ELEMENTARY EQUIVALENCE OF KAC-MOODY GROUPS

JUN MORITA AND EUGENE PLOTKIN

ABSTRACT. The paper is devoted to model-theoretic properties of Kac-Moody groups with the focus on elementary equivalence of Kac-Moody groups. We show that elementary equivalence of (untwisted) affine Kac-Moody groups implies coincidence of their generalized Cartan matrices and the elementary equivalence of their ground fields. We study also the Diophantine problem in affine Kac-Moody groups. We show that for the loop group the Diophantine problem is polynomial time equivalent (more precisely, Karp equivalent) to the problem in the ground ring. Finally, we show that in affine Kac-Moody groups over finite fields the Diophantine problem is undecidable.

1. INTRODUCTION. ELEMENTARY EQUIVALENCE

Importance of logical classifications of algebraic structures goes back to the famous works of A.Tarski and A.Malcev. Here the term "algebraic structure"or just "algebra"means a set with operations, see [25]. The main problem is to figure out *what are the algebras logically equivalent to a given one.* We discuss this problem from the perspectives of Kac-Moody groups.

Given an algebra H, its *elementary theory* $Th(H)$ is the set of all sentences (closed formulas) valid on H.

Definition 1. Two groups H_1 and H_2 are said to be *elementarily equivalent* if their elementary theories coincide.

Very often we fix a class of algebras $\mathcal{C}$ and ask what are the algebras elementarily equivalent to a given algebra inside the class $\mathcal{C}$.

Last years a lot of attention was concentrated around the elementary equivalence questions for linear groups. It is a kind of folklore that a group elementarily equivalent to a linear group is linear. However, the question what is the group elementarily equivalent to a given linear group, is far from being trivial

Throughout the paper we will denote the elementary equivalent objects A_1 and A_2 by $A_1 \equiv A_2$ and isomorphic objects are denoted by $A_1 \simeq A_2$.

Recall that the classical Keisler-Shelah's isomorphism theorem states, in particular, that two algebras A and B are elementary equivalent if and only if there is an ultrafilter such that the corresponding ultrapowers of A and B are isomorphic [11], [5].

Let G be a group from a class of groups $\mathcal{C}$. In many cases one can fix the cardinality of groups from $\mathcal{C}$. We call G *elementarily rigid*, if for any group H in $\mathcal{C}$ the equivalence $G \equiv H$ implies isomorphism $G \simeq H$.

So, the rigidity question with respect to elementary equivalence looks as follows.

Problem 2. Let a class of algebras $\mathcal{C}$ and an algebra $H \in \mathcal{C}$ be given. Suppose that the elementary theories of algebras H and $A \in \mathcal{C}$ coincide. Are they *elementarily rigid*, that is, are H and A isomorphic?

In general, elementary rigidity of groups is a rather rare phenomenon. However, studying the elementary properties of linear groups resulted in various examples of such kind. Historically, the first results were obtained in [26], [43]. Recently, a series of papers on elementary equivalence of linear groups with the emphasis on Chevalley groups and arithmetic lattices has been published, see [8], [2], [3], [41], [39]. Most of the results state a kind of elementary rigidity of these groups with respect to this or that class of groups $\mathcal{C}$. The major example is the class of finitely generated groups, cf. [39], [2].

In the paper we focus our attention on the similar problems of elementary equivalence for Kac-Moody groups and obtain a sort of rigidity results.

2. Affine Kac-Moody groups

Kac-Moody groups can be viewed as infinite dimensional analogs of Chevalley groups (see [40]). One can say that they are in the same relation with respect to infinite-dimensional Kac-Moody Lie algebras (see [17],[9], [30]) as Chevalley groups are related to semi-simple finite-dimensional Lie algebras. It is useful to emphasize that from the geometric point of view the passage from Chevalley groups to Kac-Moody groups means the passage from linear algebraic groups to ind-algebraic groups (see [38]). We refer to monographs [23], [27] for the detailed exposition of the above said.

Let $\widetilde{G}(K)$ be an (untwisted) affine Kac-Moody group over the field K. We want to prove that if two affine Kac-Moody groups $\widetilde{G}_1(K_1)$ and $\widetilde{G}_2(K_2)$ are elementary equivalent then their generalized Cartan matrices coincide and the fields K_1 and K_2 are elementary equivalent.

Let A be an $n \times n$ indecomposable generalized Cartan matrix of (untwisted) affine type. By an (untwisted) affine Kac-Moody group we mean the value of the simply connected Tits functor [42], cf.[36], [18],[19], corresponding to A. So, one can write $\widetilde{G}(K) = \widetilde{G}_{sc}(A, K)$. Let R be the ring of Laurent polynomials $R = K[t, t^{-1}]$, where K be a field. Denote by $G_{sc}(R) \simeq G_{\Phi}(R)$ the simply connected Chevalley group over R, where Φ is a finite root system, associated with A. The adjoint Chevalley group is denoted by $G_{ad}^{\Phi}(R)$. As usual, the elementary subgroups of Chevalley groups are denoted by $E_{sc}^{\Phi}(R)$ and $E_{ad}^{\Phi}(R)$.

Denote by $\widetilde{G}_{ad}(A, K)$ the adjoint Kac-Moody group ([31],[13],[42]). Its elementary subgroup is $\widetilde{E}_{ad}(A, K)$. Denote by $Z(\widetilde{G}_{sc}(K))$ and $Z(\widetilde{E}_{sc}(K))$ the centers of $\widetilde{G}_{sc}(K) = \widetilde{G}_{sc}(A, K)$ and $\widetilde{E}_{sc}(K) = \widetilde{E}_{sc}(A, K)$, respectively.

We have the following global picture:

$$
\begin{array}{ccccccccc}
& & & & 1 & & 1 & & \\
& & & & \uparrow & & \uparrow & & \\
& & 1 & \to & K^{\times} & \simeq & K^{\times} & \to & 1 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
1 & \to & Z(\tilde{G}_{sc}(K)) & \to & \tilde{G}_{sc}(K) & \to & H(R) & \to & 1 \\
& & \| & & \uparrow & & \uparrow & & \\
1 & \to & Z(\tilde{E}_{sc}(K)) & \to & \tilde{E}_{sc}(K) & \to & G_{ad}^{\Phi}(R) & \to & 1 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & 1 & & 1 & & 1 & &
\end{array}
$$

Here, by definition $H(R)$ is $\widetilde{G}_{ad}(A, K)$ and:

$$\tilde{G}_{sc}(K)/Z(\tilde{G}_{sc}(K)) \simeq H(R) := \left\langle G_{ad}^{\Phi}(R),\ \zeta(u) \mid u \in K^{\times} \right\rangle \subset \mathrm{Aut}_K(\mathfrak{g}(\dot{A}) \otimes R),$$

where $G_{ad}^{\Phi}(R) = G_{sc}^{\Phi}(R)/Z(G_{sc}^{\Phi}(R))$ is the corresponding adjoint Chevalley group, and $\zeta(u) \in \mathrm{Aut}(R)$ is defined by $t^{\pm 1} \mapsto u^{\pm 1} t^{\pm 1}$. Also here, $\mathfrak{g}(\dot{A}) \otimes R$ is the corresponding loop algebra, and $\dot{A}$ is a Cartan matrix associated with A.

Then, there is a surjective homomorphism $\pi : \widetilde{G}_{sc}(A, K) \to G_{\Phi}(R)$. Its kernel $Ker\pi$ is isomorphic to $K^{\times}$. We have, $Z(\widetilde{G}_{sc}(A, K)) = Z(\widetilde{E}_{sc}(A, K))$ and

$$1 \to Z(\widetilde{E}_{sc}(A, K)) \to \widetilde{E}_{sc}(A, K) \to G_{ad}^{\Phi}(R) \to 1.$$

Then,

$$\widetilde{E}_{sc}(A, K)/Z(\widetilde{E}_{sc}(A, K)) \simeq G_{ad}^{\Phi}(R) \simeq E_{ad}^{\Phi}(R).$$

The group $G_{ad}^{\Phi}(R) = E_{ad}^{\Phi}(R) = E_{ad}^{\Phi}(K[t, t^{-1}])$ is used to be called loop group [13]. The following commutator formulas are useful. Assume $|K| \geq 4$. Then

$$[\widetilde{G}_{sc}(A, K), \widetilde{G}_{sc}(A, K)] = \widetilde{E}_{sc}(A, K), \quad [\widetilde{E}_{sc}(A, K), \widetilde{E}_{sc}(A, K)] = \widetilde{E}_{sc}(A, K).$$

In the similar manner

$$[\widetilde{G}_{ad}(A, K), \widetilde{G}_{ad}(A, K)] = \widetilde{E}_{ad}(A, K), \quad [\widetilde{E}_{ad}(A, K), \widetilde{E}_{ad}(A, K)] = \widetilde{E}_{ad}(A, K),$$

and $\widetilde{E}_{ad}(A, K)$ is isomorphic to the loop group.

Given a group G a subgroup H is called definable if there is a first order formula ϕ such that $a \in H$, where $a \in G$ if and only if a satisfies a formula ϕ.

Theorem 3. *Let $\widetilde{E}_{1_{sc}}(A_1, K_1)$ and $\widetilde{E}_{2_{sc}}(A_2, K_2)$ be two elementary (simply connected) affine Kac-Moody groups whose ranks are bigger than 2. Suppose that $\widetilde{E}_{1_{sc}}(A_1, K_1) \equiv \widetilde{E}_{2_{sc}}(A_2, K_2)$. Then $A_1 = A_2$ and $K_1 \equiv K_2$.*

Proof. Suppose that $\widetilde{E}_{1_{sc}}(A_1, K_1) \equiv \widetilde{E}_{2_{sc}}(A_2, K_2)$. Since center is a definable subgroup the group $\widetilde{E}_{sc}(A, K)/Z(\widetilde{E}_{sc}(A, K))$ is interpretable in $\widetilde{E}_{sc}(A, K)$ (see, for example, [22], page 131). Hence we have

$$\widetilde{E}_{1_{sc}}(A_1, K_1)/Z(\widetilde{E}_{sc}(A_1, K_1)) \equiv \widetilde{E}_{2_{sc}}(A_2, K_2)/Z(\widetilde{E}_{sc}(A_2, K_2))$$

since both groups are interpretable in the elementary equivalent groups by the same first order formulas. That is $\widetilde{E}_{1_{ad}}(A_1, K_1) \equiv \widetilde{E}_{2_{ad}}(A_2, K_2)$ and according to the isomorphism with Chevalley groups $E_{1_{ad}}^{\Phi}(R_1) \equiv E_{2_{ad}}^{\Psi}(R_2)$. Recall that R_i, $i = 1, 2$ are the rings of Laurent polynomials $K_i[t, t^{-1}]$, respectively, while Φ and Ψ are the root systems associated with A_1 and A_2, respectively.

Hence,
$$E_{ad}^{\Phi}(K_1[t, t^{-1}]) \equiv E_{ad}^{\Psi}(K_2[t, t^{-1}]).$$
So, by Bunina (see [8], Theorem 11) we have $\Phi = \Psi$ and
$$K_1[t, t^{-1}] \equiv K_2[t, t^{-1}].$$

The Laurent polynomial ring $K[t, t^{-1}]$ is isomorphic to the group ring of the group $\mathbb{Z}$ over the field K. Then by Theorem 2 of [20],
$$K_1 \equiv K_2.$$
Hence, $\Phi = \Psi$, $K_1 \equiv K_2$ as required.

$\square$

The converse statement to Theorem 3 is not true.

Proposition 4. *There exist fields K_1 and K_2 such that $K_1 \equiv K_2$, but the affine Kac-Moody groups $\widetilde{E}_{1_{sc}}(A, K_1)$ and $\widetilde{E}_{2_{sc}}(A, K_2)$ are not elementary equivalent.*

Proof. Let $K_1 = \mathbb{Q}$ be the field of rational numbers and let K_2 be some its ultrapower. So these fields are elementary equivalent, i.e., $K_1 \equiv K_2$. However, the polynomial rings $K_1[t]$ and $K_2[t]$ are not elementary equivalent (see Corollary 3.8 in [16]). Then the rings of Laurent polynomials $K_1[t, t^{-1}]$ and $K_2[t, t^{-1}]$ are not elementary equivalent as well, (see Lemma 2, [20]). Then, $E_{\Phi}(K_1[t, t^{-1}])$ is not elementarily equivalent to $E_{\Phi}(K_2[t, t^{-1}])$ by [8]. Since $\widetilde{E}_{ad}(A, K_i) \simeq E_{ad}^{\Phi}(K_i[t, t^{-1}])$ the Kac-Moody groups $\widetilde{E}_{1_{sc}}(A, K_1)$ and $\widetilde{E}_{2_{sc}}(A, K_2)$ are not elementarily equivalent.

$\square$

Proposition 5. *Let K_1 and K_2 be algebraically closed fields of characteristic zero. Then the loop groups $\widetilde{E}_{1_{ad}}(A_1, K_1)$ and $\widetilde{E}_{2_{ad}}(A_2, K_2)$ of rank bigger than one are elementarily equivalent if and only if $A_1 = A_2$, $tr.deg_{\mathbb{Q}}(K_1)$ and $tr.deg_{\mathbb{Q}}(K_2)$ either both infinite or both finite and equal. Here $tr.deg_{\mathbb{Q}}(K)$ stands for the transcendence degree of the field K over $\mathbb{Q}$.*

Since algebraically closed fields are elementarily equivalent if and only if they have the same characteristic this means that for elementarily equivalent zero characteristic fields $\widetilde{E}_1(A_1, K_1) \equiv \widetilde{E}_2(A_2, K_2)$ if and only if $A_1 = A_2$, $tr.deg_{\mathbb{Q}}(K_1)$ and $tr.deg_{\mathbb{Q}}(K_2)$ either both infinite or both finite and $tr.deg_{\mathbb{Q}}(K_1) = tr.deg_{\mathbb{Q}}(K_2)$.

The proof of Proposition 5 follows from the proof of Theorem 3 and the following

Lemma 6. *Let K_1 and K_2 be algebraically closed fields of characteristic zero. Then $K_1[t, t^{-1}] \equiv K_2[t, t^{-1}]$ if and only if $tr.deg_{\mathbb{Q}}(K_1)$ and $tr.deg_{\mathbb{Q}}(K_2)$ either both infinite or both finite and $tr.deg_{\mathbb{Q}}(K_1) = tr.deg_{\mathbb{Q}}(K_2)$.*

Proof. Let $K_1[t, t^{-1}] \equiv K_2[t, t^{-1}]$. Then $K_1[t] \equiv K_2[t]$ (see Lemma 2, [20]). Then $tr.deg_{\mathbb{Q}}(K_1)$ and $tr.deg_{\mathbb{Q}}(K_2)$ either both infinite or both finite and $tr.deg_{\mathbb{Q}}(K_1) = tr.deg_{\mathbb{Q}}(K_2)$, (see Corollary 3.11, [16]). Note, that this direction is valid for arbitrary infinite fields.

Let $tr.deg_{\mathbb{Q}}(K_1)$ and $tr.deg_{\mathbb{Q}}(K_2)$ either both infinite or both finite and
$$tr.deg_{\mathbb{Q}}(K_1) = tr.deg_{\mathbb{Q}}(K_2).$$
Then Proposition 3.19 of [16] remains true for the rings of Laurent polynomials and implies $K_1[t, t^{-1}] \equiv K_2[t, t^{-1}]$.

$\square$

Question 7. Does Proposition 5 still valid if we replace $\widetilde{E}_{i_{ad}}(A_i, K_i)$ by $\widetilde{E}_{i_{sc}}(A_i, K_i)$, in it, $1 \leq i \leq 2$?

3. Diofantine Problem for Affine Kac-Moody groups

In this section we mostly follow the philosophy described in the paper [29]. The *Diophantine problem* (also called the *Hilbert's tenth problem* or the *generalized Hilbert's tenth problem*) in a countable algebraic structure $\mathcal{A}$, denoted $\mathcal{D}(\mathcal{A})$, asks whether there exists an algorithm that, given a finite system S of equations in finitely many variables and coefficients in $\mathcal{A}$, determines if S has a solution in $\mathcal{A}$ or not. In particular, if R is a countable ring then $\mathcal{D}(R)$ asks whether the question if a finite system of polynomial equations with coefficients in R has a solution in R is decidable or not.

By definition the Diophantine problem in a structure $\mathcal{A}$ *reduces to* the Diophantine problem in a structure $\mathcal{B}$, symbolically $\mathcal{D}(\mathcal{A}) \leqslant \mathcal{D}(\mathcal{B})$, if there is an algorithm that for a given finite system of equations S with coefficients in $\mathcal{A}$ constructs a system of equations S^* with coefficients in $\mathcal{B}$ such that S has a solution in $\mathcal{A}$ if and only if S^* has a solution in $\mathcal{B}$. If the reducing algorithm is polynomial-time then the reduction is termed polynomial-time (or Karp reduction). If our structures in question are uncountable one needs to restrict the Diophantine problems to equations with coefficients from a fixed countable subsets of $\mathcal{A}$ and $\mathcal{B}$, see [29] for details.

Definition 8. A subset (in particular a subgroup) H of a group G is Diophantine in G if it is definable in G by a formula of the type

$$\Phi(x) = \exists y_1 \ldots \exists y_n \left(\bigwedge_{i=1}^{k} w_i(x, y_1, \ldots, y_n) = 1 \right),$$

where $w_i(x, y_1, \ldots, y_n)$ is a group word on $x; y_1; \ldots; y_n$.

Such formulas are called *Diophantine* (in number theory) or *positive-primitive* (in model theory). Following [14], we say that

Definition 9. A structure $\mathcal{A}$ is *e-interpretable* (or *interpretable by equations*, or *Diophantine interpretable*) in a structure $\mathcal{B}$ if $\mathcal{A}$ is interpretable in $\mathcal{B}$ by Diophantine formulas.

The main point of this definition is that if $\mathcal{A}$ is e-interpretable in $\mathcal{B}$ then the Diophantine problem in $\mathcal{A}$ reduces in polynomial time (*Karp reduces*) to the Diophantine problem in $\mathcal{B}$. Hence, if the Diophantine problem in $\mathcal{B}$ is decidable, then the Diophantine problem in $\mathcal{A}$ should be decidable as well. This is a traditional way to prove that some Diophantine problem is undecidable.

Originally, Hilbert formulated the Diophantine problem for the ring of integers $\mathbb{Z}$. It was solved in the negative by Matiyasevich [28] building on the work of Davis, Putnam, and Robinson [12]. Subsequently, the Diophantine problem has been studied in a wide variety of commutative rings R, where it was shown to be undecidable by reducing $\mathcal{D}(\mathbb{Z})$ to $\mathcal{D}(R)$. For further information on the Diophantine problem in different rings and fields of number-theoretic flavour see [35], [34].

Let, as usual, $\widetilde{G}_{sc}(A, K)$ be an affine Kac-Moody group, and $G_{ad}^{\Phi}(R) \simeq E_{ad}^{\Phi}(R) = E_{ad}^{\Phi}(K[t, t^{-1}])$ be the corresponding loop group. This is a Chevalley group over the

ring of Laurent polynomials and our first goal is to study the Diophantine problem for this group.

We shall show that the ring $R = K[t, t^{-1}]$ is *e-interpretable* in the loop group $G_{ad}^{\Phi}(K[t, t^{-1}])$. Let X_α be a one-parametric subgroup generated by elementary unipotents $x_\alpha(t)$, where $\alpha \in \Phi, t \in R$. Then

Theorem 10. *Ring $R = K[t, t^{-1}]$ is e-interpretable in the group $G_{ad}^{\Phi}(K[t, t^{-1}])$, rank $\Phi \geq 2$, on every one-parametric subgroup X_α, $\alpha \in \Phi$.*

Proof. The proof follows from Proposition 3, Proposition 4 and Theorem 4 from [7]. $\square$

Corollary 11. *Field K is e-interpretable in the group $G_{ad}^{\Phi}(K[t, t^{-1}])$, rank $\Phi \geq 2$, on every one-parametric subgroup X_α, $\alpha \in \Phi$.*

Proof. By Lemma 5 in [20] (see also [21], Lemma 8) the field K is Diophantine in $K[t, t^{-1}]$. Then the result follows from Theorem 10 and the transitivity property of e-interpretability, see Proposition 2.4 of [15]. $\square$

Corollary 12. *Let $G = \widetilde{E}_{sc}(A, K)$ be an elementary affine Kac-Moody group. Let K be either a finite field $\mathbb{F}_q$ and rank of irreducible affine root system is ≥ 2 or K be a Dedekind ring of the arithmetic type and rank ≥ 3. Then K is e-interpretable in $G = \widetilde{E}_{sc}(A, K)$.*

Proof. It is known that for K specified in the conditions of the theorem the group $G = \widetilde{E}_{sc}(A, K)$ is finitely generated, see [4] and [37] for finite fields and [1] for Dedekind rings of arithmetic type. Let $a_1, \ldots, a_k$ be generators of G. Then the center $Z(G)$ of G is e-defined in G by the system of equations $[x, a_i] = 1$, where $i = 1, \ldots k$. So $Z(G)$ is a Diophantine normal subgroup in G. Then $G/Z(G) \simeq E_{ad}^{\Phi}(R) = E_{ad}^{\Phi}(K[t, t^{-1}])$ is e-interpretable in G by Lemma 2.7 from [15]. In view of Corollary 11 K is e-interpretable in $E_{ad}^{\Phi}(K[t, t^{-1}])$. Once again by transitivity property K is e-interpretable in $G = \widetilde{E}_{sc}(A, K)$.

$\square$

Corollary 13. *Let $G = \widetilde{G}_{sc}(A, K)$ be an affine Kac-Moody group. Let $K = \mathbb{F}_q$, $|K| \geq 4$, be a finite field and rank of irreducible affine root system is ≥ 2. Then K is e-interpretable in $G = \widetilde{G}_{sc}(A, K)$.*

Proof. Since $|K| \geq 4$, then

$$[\widetilde{E}_{sc}(A, K), \widetilde{E}_{sc}(A, K)] = \widetilde{E}_{sc}(A, K).$$

Moreover, $\widetilde{E}_{sc}(A, K)$ has finite commutator width, see Theorem D, [24]. Hence, $\widetilde{E}_{sc}(A, K)$ is Diophantine in $\widetilde{G}_{sc}(A, K)$, see Section 2.3, [15]. So, $\widetilde{E}_{sc}(A, K)$ is e-interpretable in $\widetilde{G}_{sc}(A, K)$. By Corollary 12 the field K is e-interpretable in $\widetilde{E}_{sc}(A, K)$. Hence K is e-interpretable in $\widetilde{G}_{sc}(A, K)$. $\square$

Note that in a similar way $K[t, t^{-1}]$ is also e-interpretable in $\widetilde{G}_{sc}(A, K)$.

In fact, the proof of Theorem 10 consists of two parts. Modification of the Double Centralizer Theorem from [39] (Theorem 1.6) and its more general variant Theorem 3 from [7] imply that all one-parametric subgroups are defined by Diophantine formulas, and hence are Diophantine sets. It remains to interpret ring operations

on R inside the one-parametric subgroups X_α. We show how to do that for some generic case in the way most appropriate for generalizations in Kac-Moody groups.

Assume that Φ contains a subsystem A_2. This case can be considered as a testing one for further generalizations. Let α and β be simple roots of A_2.

We e-interpret R on $X_{\alpha+\beta}$ turning it into a ring $\langle X_{\alpha+\beta}, \oplus, \otimes \rangle$ as follows.

For $x, y \in X_{\alpha+\beta}$ we define
$$x \oplus y = x \cdot y.$$
Note that if $y = x_{\alpha+\beta}(a)$, $y = x_{\alpha+\beta}(b)$, then $xy = x_{\alpha+\beta}(a+b)$, which corresponds to the addition in R. To define $x \otimes y$ for given $x, y \in X_{\alpha+\beta}$ we need some notation. Let $x_1, y_1 \in G$ be such that
$$x_1 \in X_\alpha \text{ and } [x_1, x_\beta(1)] = x; \quad y_1 \in X_\beta \text{ and } [x_\alpha(1), y_1] = y.$$
Note that such x_1, y_1 always exist and unique, namely if $x = x_{\alpha+\beta}(a)$, $y = x_{\alpha+\beta}(b)$, then $x_1 = x_\alpha(a)$, $y_1 = x_\beta(b)$. Define
$$x \otimes y := [x_1, y_1].$$
Observe, that in this case
$$[x_1, y_1] = [x_\alpha(a), x_\beta(b)] = x_{\alpha+\beta}(ab),$$
so it corresponds to the multiplication in R.

The map $a \mapsto x_{\alpha+\beta}(a)$ gives rise to a ring isomorphism $R \to \langle X_{\alpha+\beta}, \oplus, \otimes \rangle$. The ring $\langle X_{\alpha+\beta}, \oplus, \otimes \rangle$ is e-interpretable in G. Indeed, $X_{\alpha+\beta}$ is a Diophantine set, as it was proved above. The defined addition is clearly Diophantine in G. Since we are in A_2 case the subgroups X_α and X_β are Diophantine as well and the multiplication $\otimes$ is also Diophantine.

All remaining cases are more technical, but the proof of Definability for them is of the same flavor, see [7].

Conjecture 14. *Suppose that $\widetilde{G}(K) = \widetilde{G}_{sc}(A, K)$ is an arbitrary (not necessarily affine) Kac-Moody group over a ring K. Then, the ring K is e-interpretable in the group $\widetilde{G}(K)$.*

To prove Conjecture 14 one needs to establish a version of Double Centralizer Theorem (or some other way of description of real root subgroups) and to use more complicated than in Chevalley case commutator relations described in, for example, [10], [6].

On the other hand, since the group $G_{ad}^\Phi(K[t, t^{-1}])$ is defined by a system of polynomial equations with integer coefficients it is easy to see that this group is e-interpreted in the ring $K[t, t^{-1}]$, cf., Proposition 6 in [7]. Hence, for loop groups $G = G_{ad}^\Phi(K[t, t^{-1}])$, $rk\Phi \geq 2$ the Diophantine problem in any group G is Karp equivalent to the Diophantine problem in the ring R, cf., Theorem 6 in [7].

Proposition 15. *The Diophantine problem in the affine Kac-Moody group $G = \widetilde{G}_{sc}(A, \mathbb{F}_q)$, $rk \geq 2$ is undecidable.*

Proof. Let $\mathcal{D}(G)$ be decidable. The ring $\mathbb{F}_q[t, t^{-1}]$ is e-interpretable in $G = \widetilde{G}_{sc}(A, \mathbb{F}_q)$. So, if the Diophantine problem in $G = \widetilde{G}_{sc}(A, \mathbb{F}_q)$ would be decidable then the Diophantine problem in $\mathbb{F}_q[t, t^{-1}]$ should be decidable as well. But this is not the case, see [32], [33].

$\square$

4. Data availability

No datasets were generated or analyzed during the current study. The research has a purely theoretical character related to algebra and model theory.

5. Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] D. Allcock, *Presentation of affine Kac–Moody groups over rings* Algebra Number Theory, 10 (3) (2016) 533–556.

[2] N. Avni, A. Lubotzky, C. Meiri, *First order rigidity of non-uniform higher-rank arithmetic groups*, Invent. Math. (2019) 219–240.

[3] N. Avni, C. Meiri, *On the model theory of higher rank arithmetic groups*, arXiv 2008.01793v1 (2020) 59pp.

[4] P. Abramenko, Z. Gates, *Finite presentability of Kac-Moody groups over finite fields*, arXiv:1912.05611v1 [math.GR], 14pp.

[5] J. Bell, A. Slomson, Models and ultraproducts: an introduction, North-Holland Publishing, 1969.

[6] Y. Billig, A. Pianzola, *Root strings with two consecutive real roots*, Tohoku Math. J. (2) **47**, Number 3 (1995) 391–403.

[7] E. Bunina, A. Myasnikov, E. Plotkin, *The Diophantine problem in Chevalley groups*, arXiv http://arxiv.org/abs/2304.06259.

[8] E. I. Bunina, *Isomorphisms and elementary equivalence of Chevalley groups over commutative rings*, Sb. Math. bf 210 (2019) 1067–1091.

[9] R. Carter, Lie algebras of finite and affine type, Cambridge University Press, Cambridge. 2005.

[10] L. Carbone, M. Kownacki, S. Murray, S. Srinivasan, *Commutator relations and structure constants for rank 2 Kac–Moody algebras*, J.of Algebra, **566**, (2021) 443–476.

[11] C.C. Chang, H.J. Keisler, Model theory, North Holland, Amsterdam-New-York-Oxford, 1977.

[12] M. Davis, H. Putnam, J. Robinson *The Decision Problem for Exponential Diophantine Equations*. Annals of Mathematics, **74** (3), (1961) 425–436.

[13] H. Garland, *The arithmetic theory of loop groups*, Publications Mathématiques de l'IHÉS, Tome 52 (1980) 5–136.

[14] A. Garreta, A. Miasnikov, D. Ovchinnikov *Diophantine problems in commutative rings*, arXiv:2012.09787.

[15] A. Garreta, A. Miasnikov, D. Ovchinnikov *Diophantine problems in solvable groups*, Bulletin of Mathematical Sciences 10(1), (2020) 29pp.

[16] C. Jensen, H. Lenzing, Model Theoretic Algebra With Particular Emphasis on Fields, Rings, Modules, CRC Press,(1989) 464pp.

[17] V. Kac, Infinite dimensional Lie algebras, 3rd edition, Cambridge University Press (1990).

[18] V. Kac, D. Peterson, *Defining relations of certain infinite-dimensional groups*. In The mathematical heritage of Élie Cartan (Lyon, 1984) (1985) Astérisque (Numéro Hors Série), 165–208.

[19] V. Kac, D. Peterson, *On geometric invariant theory for infinitedimensional groups*. In Algebraic groups Utrecht 1986. Proceedings of a symposium in honor of T. A. Springer, A. M. C. et al., Ed., Lecture Notes in Mathematics, 1271. Springer, (1987) pp. 109–142.

[20] O. Kharlampovich, A. Myasnikov, *What does a group algebra "know "about a free group*, Annals of Pure and Applied Logic, **169** (2018) 523–547.
[21] O. Kharlampovich, A. Myasnikov, *Equations in Algebra*, IJAC, **28**, No. 08 (2018) 1517–1533.
[22] O. Kharlampovich, A. Myasnikov, M. Sohrabi, *Rich groups, weak second-order logic, and applications*, Groups and Model Theory, Gagta Book 2, (2021) Walter de Gruyter, 127–191.
[23] S. Kumar, Kac-Moody groups, their flag varieties and representation theory, Birkhauser, Boston, 2002.
[24] B. Kunyavskii, E. Plotkin, N. Vavilov, *Bounded generation and commutator width of Chevalley groups: function case*, arXiv:1304.5052 [math.GR], to appear EJM, 55pp.
[25] A. G. Kurosh, Lectures in general algebra. Pergamon Press, Oxford, London, Edinburgh, N. Y., Paris, Frankfurt, 1965.
[26] A. I. Mal'cev, *The elementary properties of linear groups*, Certain Problems in Mathematics and Mechanics, Sibirsk. Otdelenie Akad. Nauk SSSR, Novosibirsk, (1961), 110–132. English transl., Chapter XX in A.I. Mal'tsev, The meta-mathematics of algebraic systems, Collected papers: 1936–1967, North-Holland, Amsterdam, (1971).
[27] T. Marquis, An Introduction to Kac–Moody Groups over Fields, EMS Textbooks in Mathematics, 2017.
[28] Y. Matijasevich *The Diophantineness of enumerable sets*. Doklady Akademii Nauk USSR, **191**, (1970) 279–282.
[29] A. Myasnikov, M. Sohrabi *The Diophantine problem in the classical matrix groups*. Izv. RAN. Ser. Mat., **85**(6), (2021) 205–244.
[30] R. Moody, A. Pianzola, Lie algebras with triangular decompositions, Canad. Math. Soc. Ser. Monographs Adv. Texts, Wiley-Interscience, New York, 1995.
[31] R. Moody, K. Teo, *Tits systems with crystallographic Weyl groups*, J. Algebra, **21** (1972) 178–190.
[32] P. Pappas, *A Diophantine problem for Laurent polynomial rings*, Proceedings of the AMS, Volume 93, Number 4. (1985) 713–718.
[33] T. Pheidas, *Diophantine undecidability for addition and divisibility in polynomial rings*, Fundamenta Mathematicae, 182 (2004) 205–220.
[34] T. Pheidas, K. Zahidi *Undecidability of existential theories of rings and fields: A survey.* Contemporary Mathematics, **270**, (2000) 49–106.
[35] B. Poonen *Hilbert's Tenth Problem over rings of number-theoretic interest.* Notes for Arizona Winter School on "Number theory and logic", 2003.
[36] D. Peterson, V. Kac, *Infinite flag varieties and conjugacy theorems*, Proc. Nat. Acad. Sci. USA 80, (1983) 1779–1782.
[37] B.Remy, Buildings and Kac-Moody Groups. In: Sastry, N. (eds) Buildings, Finite Geometries and Groups. Springer Proceedings in Mathematics, vol 10. Springer, New York, NY.
[38] I. Shafarevich, *On some infinite-dimensional groups*, II Math USSR Izvestija 18 (1982), pp. 185–194.
[39] D. Segal, K. Tent, *Defining R and G(R)*, arXiv:2004.13407, (2020) 30 pp.
[40] R. Steinberg. Lectures on Chevalley groups. Yale University, 1967.
[41] M. Sohrabi, A. Myasnikov, *Bi-interpretability with Z and models of the complete elementary theories of $SL_n(\mathcal{O})$, $T_n(\mathcal{O})$ and $GL_n(\mathcal{O})$, $n \geqslant 3$*, arXiv:2004.03585, (2020) 19pp.
[42] J. Tits, *Uniqueness and Presentation of Kac-Moody Groups over Fields*, Journal of Algebra, **105**, (1987) 542–573.
[43] B. Zilber, *Some model theory of simple algebraic groups over algebraically closed fields*, Colloq. Math., **48:2**, (1984) 173–180.

Jun Morita, Division of Mathematics, Faculty of Pure and Applied Sciences, University of Tsukuba, Tsukuba, Ibaraki 305-8571, JAPAN
  Email address: `morita@math.tsukuba.ac.jp`

Eugene Plotkin, Department of Mathematics, Bar-Ilan University, 5290002 Ramat Gan, ISRAEL
  Email address: `plotkin@math.biu.ac.il`