{sc}(\Phi, R).$$

In fact, for $\Phi = A_l, C_l$ this result was established already by Hyman Bass, John Milnor and Jean-Pierre Serre in [BMS]. Recently Anastasia Stavrova generalised it to isotropic reductive groups and to polynomial rings over R , see [Sta, Corollary 1.2].

2. UNIFORM BOUNDED GENERATION OF CHEVALLEY GROUPS: AROUND THE PROOF OF THEOREM A

In this section we sketch a proof of Theorem A.

Recall that the results of [Mor, MRS, Tr22, KMR] completely solve the problem of the uniform bounded elementary generation for the special linear groups $\mathrm{SL}(n, R)$, $n \geq 3$, — and when R^* is infinite, even for $\mathrm{SL}(2, R)$.

Observe that the methods of [KPV] completely reduce the proof of similar result for almost all other Chevalley groups, including even the *symplectic* groups $\mathrm{Sp}(2l, R)$, $l \geq 3$, to the case of $\Phi = A_2$.

The *only* case that does not follow rightaway from results of the above papers, is that of $\mathrm{Sp}(4, R)$. The analysis of that case is longer and far too technical, its inclusion would tilt the balance of the present paper. Therefore, below we give a complete argument only for the cases needed for the proof of Theorem B, collecting these cases in Theorem E. The proof for all other cases, with all details and explicit bounds, is contained in [KPV2].¹

2.1. Tavgen rank reduction theorem. In most cases the reduction to A_1 or A_2 is based on the following cunning observation, whose idea goes back to the work of Oleg Tavgen [Tav]. His trick was then generalised in [VSS] and [SSV]. The following final form is proven in [KPV, Theorem 3.2].

Theorem 2.1. *Let Φ be a reduced irreducible root system of rank $l \geq 2$, and R be a commutative ring. Let $\Delta_1, \dots, \Delta_t$ be some subsystems of Φ , whose union contains all fundamental roots of Φ . Suppose that for all Δ_i the elementary Chevalley group $E(\Delta_i, R)$ admits a unitriangular factorisation*

$$E(\Delta_i, R) = U^+(\Delta_i, R) U^-(\Delta_i, R) U^+(\Delta_i, R) \dots U^\pm(\Delta_i, R)$$

of length N (not depending on i). Then the elementary group $E(\Phi, R)$ itself admits unitriangular factorisation

$$E(\Phi, R) = U^+(\Phi, R) U^-(\Phi, R) U^+(\Phi, R) \dots U^\pm(\Phi, R)$$

of the same length N .

Below, we essentially apply it to two cases, when all Δ_i 's are A_1 , and when all of them are A_2 .

2.2. The case when R^* is infinite. The case where R has infinitely many units and its field of fractions is a number field is *completely* solved, with very small *absolute* constant. We cannot describe the whole chain of events here, and mention all contributors. After the initial breakthrough by Maxim Vsemirnov [Vse], which was a first unconditional result of this sort, not depending on the GRH, Aleksander Morgan, Andrei Rapinchuk and Sury [MRS] succeeded in solving the number case,

¹In the most recent preprint, Trost established the uniformly bounded generation in the symplectic case (with the estimates weaker than in [KPV2]); see [Tr23, Corollary 3.11].

with the bound $L = 9$. This bound was then improved to $L = 8$ by Bruce Jordan and Yevgeny Zaytman [JoZa] (and can be further improved in the presence of finite or real valuations in S).

Lemma 2.2. [JoZa] *For any Dedekind ring of arithmetic type R in a number field with the infinite multiplicative group R^* any element in $\mathrm{SL}(2, R)$ is a product of at most 8 elementary transvections.*

Remark 2.3. In the paper presently under way the first author, Dave Morris and Andrei Rapinchuk [KMR] improve the bound to $L = 7$ in the number case (which we believe is the best possible and cannot be further improved, in general). Also, they obtain a similar result in the function case, with the bound $L = 8$ (which, we believe, can be further improved to $L = 7$).

Theorem D ([KMR]). *For any Dedekind ring of arithmetic type R with the infinite multiplicative group R^* any element in $G_{\mathrm{sc}}(\Phi, R)$ is a product of at most $L = 8|\Phi^+|$ elementary unipotents.*

Proof. Combine Theorem 2.1 with $\Delta_i = A_1$, Lemma 2.2 and Remark 2.3. \square

Thus, if we are not interested in actual bounds, but just in uniform boundedness, one can restrict oneself to considering the Dedekind rings of arithmetic type with *finite* multiplicative groups. In the number case, these are \mathbb{Z} and the rings of integers in imaginary quadratic number fields. Note that as discovered by Alexander Trost [Tr22], in the function case for ranks ≥ 2 we do not have to distinguish between rings with finite and infinite multiplicative group.

2.3. The simply laced case and $\Phi = F_4$.

Theorem 2.4. [Tr22, Theorem 4.1] *For each $l \geq 2$, there exists a constant $L = L(l) \in \mathbb{N}$ such that for any Dedekind ring of arithmetic type R , any element in $G_{\mathrm{sc}}(A_l, R)$ is a product of at most L elementary root unipotents.*

Proof. By a theorem of Carter–Keller–Paige (redeveloped by Morris [Mor]), see [CKP, (2.4)], bounded generation for groups of type A_l , $l \geq 2$, holds for Dedekind rings R in number fields K , with a bound depending on l and also on the degree d of K . But since for all degrees $d \geq 3$ the existence of a uniform bound already follows from Theorem D, we only need to take maximum of that, and the universal bound for $d = 1, 2$.

Combining this result with the subsequent work of Trost [Tr22] on the function field case, one obtains the result. \square

Remark 2.5. In fact, in the sequel we only need the special case of the above result pertaining to $\mathrm{SL}(3, R)$, which corresponds to $\Phi = A_2$. In the function case Trost [Tr22] gave the estimate $L(2) \leq 65$. No such explicit estimate is known in the number case.

Since the fundamental systems of the simply laced systems and F_4 are covered by copies of A_2 , combining Theorem 2.1 with Theorem 2.4 one gets another stronger form of Theorem A, now without the assumption that R^* is infinite, but only in the special case of simply laced systems of rank ≥ 2 and F_4 . This is the only part of Theorem A on which Theorem B relies.

Theorem E. *Let Φ be simply laced of rank ≥ 2 or $\Phi = F_4$ and R be any Dedekind ring of arithmetic type. Then $G_{\text{sc}}(\Phi, R)$ is a product of at most $L(2) \cdot |\Phi^+|$ elementary unipotents.*

The bound here is very rough, since $L(2)$ is the number of *elementary* factors, the number of unitriangular ones can be much smaller. Also, the use of stability allows to get much better bounds, of the type $L = L(2) + 4|\Phi^+|$, where some multiple of $|\Phi^+|$ occurs as a summand, not as a factor.

2.4. Idea of the rest of the proof of Theorem A. To establish Theorem A for the cases of Chevalley groups $G(\Phi, R)$ not covered by Theorem E, where Tavgen's trick played a crucial role, we use the arguments based on the surjective stability of K_1 -functor, in the spirit of [KPV]. This way, we obtain the following reduction theorem.

Lemma 2.6. *Let R be a Dedekind ring of arithmetic type. Then (uniform) bounded generation of the groups $G(\Phi, R)$, $\Phi \neq C_2$ follows from (uniform) bounded generation of the group $G(A_2, R)$. \square*

For instance, the case $G(G_2, R)$ follows from [KPV, Proposition 4.3]. By stability arguments, $G(B_l, R)$ and $G(C_l, R)$ ($l \geq 3$) reduce to $G(B_3, R)$ and $G(C_3, R)$, respectively. The group $G(B_3, R)$ is treated as in [KPV, Section 6.2]. Somewhat surprisingly, the case $G(C_3, R)$ can be reduced to $G(A_2, R)$, along the lines of [KPV, Sections 5 and 6], using arithmetic lemmas of Carter and Keller [CaKe] in the number case and of Trost [Tr22] in the function case. Actually, the idea of such a reduction was contained already in Zakiryanov's thesis, see [Zak], but the authors of [KPV] have not realised this fact before rediscovering the same idea in the general case in March 2023².

Note that in all cases we get absolute constants as bounds, depending only on $L(2)$ appearing in Theorem 2.4. Thus these bounds are explicit in the function case and implicit in the case where R is a quadratic imaginary ring.

This finishes the proof of Theorem A in all cases except C_2 which turns out to be much more involved. It does not reduce to $G(A_2, R)$ and is settled in [KPV2] by the same methods as in [Mor, Tr21, Tr22, Tr23], using some results of [KPV]. As in all other cases, the obtained bounds are in the form of absolute constants, which are explicit in the function case.

²It is wrongly claimed in [Zak] that $\text{Sp}(4, \mathbb{Z})$ is not boundedly generated. As a result this work has not been given the credit it deserves. In particular, it should have been cited in the historical survey of [KPV].

3. K_2 MODELED ON CHEVALLEY GROUPS

In this section we collect the classical results on $K_2(\Phi, -)$ which we will use in this paper.

There is a natural map from $\text{St}(\Phi, R)$ to $G_{\text{sc}}(\Phi, R)$ sending generators of the Steinberg group $x_\alpha(r)$ to elementary root unipotents $x_\alpha(r)$ of the Chevalley group, $\alpha \in \Phi$, $r \in R$. Following [St78], we denote

$$K_2(\Phi, R) = \text{Ker}(\text{St}(\Phi, R) \rightarrow G_{\text{sc}}(\Phi, R)).$$

Let R be a commutative ring. Following Steinberg, for $\alpha \in \Phi$, $u \in R^*$ we define the elements

$$(3) \quad h_\alpha(u) = w_\alpha(u)w_\alpha(-1)$$

of the Steinberg group $\text{St}(\Phi, R)$, where $w_\alpha(u)$ is defined in § 1 (2), see [St71].

Further, for two invertible elements $u, v \in R^*$ we define the **Steinberg symbol**

$$\{u, v\}_\alpha = h_\alpha(uv)h_\alpha(u)^{-1}h_\alpha(v)^{-1}.$$

The following fact is well-known, see, for instance, [St73, Proposition 1.3].

Lemma 3.1. *For a ring R and a reduced irreducible root system Φ elements $\{u, v\}_\alpha$ for $u, v \in R^*$, $\alpha \in \Phi$, are central in $\text{St}(\Phi, R)$ and belong to $K_2(\Phi, R)$.*

The following classical result is due to Matsumoto [Mat, Corollaire 5.11].

Theorem 3.2. *Let k be a field, Φ be a reduced irreducible root system.*

- 1) *The group $K_2(\Phi, k)$ is generated by $\{u, v\}_\alpha$ for any fixed long root $\alpha \in \Phi$, and all $u, v \in R^*$.*
- 2) *Let Φ be a non-symplectic reduced irreducible root system (i.e., $\Phi \neq A_1, B_2, C_l$). Consider any embedding $A_2 \hookrightarrow \Phi$ on long roots. Then the induced map*

$$K_2(A_2, k) \rightarrow K_2(\Phi, k)$$

is in fact an isomorphism.

- 3) *For a symplectic reduced irreducible root system $C_1 = A_1$, $C_2 = B_2$, or C_l , $l \geq 3$, consider any embedding $A_1 \hookrightarrow C_l$ on long roots. Then the induced map*

$$K_2(A_1, k) \rightarrow K_2(C_l, k)$$

is in fact an isomorphism.

Remark 3.3. In fact, Matsumoto describes $K_2(\Phi, k)$ in terms of generators and relations in [Mat, Corollaire 5.11]. In modern terms, Matsumoto proved that $K_2(\Phi, k)$ coincides with Milnor $K_2^M(k)$ for non-symplectic Φ and with Milnor–Witt $K_2^{\text{MW}}(k)$ for symplectic Φ . However, we will not need the explicit description of relations in this paper.

We will also need the following stabilisation results. The next statement is a particular case of the Suslin–Tulenbaev theorem, see [SuTu, Corollary 4.2].

Theorem 3.4. *Let R be a Dedekind domain, then the natural map*

$$K_2(A_l, R) \rightarrow K_2(A_{l+1}, R)$$

is surjective for $l \geq 2$ and injective for $l \geq 3$.

For the root systems other than A_l we only have the surjective stability part, established by Stein [St78, Corollary 3.2, Theorem 4.1].

Theorem 3.5. *Let R be a Dedekind domain. Consider the following embeddings of root systems $\Psi \hookrightarrow \Phi$:*

- *natural embedding $D_l \hookrightarrow D_{l+1}$ for $l \geq 3$;*
- *natural embedding $E_l \hookrightarrow E_{l+1}$ for $l = 6, 7$;*
- *the embedding $D_5 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\} \hookrightarrow E_6$ (with numbering according to Bourbaki [B, Table V]).*

Then the induced map

$$K_2(\Psi, R) \rightarrow K_2(\Phi, R)$$

is surjective.

On the other hand, for Dedekind rings of arithmetic type with infinite multiplicative groups the bounds in surjective/injective stability can be improved by 1. This was done by van der Kallen [Ka81, Theorem 1].

Theorem 3.6. *Let R be a Dedekind ring of arithmetic type with infinitely many units. Then the natural map*

$$K_2(A_l, R) \rightarrow K_2(A_{l+1}, R)$$

is surjective for $l \geq 1$ and injective for $l \geq 2$.

Finally, we will need also another result by Stein claiming that surjective stability implies centrality of K_2 [St71, Theorem 5.1].

Lemma 3.7. *Let Π denote a set of simple roots in a reduced root system Φ . For $\alpha \in \Pi$, let $\Psi \subseteq \Phi$ be the subsystem generated by $\Pi \setminus \alpha$. Then*

$$K_2(\Phi, R) \cap \text{Im}(\text{St}(\Psi, R) \rightarrow \text{St}(\Phi, R))$$

is a central subgroup of $\text{St}(\Phi, R)$ for any commutative ring R .

4. STABLE LINEAR K_2

Recall that

$$K_2(R) = \lim_{l \rightarrow \infty} K_2(A_l, R)$$

is the usual stable linear K_2 -functor for any ring R (cf. [W, Chapter III, Section 5]).

For a place v of a field F let κ_v denote the corresponding residue class field and

$$\partial_v: K_2(F) \rightarrow \kappa_v^*$$

the corresponding residue homomorphism (also called *tame symbol*) sending generators $\{x, y\}_\alpha$ of $K_2(F)$ (see Theorem 3.2) to

$$(-1)^{v(x)v(y)} \overline{\left(\frac{y^{v(x)}}{x^{v(y)}}\right)} \in \kappa_v^*,$$

see [W, Chapter III, Lemma 6.3]. We will need the following result due to Christophe Soulé, see, for instance, [W, Chapter V, Theorem 6.8].

Theorem 4.1. *Let R be a Dedekind domain whose field of fractions F is a global field. Then there is an exact sequence*

$$0 \rightarrow K_2(R) \rightarrow K_2(F) \xrightarrow{\oplus \partial_{\mathfrak{p}}} \bigoplus_{\mathfrak{p}} (R/\mathfrak{p})^* \rightarrow 0,$$

where the first arrow is induced by the natural inclusion $R \hookrightarrow F$, and the sum is taken over all non-zero prime ideals \mathfrak{p} of R .

Our proof of Theorem B heavily relies on the following classical result. In the function case this is due to Bass and Tate [BaTa, Chapter II, Theorem 2.1]. In the number case this was first established by Garland [Gar] by analytic methods (see also [BaTa, Chapter II, Remark after Theorem 2.1]).

Theorem 4.2. *Let F be a global field. Then*

$$H_2 = \text{Ker} \left(K_2(F) \xrightarrow{\oplus \partial_v} \bigoplus \kappa_v^* \right),$$

where the sum is taken over all finite places of F , is finite.

Remark 4.3. The case $\text{char } F = p > 0$ of Theorem 4.2 was generalised to higher K-theory by Harder [Har, Korollar 3.2.3]. For the number field case see [Bor, Proposition 12.2].

The following corollary is well-known for specialists, however we provide an argument for the convenience of the reader.

Corollary 4.4. *Let R be a Dedekind ring of arithmetic type defined by the set of places S . Then $K_2(R)$ is finite.*

Proof. By Proposition 4.1 we get an exact sequence

$$0 \rightarrow K_2(R) \rightarrow K_2(F) \xrightarrow{\oplus \partial_v} \bigoplus_{v \notin S} \kappa_v^* \rightarrow 0.$$

Therefore we may consider $K_2(R)$ as a subgroup of $K_2(F)$, and restricting ∂_v to it we get an exact sequence

$$0 \rightarrow H_2 \rightarrow K_2(R) \xrightarrow{\oplus \partial_v} \bigoplus_{\substack{v \in S \\ \text{finite}}} \kappa_v^* \rightarrow 0,$$

where H_2 is the group from Proposition 4.2. However, S is a finite set, κ_v is a finite field for a finite place v , and H_2 is finite by Proposition 4.2. \square

We will denote by $I(k)$ the fundamental ideal of the Witt ring of symmetric bilinear forms $W(k)$ of a field k , see [MH]. Following [Sus, MoRe] we denote $K_2\mathrm{Sp}(R) = \varinjlim_l K_2(\mathbb{C}_l, R)$. We will need the following result due to Suslin [Sus, Theorem 6.5].

Lemma 4.5. *For any field k there is an exact sequence*

$$0 \rightarrow I^3(k) \rightarrow K_2\mathrm{Sp}(k) \rightarrow K_2(k) \rightarrow 0.$$

5. COMPARISON THEOREMS FOR K_2 : PROOF OF THEOREM B

5.1. Simply laced root systems. The aim of this section is to prove the following result, which, together with Theorem A (or, in fact, already with its special case, Theorem E), implies Theorem B. We believe it is very interesting in its own right, and may have further applications.

Theorem F. *Let R be a Dedekind ring of arithmetic type, and let $\Phi \neq A_1$ be a simply laced reduced irreducible root system (i.e., $\Phi = A_l, D_l, E_l, l \neq 1$). Assume additionally that*

$$\text{either } \mathrm{rk}(\Phi) \geq 3 \quad \text{or} \quad R \text{ has infinitely many units.}$$

Then $K_2(\Phi, R)$ is a central subgroup of $\mathrm{St}(\Phi, R)$, and, moreover,

$$K_2(\Phi, R) = K_2(R),$$

in particular, $K_2(\Phi, R)$ is finite.

Remark 5.1. For a root system Φ one can choose a lattice Λ between the root lattice $Q(\Phi)$ and the weight lattice $P(\Phi)$ and construct the Chevalley–Demazure group scheme $G = G_\Lambda(\Phi, -)$, see [VaPl, §7]. The corresponding K_2 -functor K_2^G modeled on the Chevalley group G is defined as the kernel of the map $\mathrm{St}(\Phi, -) \rightarrow G$. For $\Lambda = P(\Phi)$ we get $K_2^G = K_2(\Phi, -)$ and for a general Λ we have a left exact sequence of groups

$$1 \rightarrow M(R) \rightarrow G_{\mathrm{sc}}(\Phi, R) \rightarrow G$$

where M is either μ_n for some n or $\mu_2 \times \mu_2$ (see [SGA3, Exposé XXII, Remarque 4.1.8], [B, Tables I–IX]) which induces the following short exact sequence:

$$1 \rightarrow K_2(\Phi, R) \rightarrow K_2^G(R) \rightarrow M(R) \rightarrow 1$$

(see [LSV23, Proof of Theorem 1.1] for more details). In particular, finiteness of $K_2(\Phi, R)$ implies finiteness of $K_2^G(R)$ for all Λ .

Clearly, centrality of $K := K_2(\Phi, R)$ implies centrality of $K' := K_2^G(R)$ as well. Indeed, $[K', S] \subseteq K$ by centrality of $M(R)$, and since $S := \mathrm{St}(\Phi, R)$ is perfect,

$$[K', S] = [K', [S, S]] \subseteq [[K', S], S] \subseteq [K, S] = 1.$$

The above theorem was probably never published, but it may be mostly known to experts. In any case, for $\Phi = D_l$, $l \geq 5$, it follows from a result by Panin [Pan, Theorem 6.1]. Moreover, in the same paper Panin proves in [Pan, Theorem 9.1] a similar stabilisation result also for higher orthogonal K-theory.

However, to cover also $\Phi = D_4, E_6, E_7$ and E_8 we start with the following result.

Lemma 5.2. *Let R be a Dedekind ring of arithmetic type, and let Φ denote a simply laced reduced irreducible root system of $\text{rk}(\Phi) \geq 3$. Then there exists an embedding $A_3 \hookrightarrow \Phi$ such that the induced map*

$$K_2(A_3, R) \rightarrow K_2(\Phi, R)$$

is an isomorphism.

Proof. By Theorem 3.5 (using $A_3 = D_3$) we conclude that there exists an embedding $A_3 \hookrightarrow \Phi$ such that the induced map

$$K_2(A_3, R) \rightarrow K_2(\Phi, R)$$

is surjective. Let F be the field of fractions of R . By Theorem 3.2 we conclude that the induced map

$$K_2(A_3, F) \rightarrow K_2(\Phi, F)$$

is an isomorphism. Moreover, by Theorem 3.4 we have $K_2(A_3, R) = K_2(R)$, and therefore the natural map

$$K_2(A_3, R) \rightarrow K_2(A_3, F)$$

is injective by Theorem 4.1. Consider the following commutative diagram:

$$\begin{array}{ccc} K_2(A_3, R) & \hookrightarrow & K_2(A_3, F) \\ \downarrow & & \downarrow \cong \\ K_2(\Phi, R) & \longrightarrow & K_2(\Phi, F). \end{array}$$

The claim follows by a simple diagram chase. □

Now we are all set to finish the proof of Theorem F.

Proof of Theorem F. For $\text{rk}(\Phi) \geq 3$ consider the embedding $A_3 \hookrightarrow \Phi$ from Lemma 5.2 and use that $K_2(A_3, R) = K_2(R)$ by Theorem 3.4 to get the equality

$$K_2(\Phi, R) = K_2(R).$$

To prove centrality, use the surjectivity of the map $K_2(A_3, R) \rightarrow K_2(\Phi, R)$ for $\text{rk}(\Phi) \geq 4$ or the surjectivity of the map $K_2(A_2, R) \rightarrow K_2(A_3, R)$ from Theorem 3.4 together with Lemma 3.7. The finiteness of $K_2(R)$ follows from Corollary 4.4.

For $\Phi = A_2$ the claim follows from Theorem 3.6 (together with Lemma 3.7 and Corollary 4.4). □

Theorem B is a direct consequence of Theorem A and Theorem F.

5.2. Multiply laced root systems. As we see in the next section, for multiply laced root systems Φ the equality

$$K_2(\Phi, R) = K_2(R)$$

may hold for some Dedekind rings R .

However, the following counter-example shows that it certainly fails in the symplectic case, in general.

Lemma 5.3. *For $l \geq 3$ one has*

$$K_2(C_l, \mathbb{Z}) \neq K_2(\mathbb{Z}).$$

Proof. Recall that $K_2(\mathbb{Z}) = \mathbb{Z}/2$ (see, e.g., [M, Corollary 10.2]). However, the map $K_2(C_l, \mathbb{Z}) \rightarrow K_2(C_{l+1}, \mathbb{Z})$ is surjective by [St78, Corollary 3.2], and

$$\lim_l K_2(C_l, \mathbb{Z}) = K_2\mathrm{Sp}(\mathbb{Z}) = \mathbb{Z},$$

see, e.g., [Sch, Theorem 2.1]. □

Thus, there is no hope to prove the symplectic analogue of Theorem B along the same lines. This does not mean that bounded generation of $\mathrm{St}(C_l, R)$, $l \geq 3$, fails. But if it holds, its proof would require some completely different ideas.

6. K_2 FOR (LAURENT) POLYNOMIAL RINGS: PROOF OF THEOREM C

In this section we consider the polynomial rings $R = \mathbb{F}_q[X]$ and $R = \mathbb{F}_q[X, X^{-1}]$. Bounded generation of the Chevalley groups themselves in these cases is proven in [KPV]. On the other hand, for these rings $K_2(\Phi, R)$ is generated by the usual Steinberg symbols $\{u, v\}_\alpha$, $u, v \in R^*$, which allows one to explicitly calculate it. Observe that this is rarely the case for more general Dedekind domains, where one needs higher symbols.

Recall that $I(k)$ denotes the fundamental ideal of the Witt ring of symmetric bilinear forms of a field k (see [MH]). We will use the following well-known facts.

The first statement below is the [second] Steinberg theorem, it is proven, e.g., in [S] or in [M, Corollary 9.13]. For the second statement see, e.g., [MH, Chapter IV, Lemma 1.5].

Lemma 6.1. *Let \mathbb{F}_q be a finite field. Then*

- 1) $K_2(\mathbb{F}_q) = 0$;
- 2) $I^2(\mathbb{F}_q) = 0$.

The next result is an immediate corollary of a result by Rehmann [Reh].

Lemma 6.2. *Let $R = \mathbb{F}_q[X]$ be the polynomial ring over a finite field, Φ any reduced irreducible root system. Then*

$$K_2(\Phi, R) = K_2(R) = 0.$$

Proof. For any field k the natural embedding induces an isomorphism

$$K_2(\Phi, k) \cong K_2(\Phi, k[t])$$

by [Reh, Korollar zu Satz 1]. It remains to use that $K_2(\mathbb{F}_q) = 0 = I^3(\mathbb{F}_q)$ by Lemma 6.1, and apply Lemma 4.5. \square

The key role in the proof of Theorem C is played by the following observation of the second author and Sinchuk, see [LS20, Lemma 2.2], which in turn relies on deep results of Hurrelbrink, Abe and Morita.

Lemma 6.3. *For an arbitrary field k and a non-symplectic root system Φ there is an exact sequence of abelian groups*

$$0 \rightarrow K_2(\Phi, k) \rightarrow K_2(\Phi, k[X, X^{-1}]) \rightarrow k^* \rightarrow 0$$

split by the map

$$\{X, -\}_\alpha: k^* \rightarrow K_2(\Phi, k[X, X^{-1}])$$

for any fixed long root $\alpha \in \Phi$. In particular, the natural embedding induces an injective map

$$K_2(\Phi, k[X, X^{-1}]) \hookrightarrow K_2(\Phi, k(X)).$$

Proof. Since $K_2(\Phi, F) = K_2(F)$ for any field F by Theorem 3.2, the second statement follows from the first one.

Indeed, the map $K_2(k) \rightarrow K_2(k(X))$ is injective, e.g., by Milnor's theorem [W, Chapter III, Example 6.1.2, Theorem 7.4], and $k^* \rightarrow K_2(k(X))$ is injective as a splitting to the residue homomorphism ∂_X corresponding to an order of the zero or the pole at $X = 0$ (see Section 4, cf. also [LS20, Proof of Lemma 2.2]).

The first statement is proven for $\Phi \neq G_2$ in [Hur, Satz 3] (cf. [LS20, Lemma 2.2], Theorem 3.2). For $\Phi = G_2$ consider the following commutative diagram

$$\begin{array}{ccc} K_2(A_2, k[X, X^{-1}]) & \longrightarrow & K_2(G_2, k[X, X^{-1}]) \\ \downarrow & & \downarrow \\ K_2(A_2, k(X)) & \xrightarrow{\cong} & K_2(G_2, k(X)), \end{array}$$

where the horizontal arrows are induced by the natural embedding $A_2 \hookrightarrow G_2$ as a set of long roots.

Since $K_2(G_2, k[X, X^{-1}])$ is generated by $\{u, v\}_\alpha$ for $u, v \in k[X, X^{-1}]^*$, and $\alpha \in G_2$ a fixed long root by [AbMo, Corollary 6], we conclude that the top horizontal arrow is surjective. The bottom horizontal arrow is an isomorphism by Theorem 3.2. The injectivity of left horizontal arrow is already discussed above. Therefore, a simple diagram chase shows that the top horizontal arrow is in fact an isomorphism. \square

Modulo this results we can now summarise the Hurrelbrink, Abe–Morita, and Morita–Rehmann as follows.

Lemma 6.4. *Let $R = \mathbb{F}_q[X, X^{-1}]$ be the Laurent polynomial ring over a finite field, Φ any reduced irreducible root system. Then $K_2(\Phi, R)$ is a central subgroup of $\text{St}(\Phi, R)$, and, moreover,*

$$K_2(\Phi, R) = K_2(R) = \mathbb{F}_q^*.$$

Proof. For any field k there is an isomorphism

$$K_2(\Phi, k[X, X^{-1}]) \cong K_2(k) \oplus k^*$$

for $\Phi \neq C_l$ by Lemma 6.3 (and Theorem 3.2), and

$$K_2(C_l, k[X, X^{-1}]) \cong K_2\text{Sp}(k) \oplus P(k)$$

where $P(k) = k^* \oplus I^2(k)$ for $l \geq 1$ by [MoRe, Theorem B]. It remains to observe that $I^2(\mathbb{F}_q) = 0 = K_2(\mathbb{F}_q)$ by Lemma 6.1, and therefore (using Lemma 4.5) one has

$$K_2(\Phi, R) = K_2(R) = \mathbb{F}_q^*.$$

To prove the first statement, observe that $K_2(\Phi, k[X, X^{-1}])$ is generated by the Steinberg symbols $\{u, v\}_\alpha$ for $u, v \in k[X, X^{-1}]^*$ by [AbMo, Corollary 6], in particular, it is a central subgroup of $\text{St}(\Phi, k[X, X^{-1}])$ by Lemma 3.1. \square

Now Theorem C is a direct consequence of [KPV, Theorem A and Theorem C] and Lemmata 6.2, 6.4.

7. CONCLUDING REMARKS

Here we mention some eventual generalisations of the results of the present paper.

- Let $I \trianglelefteq R$ be an ideal of R . In the present paper we addressed the *absolute* case $I = R$ alone. However, it makes sense to ask similar questions for the *relative* case, in other words for the relative elementary subgroups $E(\Phi, R, I)$ of level $I \trianglelefteq R$. (Unlike the absolute case, $E(\Phi, R, I)$ does not necessarily coincide with the congruence subgroups $G(\Phi, R, I)$ of the same level.)

The expectation is that for $E(\Phi, R, I)$ one can get similar *uniform* bounds in terms of the elementary conjugates

$$x_{-\alpha}(r)x_\alpha(s)x_{-\alpha}(-r), \quad \alpha \in \Phi, \quad s \in I, \quad r \in R.$$

Otherwise, one could look at the *true = unrelativised* elementary subgroup $E(\Phi, I)$ of level I generated by $x_\alpha(s)$, $\alpha \in \Phi$, $s \in I$, and ask a similar question in terms of the elementary generators of level I .

Problem 1. *Establish analogues of Theorem A for the elementary groups $E(\Phi, I)$ and $E(\Phi, R, I)$ of level I , with uniform bounds not depending on either R or I .*

Some partial results in this direction for classical groups are obtained by Sergei Sinchuk and Andrei Smolensky [SiSm] and by Pavel Gvozdevsky [Gv23], but their bounds are not uniform. The most recent achievements are discussed in the last paragraph of [KPV2]. As a more remote goal one could think of generalisations to birelative subgroups, see [HSVZ] for this context.

- It is known that bounded elementary generation is closely related to many other flavours of bounded generation, including, in particular, finite width in commutators.

Namely, Alexei Stepanov [Ste] has discovered that there exists a *universal* bound $L = L(\Phi)$, depending on Φ alone, such that the commutators $[x, y]$, $x \in G(\Phi, R)$, $y \in E(\Phi, R)$ have elementary width $\leq L$ over an *arbitrary* commutative ring R . (Previously in [SiSt] and [StVa] similar results were proven for finite-dimensional rings, with the bound L depending on Φ and dimension $\dim(R)$).

Thus, for all Chevalley groups bounded elementary generation and bounded commutator width are equivalent! Morally, this says that there are very few commutators in $x \in G(\Phi, R)$, not much more than elementary generators.

But of course the actual bound for commutator width will be much smaller than the elementary width. So far, using the results of Smolensky [Smo] we were able to prove that for a Dedekind ring of arithmetic type R with the infinite multiplicative group R^* every element of $G_{\text{sc}}(\Phi, R)$ is a product of not more than 4,5,6 or 7 commutators, depending on the type Φ and on whether R is a number ring or a function ring, this result is contained in [KPV2].

- Above, Theorem F is stated only for *simply laced* root systems. But there is very strong evidence that suggests that the same is true for all *non-symplectic* root systems. We strongly believe in the following statement and are tempted to call it a conjecture.

Problem 2. *Let R be the Dedekind ring of arithmetic type with infinitely many units and Φ be a reduced irreducible non-symplectic root system (i.e., $\Phi \neq A_1, B_2, C_l$). Then $K_2(\Phi, R)$ is a finite central subgroup of $\text{St}(\Phi, R)$.*

In particular, $\text{St}(\Phi, R)$ is boundedly generated by the set $X = \{x_\alpha(r) \mid r \in R, \alpha \in \Phi\}$.

Remark 7.1. The centrality of $K_2(\Phi, R)$ in fact holds for any commutative ring R and any reduced irreducible root system Φ of rank at least 3. This result was first proven in [Ka77] for $\Phi = A_l$, and then in [Lav, Sin, LS17, Vor, LSV24] for the other root systems. However, if Φ has rank 2 then as shown in [Wen] centrality may fail even for some very nice rings.

As Lemma 5.3 shows, one cannot expect an analogue of this to hold in the symplectic case. However, one can still hope that $\text{St}(C_l, R)$, $l \geq 3$, is boundedly generated and could try to approach it by other means. We state the following problem.

Problem 3. *Let R be a Dedekind ring of arithmetic type with infinitely many units. Is $\mathrm{St}(C_l, R)$ boundedly elementarily generated?*

• Yet another aspect is that Theorem B is much weaker than Theorem A in that the bound depends on the size of $K_2(R)$. The natural question arises, whether there is a uniform bound in this case too? However, it seems that an answer to this question is presently out of range, and in any case should involve some hard core arithmetic.

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DEPT. OF MATHEMATICS, BAR-ILAN UNIVERSITY, RAMAT GAN, ISRAEL
Email address: kunyav@macs.biu.ac.il

MATHEMATISCHES INSTITUT, UNIVERSITÄT MÜNCHEN, MÜNCHEN, GERMANY
Email address: avlavrenov@gmail.com

DEPT. OF MATHEMATICS, BAR-ILAN UNIVERSITY, RAMAT GAN, ISRAEL
Email address: plotkin@macs.biu.ac.il

DEPT. OF MATHEMATICS AND COMPUTER SCIENCE, ST PETERSBURG STATE UNIVERSITY,
 ST PETERSBURG, RUSSIA
Email address: nikolai-vavilov@yandex.ru