THE GROUP $\mathrm{SL}_2(\mathbb{C})$: WORD MAPS AND RELATED TOPICS

NIKOLAI GORDEEV AND EUGENE PLOTKIN

Introduction

Let G be a group, and let $F_n = \langle x_1, \dots, x_n \rangle$ be the free group of rank n. For every word $w \in F_n$, the *word map* is defined as $\widetilde{w} : G^n \to G$, where $\widetilde{w}(g_1, \dots, g_n) = w(g_1, \dots, g_n)$.

In group theory, many problems are associated with word maps. For instance, the well-known Burnside problem asks about the finiteness of the quotient group $F_k/\langle\langle \mathrm{Im}\, w \rangle\rangle$, where $n = 1$, $F_n = \langle x \rangle$, $G = F_k$, $k > 1$, $w = x^m$ for some $m \in \mathbb{N}$, and $\langle\langle \mathrm{Im}\, w \rangle\rangle$ is the normal subgroup generated by the image of the word map $w : F_k \to F_k$. Another example is the Ore problem: is $G \stackrel{?}{=} \mathrm{Im}\, w$, where $n = 2$, $w = [x, y]$, and G is a finite simple group.

The surjectivity of word maps. The question $G \stackrel{?}{=} \mathrm{Im}\, w$ has been studied for various types of words and groups in recent years. The particular interest here is the case when G is a simple algebraic group. One of the first results in this direction is A. Borel's theorem ([Bo1]) which states: for any non-trivial word $w \in F_n$ and any semisimple algebraic group G the word map $\widetilde{w} : G^n \to G$ is dominant. This means that the image $\mathrm{Im}\, \widetilde{w}$ of the word map $\widetilde{w}$ contains a non-empty open subset of G, that is, it contains "almost all" elements of G. However, in the same paper Borel presented the simplest counterexample to the surjectivity of word maps. Namely, if $n = 1$, $w = x^2$ and $G = \mathrm{SL}_2(\mathbb{C})$, then $-u \notin \mathrm{Im}\, \widetilde{w}$ where $u \in \mathrm{SL}_2(\mathbb{C})$ is a non-trivial unipotent matrix.

Certainly we can avoid such kind of counter-examples on $\mathrm{SL}_2(\mathbb{C})$ if we consider the group $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/Z(\mathrm{SL}_2(\mathbb{C}))$. It has been shown (see [BZ]) that every non-central semisimple element of $\mathrm{SL}_2(\mathbb{C})$ is contained in $\mathrm{Im}\, \widetilde{w}$ for any $w \neq e$. This implies that every element of $\mathrm{PGL}_2(\mathbb{C})$, except possibly non-trivial unipotent elements, is in $\mathrm{Im}\, \widetilde{w}$. We have examples of surjective word maps $\widetilde{w} : \mathrm{PGL}_2(\mathbb{C})^n \to \mathrm{PGL}_2(\mathbb{C})$ (see [BZ], [GKP1]-[GKP4], [GG], [JS]). In particular, there exist series of such words, satisfying $w_k \in F_n^{(k)} \setminus F_n^{(k+1)}$, where $\{F_n^{(k)}\}_{k=0}^{\infty}$ is the derived series of F_n.

It remains unproven that $\mathrm{Im}\, \widetilde{w} = \mathrm{PGL}_2(\mathbb{C})$ for every $w \neq e$, even for $n = 2$. Specifically, it is not established whether substituting 2×2 complex matrices for x, y in the word $x^{l_1} y^{m_1} \cdots x^{l_r} y^{m_r}$ can yield a unipotent matrix. This unresolved question is sometimes referred to as *The Shame Problem.*

So, the major problem which stands behind all considerations of the current paper is the following old open question:

Problem 0.1. *Is it true that a word map* $\widetilde{w} : \mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$*, where* $w = w(x, y) \neq 1$*, is surjective? In other words is it true that every equation* $w(x, y) = a$*, where* a *is a matrix from* $PGL_2(\mathbb{C})$*, has a solution?*

Unfortunately, nowadays after many years of intensive research and numerous attacks we are forced to state the absence of the decisive approach to this problem. Despite the apparent ease of formulation, the problem is deep, difficult and toxic. We need to feel better how the geometry of verbal varieties behave in various cases. Hopefully this will give rise to still unearthed totally new ideas. This paper deals with two methods of geometrical nature which look promising. Generally speaking, our goal is to study algebraic properties of groups related to specific word varieties.

The situation with the problem $\operatorname{Im} w \stackrel{?}{=} G$ for the general case of simple algebraic groups G is much more complicated. Unlike the case of $\mathrm{SL}_2(\mathbb{C})$, no general results exist for semisimple elements (not even for $\mathrm{SL}_3(\mathbb{C})$). Counterexamples to surjectivity for word maps are currently limited to cases involving powers w^l of words (see [GKP3]). Furthermore, we have only a few types of word maps for that we may guarantee the surjectivity of word maps on simple algebraic groups (see [GKP3], [G]). Meanwhile, the general conjecture is as follows.

Problem 0.2. *Is it true that a word map* $\widetilde{w} : \mathrm{PGL}_n(\mathbb{C}) \times \mathrm{PGL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$*, where* $w = w(x, y) \neq 1$*, is surjective? In other words is it true that every equation* $w(x, y) = a$*, where* a *is an element of* $\mathrm{PGL}_n(\mathbb{C})$*, has a solution?*

We cannot expect surjectivity of word maps $\widetilde{w} : G^n \to G$ for every simple algebraic group G of adjoint type. Say $\widetilde{w} : \mathrm{PSp}_n(\mathbb{C}) \to \mathrm{PSp}_n(\mathbb{C})$ where $w = x^2$ is not surjective [GKP3]. Possibly we have to exclude words $w = \omega^k, \omega \in F_n, k > 1$.

A gloomy picture with word maps on algebraic groups prompts us to find new approaches which use more deep connections between the group and the topological nature of algebraic groups. Any additional information can be an essential help. In particular, methods of AI can be useful in this concern. For instance, solutions of the equation $w(x, y) = 1$ for various types of w, or at least dimensions of the irreducible components of the variety $w(x, y) = 1$ can deliver a yet missing hint towards the solution of Problems 0.1 and 0.2.

The variety of representations. The problem of the surjectivity of word maps on the group $G = \mathrm{SL}_2(\mathbb{C})$ is closely tied to the structure of *the variety of representations* of finitely generated groups with one relation (see [GKP1]).

For a group G and a word map $\widetilde{w} : G^n \to G$ let
$$\mathcal{W}_w := \widetilde{w}^{-1}(e)$$
(here e is the identity of G). Let $\Gamma_w := F_n/\langle\langle w \rangle\rangle$ be the group of n-generators $\langle \bar{x}_1, \ldots, \bar{x}_n \rangle$ and one relation w. Here $\langle\langle w \rangle\rangle := \langle f w f^{-1} \mid f \in F_n \rangle$ is the normal subgroup of F_n generated by the conjugates of w, and the generators of Γ_w are $\bar{x}_i \equiv x_i (\operatorname{mod} \langle\langle w \rangle\rangle)$. Element $\mathfrak{g} = (g_1, \ldots, g_n) \in \mathcal{W}_w$ corresponds to the homomorphism $\rho_{\mathfrak{g}} \in \operatorname{Hom}(\Gamma_w, G)$ where $\rho_{\mathfrak{g}}(\bar{x}_i) = g_i$. On the other hand, every homomorphism $\rho \in \operatorname{Hom}(\Gamma_w, G)$ corresponds to the element $\mathfrak{g}_\rho = (\rho(\bar{x}_1), \rho(\bar{x}_2), \ldots, \rho(\bar{x}_n)) \in \mathcal{W}_w$. Thus we have one-to-one correspondence between the elements of the algebraic set $\mathcal{W}_w$ in G^n and the elements of the set of homomorphisms $\operatorname{Hom}(\Gamma_w, G)$.

If G is a simple algebraic group, then the set $\mathcal{W}_w$ is a Zariski closed subset of G^n. In general, it is a reducible set. The set $\mathcal{W}_w$ is called *the variety of representations of the group* Γ_w, see [LM], [PRR]. In the case when $G = \mathrm{SL}_2(\mathbb{C})$ we have $\dim \mathcal{W}_w \leq 5$ and if there is an irreducible component $\mathcal{W}_w^i$ of $\mathcal{W}_w^i$ such that $\dim \mathcal{W}_w^i < 5$ then the image $\operatorname{Im} \widetilde{w}$ contains a non-trivial unipotent element of $G = \mathrm{SL}_2(\mathbb{C})$ (and therefore the induced map $\widetilde{w} : \mathrm{PGL}_2(\mathbb{C})^n \to \mathrm{PGL}(\mathbb{C})$ is surjective; see [GKP1]). However, the inverse assertion:

$$\operatorname{Im} \widetilde{w} \ \textit{contains a non-trivial unipotent element} \Rightarrow \dim \mathcal{W}_w^i < 5 \ \ \textit{for some } i,$$

is an open problem.

The paper is organized as follows. Its first part is devoted to the variety of representations. We define a word equivalence $\asymp_G$ related to w and G, which, in its turn, naturally implies the order relation on the set of words. Finally, all these notations are used in the definition of the radical $\sqrt[G]{w}$ of the word w with respect to the group G.

It is our pleasure to note that this geometric insight resembles general ideas of universal geometry introduced by B.Plotkin. He considers a kind of Galois correspondence between sets of equations over free algebras in some variety of algebras $\mathcal{A}$ and their solutions in the Affine space $G^n \equiv Hom(F_n, G)$, F_n is a free in $\mathcal{A}$ algebra, $G \in \mathcal{A}$. Given a system of words T one can consider its solution $A = T' \in G^n$ and the set $Rad_G(T)$ of all words having the same solution A. Two algebras G_1 and G_2 are called geometrically equivalent if for every system T we have $Rad_{G_1}(T) = Rad_{G_2}(T)$ [Pl1], [Pl2]. It is a rare case when one can explicitly describe radicals $Rad_G(T)$. In these cases we can speak about explicit *Nullstellensatz.*

It would be a great advancement to get a classification of equivalence classes of words with respect to equivalence $\asymp_G$ and semisimple algebraic groups G over complex numbers or, at least, with respect to $G = PGL_2(\mathbb{C})$. Even more ambitious question is to obtain a kind of *Nullstellensatz* in this case.

Section 2 is focused on simple algebraic groups and, first of all, on the irreducible components of the variety of representations. Let, for simplicity, $G = SL_2(\mathbb{C})$. Let $w \in F_n$ be a word and let $\mathcal{W}_w$ be the corresponding variety of representations.This variety splits into a finite number of the irreducible components $\mathcal{W}_w^i$. To each component $\mathcal{W}_w^i$ one can associate its radical $\sqrt[G]{w}^{\,i} \leq F_n$ such that the quotient group $\Delta_w^i := F_n / \sqrt[G]{w}^{\,i}$ is isomorphic to the group $\langle g_1, \dots, g_n \rangle$ for "almost all points" $\mathfrak{g} = (g_1, \dots, g_n) \in \mathcal{W}_w^i$ and for every point $\mathfrak{g} = \langle g_1, \dots, g_n \rangle \in \mathcal{W}_w^i$ there exists the epimorphism $\Delta_w^i \to \langle g_1, \dots, g_n \rangle$. The group Δ_w^i is called *the group of general position of the component* $\mathcal{W}_w^i$ *or* $\mathcal{W}_w^{i\star}$*-group.* In some sense the groups Δ_w^i are responsible for groups having faithful representation in G, see Proposition 2.1 for the precise meaning. The rest of the Section deals with the example $G = \mathrm{SL}_2(\mathbb{C}), w = [x, [x^2, yxy^{-1}]] \in F_2(x, y)$. The results are accumulated in remarks and the diagram before Section 3 and reveal a quite complicated picture.

The last Section considers the similar problems from the positions of matrix calculation.

Notations and Terminology. Here:

- $\mathbb{N}, \mathbb{Z}, \mathbb{C}$ represent the set of natural numbers, the ring (group) of integers, and the field of complex numbers, respectively.
- $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ is the cyclic group of order m.
- $\mathbb{Z}^m = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (m times); $\mathbb{Z}^\infty = \oplus_{i=1}^\infty A_i$, where $A_i \approx \mathbb{Z}$ for every $i \in \mathbb{N}$.

For any groups Γ and H:

- $e \in \Gamma$ denotes the identity element of the group Γ.
- $\operatorname{ord} g$ represents the order of the element $g \in \Gamma$.
- $Z(\Gamma)$ is the center of the group Γ.
- $\Delta = H \cdot \Gamma$ denotes a semidirect product, where $H \lhd \Delta$ and $\Delta = (h, \gamma) \mid h \in H,\ \gamma \in \Gamma$.
- $H * \Gamma$ is the free product of the groups H and Γ.

If G is a simple algebraic group, then B, T, U denote:

- B: a fixed Borel subgroup.
- T: a maximal torus, where $T \leq B$.
- U: the maximal unipotent subgroup, where $U \leq B$ and $B = TU$.

For the group $\mathrm{SL}_2(\mathbb{C})$, we denote:

- B: the subgroup of upper triangular matrices.
- T: the subgroup of diagonal matrices.
- U: the group of upper triangular matrices with eigenvalues 1.

Given field K denote:

- $\mathrm{M}_m(K)$: the set of all $m \times m$ matrices with the entries $m_{i,j} \in K$.

Let K be a field and let X be an algebraic variety over K.
- $\overline{Y}$: closure (with respect to Zariski topology) of a subset $Y \subset X$.
- $\dim Y$: the dimension of a closed subset $Y \subset X$.

If K has a big enough transcendence degree over the prime field $F \leq K$ (in particular, $\operatorname{tr}\deg_F K = \infty$), then the set

$$X \setminus \Big(\bigcup_{i=1}^{\infty} X_i\Big),$$

where $X_i \subsetneqq X$ is a proper closed subset for every i, is a dense non-empty subset of X(see [Bo1]). We call such sets *countably open* or *c-open*.

1. G-equivalence and G-orders on words

1.1. G-equivalence on words.

Definition 1.1. *We say that the words $w_1, w_2 \in F_n$ are G-equivalent if $\mathcal{W}_{w_1} = \mathcal{W}_{w_2}$. If $w_1, w_2 \in F_n$ are equivalent words then we will write*

$$w_1 \asymp_G w_2.$$

The simple criterium of the equivalence of words is the following

Proposition 1.2. *Let G be a group, $w \in F_n$, and let $\varsigma : F_n \to F_n$ be an automorphism of F_n that stabilizes the normal subgroup $\langle\langle w \rangle\rangle \lhd F_n$. Then*

$$w \asymp_G \varsigma\big(w^{\pm 1}\big).$$

Proof. Since $\varsigma\big(\langle\langle w \rangle\rangle\big) = \langle\langle w \rangle\rangle$ we may consider the map ς as an automorphism of the group $\Gamma_w = F_n/\langle\langle w \rangle\rangle$. If $\rho : \Gamma_w \to G$ is a homomorphism then

$$\varsigma(\rho) : \Gamma_w \to G \ \text{ where } \ \varsigma(\rho)(\gamma) := \rho\big(\varsigma(\gamma)\big) \ \text{ for every } \gamma \in \Gamma_w$$

is also a homomorphism. Hence

$$\mathrm{Hom}(\Gamma_w, G) = \varsigma\big(\mathrm{Hom}(\Gamma_w, G)\big) = \mathrm{Hom}(\Gamma_{\varsigma(w)}, G) \Rightarrow \mathcal{W}_w = \mathcal{W}_{\varsigma(w)} \Rightarrow w \asymp_G \varsigma(w).$$

The equivalence $w \asymp_G w^{\pm 1}$ is obvious. □

Remark 1.3. *The condition $\varsigma\big(\langle\langle w\rangle\rangle\big) = \langle\langle w\rangle\rangle$ is essential. Indeed, let $w = xy \in F_2$ and G be any group G which has an element g of order 3. Let $\varsigma : F_2 \to F_2$ be the automorphism such that $\varsigma(x) = x,\ \varsigma(y) = xy$. Then $\varsigma(w) = x^2y$ and $(g, g) \in \mathcal{W}_{\varsigma(w)},\ (g, g) \notin \mathcal{W}_w$.*

Example 1.4. *Let $G = \mathbb{Z}/2\mathbb{Z}$, $w_1 = x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k},\ \ w_2 = x_{j_1}^{b_1} x_{j_2}^{b_2} \cdots x_{j_l}^{b_l}$. Then*

$$w_1 \asymp_G w_2 \Leftrightarrow \mathcal{W}_{w_1} = \mathcal{W}_{w_1} \Leftrightarrow a_i \equiv b_i (\mathrm{mod}\, 2) \ \text{ for every } \ i = 1, \ldots, n.$$

Example 1.5. *Let $G = \mathrm{SU}_2(\mathbb{C})$ and let $||\ ||$ be the norm on G. Then for every $0 < \epsilon \in R$ there exists a word $\omega \in F_2 = \langle x, y\rangle$ such that*

$$||E_2 - \widetilde{\omega}(g_1, g_2)|| < \frac{1}{2} \ \text{ for every } \ (g_1, g_2) \in G^2 \tag{1.1}$$

(see [T]*). Then*

$$[x, [x, \omega]] \asymp_G [x, \omega].$$

Proof.

Lemma 1.6. *For every $\alpha \in \mathbb{C},\ |\alpha| = 1$, we have*

$$\dot{w}_\alpha := \begin{pmatrix} 0 & \alpha \\ -\bar{\alpha} & 0 \end{pmatrix} \notin \mathrm{Im}\ \widetilde{\omega}, \ \text{ for every } \ \alpha \in \mathbb{C},\ |\alpha| = 1.$$

Proof. Indeed,

$$||E_2 - \dot{w}_\alpha|| = \left\| \begin{pmatrix} 1 & -\alpha \\ \bar{\alpha} & 1 \end{pmatrix} \right\| = \frac{1}{\sqrt{2}} \sqrt{1^2 + 1^2 + |-\alpha|^2 + |\bar{\alpha}|^2} = \sqrt{2} > \frac{1}{2}.$$

□

Suppose $[x, [x, \omega]] \not\asymp_G [x, \omega]$. Then there exists a pair $(\sigma, \tau) \in G^2$ such that

$$[\sigma, \widetilde{\omega}(\sigma, \tau)] \neq E_2, \ \ [\sigma, [\sigma, \widetilde{\omega}(\sigma, \tau)]] = E_2. \tag{1.2}$$

Note, that σ in (1.2) is a non-central element. Thus we may assume $\sigma = \begin{pmatrix} s & 0 \\ 0 & \bar{s} \end{pmatrix}$ where $|s| = 1, s \neq \pm 1$. Then condition (1.2) implies

$$[\sigma, \widetilde{\omega}(\sigma, \tau)] = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \ \text{ where } \ |z| = 1, z \neq \pm 1. \tag{1.3}$$

(Indeed, $[\sigma, [\sigma, \widetilde{\omega}(\sigma, \tau)]] = E_2$ and therefore $[\sigma, \widetilde{\omega}(\sigma, \tau)] \neq E_2$ is a diagonal matrix. Suppose $[\sigma, \widetilde{\omega}(\sigma, \tau)] = -E_2$. Then $||E_2 - [\sigma, \widetilde{\omega}(\sigma, \tau)]|| = ||E_2 - (-E_2)|| = 2||E_2|| = 2$. But

$$||E_2 - \widetilde{\omega}(\sigma, \tau)|| < \frac{1}{2} \Rightarrow ||E_2 - [\sigma, \widetilde{\omega}(\sigma, \tau)]|| \leq 2||E_2 - \widetilde{\omega}(\sigma, \tau)|| < 1.$$

That is a contradiction.)

Now let

$$\omega(\sigma, \tau) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{SU}_2(\mathbb{C}).$$

Then

$$[\sigma,\omega(\sigma,\tau)] = \begin{pmatrix} s & 0 \\ 0 & \bar{s} \end{pmatrix} \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} \bar{s} & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix} =$$

$$\begin{pmatrix} a & s^2 b \\ -\bar{b}\bar{s}^2 & \bar{a} \end{pmatrix} \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 s^2 & ab(s^2-1) \\ \bar{a}\bar{b}(1-\bar{s}^2) & |a|^2+|b|^2\bar{s}^2 \end{pmatrix} \overset{(1.2)}{=} \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} a = 0 \Rightarrow \omega(\sigma,\tau) = \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} \text{ (this case is impossible by Lemma 1.6),} \\ \text{or} \\ b = 0 \Rightarrow \omega(\sigma,\tau) = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \Rightarrow [\sigma,\omega(\sigma,\tau)] = E_2 \quad \big(\text{a contradiction with } (1.3)\big). \end{cases}$$

Thus we have proved $[x,[x,\omega]] \asymp_G [x,\omega]$.

□

Example 1.7. *Let* $G = \mathrm{SL}_2(\mathbb{C})$, $\omega \in F_2 = \langle x, y\rangle$. *Then*

$$[x,[x^2,\omega x\omega^{-1}]] \asymp_G [x^2,\omega x\omega^{-1}].$$

Proof. Suppose $[x,[x^2,\omega x\omega^{-1}]] \not\asymp_G [x^2,\omega x\omega^{-1}]$. Then there exists a pair $(\sigma,\tau) \in G^2$ such that

$$\zeta := [\sigma^2, \omega\sigma\omega^{-1}] \neq E_2, \ \ [\sigma,\zeta] = E_2. \tag{1.4}$$

Case I. σ is a semisimple element. We may assume $\sigma = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$ where $s \neq \pm 1, \neq \pm i$. Then condition (1.4) implies

$$[\sigma^2, \underbrace{\omega\sigma\omega^{-1}}_{=:\mu\notin T}] = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \text{ where } z \neq 1 \Rightarrow \mu = \begin{pmatrix} 0 & r \\ -r^{-1} & 0 \end{pmatrix}. \tag{1.5}$$

However,

$$\mathrm{tr}\begin{pmatrix} 0 & r \\ -r^{-1} & 0 \end{pmatrix} = 0 = \mathrm{tr}\left(\omega \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \omega^{-1}\right) \Rightarrow s = \pm i$$

that contradicts to our assumption.

Case II. $\pm\sigma$ is a unipotent element. We may assume $\sigma = \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in U$. Then

$$\zeta = [\underbrace{\sigma^2}_{\in U}, \underbrace{\omega\sigma\omega^{-1}}_{:=\mu}] = \pm\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \text{ where } z \neq 0 \Rightarrow \mu \in B.$$

$\big($Indeed, if $\mu \notin B$ then $\mu \in B\dot{w}B$ and $\mu = u_1\dot{w}tu_2$ where $u_1, u_2 \in U$ and $t \in T$. Hence

$$\mu\sigma^{-2}\mu^{-1} = (u_1\dot{w}tu_2)\overbrace{\sigma^{-2}}^{=u\in U, u\neq 1}(u_2^{-1}t^{-1}\dot{w}^{-1}u_1^{-1}) = u_1\underbrace{(\dot{w}\overbrace{tu_2\sigma^{-2}u_2^{-1}t^{-1}}^{=u'\in U, u'\neq 1}\dot{w}^{-1})}_{=\pm\begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}, \delta\neq 0}u_1^{-1} \notin B.$$

It is a contradiction with the equality $\zeta = [\sigma^2, \mu] = \sigma^2(\mu\sigma^{-2}\mu^{-1}) = \pm\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}$ where $z \neq 0$.$\big)$ Hence

$$B \ni \mu = \omega \underbrace{\sigma}_{=\pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}} \omega^{-1} = \pm\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix} \tag{1.6}$$

Thus we have a contradiction between (1.6) and (1.4).

□

1.2. G-order on the words.

Definition 1.8. *We say that a word $w_1 \in F_n$ is G-deeper than the word $w_2 \in F_n$ if $\mathcal{W}_{w_1} \supset \mathcal{W}_{w_2}$. In this case we will write*

$$w_1 \succcurlyeq_G w_2.$$

If $w_1 \asymp_G w_2$, then it holds that $w_1 \succcurlyeq_G w_2$.

Proposition 1.9. *Let $w, \omega \in F_n$. Then:*

i. $\omega \in \langle\langle w\rangle\rangle \Rightarrow \omega \succcurlyeq_G w$; in particular, $w^a \succcurlyeq_G w,\ [w, w'] \succcurlyeq_G w$ for every $a \in \mathbb{Z},\ w' \in F_n$;

ii. if $\rho_{\mathfrak{g}} : \Gamma_w \to G$ is a faithful representation for some $\mathfrak{g} \in \mathcal{W}_w$, then

$$\langle\langle w\rangle\rangle = \{\omega \in F_n \quad | \quad \omega \succcurlyeq_G w\}.$$

Proof.

i. The set $\mathcal{W}_w$ is invariant under the conjugations by elements of G and therefore $\mathcal{W}_w = \mathcal{W}_{fwf^{-1}}$ for every $f \in F_n$. Then

$$\omega = \prod_{i=1}^{m}(f_i w f_i^{-1}) \Rightarrow \mathcal{W}_\omega \supset \mathcal{W}_w.$$

Now the inequalities $w^a \succcurlyeq_G w,\ [w, w'] \succcurlyeq_G w$ follow directly from the definitions of $\succcurlyeq_G$ and $\langle\langle w\rangle\rangle$.

ii. Let $\rho_{\mathfrak{g}} : \Gamma_w \to G$ be a faithful representation for some $\mathfrak{g} = (g_1, \dots, g_n) \in \mathcal{W}_w$. Suppose $\omega \succcurlyeq_G w$ for some word $\omega \notin \langle\langle w\rangle\rangle$. Since $\rho_{\mathfrak{g}}$ is a faithful representation then $\langle g_1, \dots, g_n\rangle \approx \Gamma_w = F_n/\langle\langle w\rangle\rangle$ and therefore $\omega(g_1, \dots, g_n) \neq e$. On the other hand, $\omega \succcurlyeq_G w$. Hence $\omega(g_1, \dots, g_n) = e$. It is a contradiction.

□

Remark 1.10. *If $w_1 \not\succcurlyeq_G w_2$ then $[w_1, w_2] \not\asymp_G w_1$.*

Indeed, if $\mathcal{W}_{w_1} \not\supseteq \mathcal{W}_{w_2}$ then there exists an element $\mathfrak{g} = (g_1, \dots, g_n) \in \mathcal{W}_{w_2} \setminus \mathcal{W}_{w_1}$. Hence

$$\widetilde{w}_1(\mathfrak{g}) \neq e,\ \ \widetilde{w}_2(\mathfrak{g}) = e \Rightarrow \mathcal{W}_{[w_1, w_2]} \neq \mathcal{W}_{w_1}.$$

Definition 1.11. *The subgroup*

$$\sqrt[G]{w} = \{\omega \in F_n \quad | \quad \omega \succcurlyeq_G w\} \leq F_n$$

will be called the G-radical of the word w.

Remark 1.12. *Obviously, $\sqrt[G]{w}$ is a normal subgroup of F_n and $\langle\langle w\rangle\rangle \lhd \sqrt[G]{w}$.*

2. Case of a simple algebraic group G over $\mathbb{C}$

2.1. **Irreducible components of the representation variety $\mathcal{W}_w$.** Let $\mathcal{G}$ be a simple algebraic group which is defined over $\mathbb{C}$ and $G = \mathcal{G}(\mathbb{C})$.

Let $\widetilde{w} : G^n \to G$ be a word map. Then $\mathcal{W}_w \subset G^n$ is a Zariski closed subset which consists of the union of finite number of irreducible components

$$\mathcal{W}_w = \bigcup_{i=1}^{m} \mathcal{W}_w^i$$

(see [GKP1]). We define

$$\sqrt[G]{w}^{\,i} \overset{\text{def}}{=} \{\omega \in F_n \ \mid \ \mathcal{W}_\omega \supset \mathcal{W}_w^i\}.$$

Theorem 2.1.

i. The set $\sqrt[G]{w}^{\,i} \subset F_n$ is a normal subgroup of F_n and $\sqrt[G]{w} \lhd \sqrt[G]{w}^{\,i}$ for every i.

ii. The set $\mathcal{W}_w^{i\star} \overset{def}{=} \{\mathfrak{g} = (g_1, \dots, g_n) \in \mathcal{W}_w^i \ \mid \ \langle g_1, \dots, g_n\rangle \approx F_n/\sqrt[G]{w}^{\,i}\}$ is a non-empty c-open subset of $\mathcal{W}_w^i$.

iii. For every $\mathfrak{g} = \langle g_1, \dots, g_n\rangle \in \mathcal{W}_w^i$ there exists the epimorphism $F_n/\sqrt[G]{w}^{\,i} \to \langle g_1, \dots, g_n\rangle$.

iv. The homomorphism $F_n/\sqrt[G]{w} \overset{\prod_{i=1}^m \lambda_i}{\longrightarrow} \prod_{i=1}^m F_n/\sqrt[G]{w}^{\,i}$, where $\lambda_i : F_n/\sqrt[G]{w} \to F_n/\sqrt[G]{w}^{\,i}$ are natural epimorphisms, is an injection.

v. A faithful representation $\rho : F_n/\sqrt[G]{w} \to G$ exists if and only if the homomorphism $\lambda_i : F_n/\sqrt[G]{w} \to F_n/\sqrt[G]{w}^{\,i}$ is an isomorphism for some i.

Proof.

i. It follows directly from the definitions and Proposition 1.2.

ii. Let $\omega \notin \sqrt[G]{w}^{\,i}$. Then there exists a point $\mathfrak{g} \in W_w^i$ such that $\omega(\mathfrak{g}) \neq e$. Hence $\mathcal{W}_\omega \cap \mathcal{W}_w^i$ is a proper closed subset of $\mathcal{W}_w^i$. Then the set

$$\mathcal{W}_w^i \setminus \Big(\bigcup_{\omega \notin \sqrt[G]{w}^{\,i}} \mathcal{W}_\omega \Big)$$

is a non-empty c-open subset of $\mathcal{W}_w^i$ and this set consists of all elements $\mathfrak{g} = (g_1, \dots, g_n) \in \mathcal{W}_w^i$ such that $\langle g_1, \dots, g_n\rangle \approx F_n/\sqrt[G]{w}^{\,i}$.

iii. Since $\omega(\mathfrak{g}) = e$ for every $\mathfrak{g} = (g_1, \dots, g_n) \in \mathcal{W}_w^i$ and for every $\omega \in \sqrt[G]{w}^{\,i}$, the group $\langle g_1, \dots, g_n\rangle$ is the quotient group of $F_n/\sqrt[G]{w}^{\,i}$.

iv. Let

$$\lambda : F_n/\sqrt[G]{w} \overset{\prod_{i=1}^m \lambda_i}{\longrightarrow} \prod_{i=1}^{m} F_n/\sqrt[G]{w}^{\,i}$$

and let $\omega \in F_n$ be a word such that $\overline{\omega} \in \operatorname{Ker} \lambda$ where $\overline{\omega} = \omega (\operatorname{mod} \sqrt[G]{w})$. Then $\overline{\omega} \in \operatorname{Ker} \lambda_i$ for every i where $\lambda_i : F_n/\sqrt[G]{w} \to F_n/\sqrt[G]{w}^{\,i}$. Hence

$$\omega \in \sqrt[G]{w}^{\,i} \text{ for every } i \Rightarrow \omega \in \sqrt[G]{w} \Rightarrow \overline{\omega} = e.$$

v. Suppose $\rho : F_n/\sqrt[G]{w} \to G$ is a faithful representation. Let

$$\mathfrak{g} = (\rho(\bar{x}_1), \dots, \rho(\bar{x}_n)) = (g_1, \dots g_n) \text{ where } g_i = \rho(x_i).$$

Then $\langle g_1, \ldots, g_n \rangle = \rho\big(F_n/\sqrt[G]{w}\big) \approx F_n/\sqrt[G]{w}$. Hence $\omega(\mathfrak{g}) = e$ for every $\omega \in \sqrt[G]{w}$. In particular, $w(\mathfrak{g}) = e$ and therefore $\mathfrak{g} \in \mathcal{W}_w$. Let $\mathcal{W}_w^i$ be an irreducible component that contains $\mathfrak{g}$. We have $\omega(\mathfrak{g}) = e$ for every $\omega \in \sqrt[G]{w}^{\,i}$. However, $\omega'(\mathfrak{g}) \neq e$ if $\omega' \notin \sqrt[G]{w}$. Hence $\sqrt[G]{w} = \sqrt[G]{w}^{\,i}$ and we have the isomorphism $\lambda_i : F_n/\sqrt[G]{w} \to F_n/\sqrt[G]{w}^{\,i}$. The inverse statement is obvious.

□

Definition 2.2. *The group $\Delta_w^i := F_n/\sqrt[G]{w}^i$ is called the group of general position of the component $\mathcal{W}_w^i$ or $\mathcal{W}_w^{i\star}$-group.*

2.2. **An example.** Consider the case from Example 1.7:

$$w_1 = [x^2, yxy^{-1}],\ w_2 = [x, yxy^{-1}],\ w_3 = [x, y].$$

Hence $w \asymp_G w_1$ (by Example 1.7). Also, it follows directly from the definitions

$$w_1 \succcurlyeq_G w_2 \succcurlyeq_G w_3.$$

Put

$$\mathcal{V}_3 := \overline{\{g(t_1, t_2)g^{-1} \ \mid\ g \in G,\ (t_1, t_2) \in T \times T\}} \subset G \times G,$$
$$\mathcal{V}_2 := \overline{\{g(t_1, wt_2)g^{-1} \ \mid\ g \in G,\ (t_1, t_2) \in T \times T\}} \subset G \times G,$$
$$\mathcal{V}_2^{\pm} := \overline{\{g(\pm u, t)g^{-1} \ \mid\ g \in G,\ (u, t) \in U \times T\}} \subset G \times G,$$
$$\mathcal{V}_1 := C_4 \times G,\ \text{where}\ C_4\ \text{is the conjugacy class of the order 4.}$$

Proposition 2.3. *The sets $\mathcal{V}_3,\ \mathcal{V}_2,\ \mathcal{V}_2^{\pm},\ \mathcal{V}_1$ are closed irreducible subsets of $\mathcal{W}_{w_1}$ and*

$$\mathcal{W}_{w_3} = \mathcal{V}_3,\ \mathcal{W}_{w_2} = \mathcal{V}_3 \cup \mathcal{V}_2 \cup \mathcal{V}_2^+ \cup \mathcal{V}_2^-,\ \mathcal{W}_{w_1} = \mathcal{V}_3 \cup \mathcal{V}_2 \cup \mathcal{V}_2^+ \cup \mathcal{V}_2^- \cup \mathcal{V}_1.$$

Proof. $\mathcal{V}_1$ is the product of two closed irreducible subsets of G. $\mathcal{V}_2,\ \mathcal{V}_2^{\pm},\ \mathcal{V}_3$ are closures of the images of closed irreducible sets $T \times T$, $T \times wT$, $U \times T$, $-U \times T$ and G with respect to the map $\varphi : G \times G \times G \to G \times G$ where $\varphi(x, y, z) = z(x, y)z^{-1}$. Thus, $\mathcal{V}_1,\ \mathcal{V}_2,\ \mathcal{V}_2^{\pm},\ \mathcal{V}_3$ are closed irreducible subsets of $G \times G$.

The first equality $\mathcal{W}_{w_3} = \mathcal{V}_3$ is well-known (see [GKP1]).

Consider the set $\mathcal{W}_{w_2}$. Suppose $(x, y) \in \mathcal{W}_{w_2} \setminus \mathcal{V}_3$. Then $x \neq \pm e$. Let x be a semisimple element. We may assume $x \in T,\ x \neq \pm e,\ y \notin T$. Then

$$[x, yxy^{-1}] = e \Rightarrow yxy^{-1} \in T \overset{y \notin T}{\Longrightarrow} y \in wT \Rightarrow (x, y) \in \mathcal{V}_2 \Rightarrow$$
$$\Rightarrow xyxy^{-1} = e\ \text{ and } y^4 = e.$$

Let $\pm x$ be a unipotent element. We may assume $x = \pm u$ where $u \in U, u \neq e$. Let $x = u$. Then

$$[x, yxy^{-1}] = e \Rightarrow \pm yuy^{-1} \in U \Rightarrow y = b \in B \Rightarrow (x, y) \in (u, B).$$

On the other hand,

$$(u, B) = \overline{\{v(u, t)v^{-1} \ \mid\ t \in T,\ v \in U\}} \subset \mathcal{V}_2^+.$$

Hence $(u, b) \in \mathcal{V}_2^+ \subset \mathcal{W}_{w_2}$. The same arguments show $(-u, b) \in V_2^- \subset \mathcal{W}_{w_2}$. Thus

$$\mathcal{W}_{w_2} = \mathcal{V}_3 \cup \mathcal{V}_2 \cup \mathcal{V}_2^+ \cup \mathcal{V}_2^-.$$

Let $(x,y) \in \mathcal{W}_{w_1} \setminus \mathcal{W}_{w_2}$. Then

$$\begin{cases} [x^2, yxy^{-1}] = e \Leftrightarrow [y^{-1}x^2y, x] = e, \\ [x, yxy^{-1}] \neq e, \\ x^2 \neq e. \end{cases} \Rightarrow \begin{cases} [x^2, y] = e, \\ x^4 = e. \end{cases}$$

Hence $(x,y) \in C_4 \times G$.

$\square$

The equivalence $w \asymp_G w_1$ and the Proposition 2.3 imply that the closed irreducible components of $\mathcal{W}_w$ are exactly the sets $\mathcal{V}_i$. Put

$$\mathcal{W}_w^3 = \mathcal{V}_3,\ \mathcal{W}_w^2 = \mathcal{V}_2,\ \mathcal{W}_w^{2+} = \mathcal{V}_2^+,\ \mathcal{W}_w^{2-},\ \mathcal{W}_w^1 = \mathcal{V}_1.$$

Thus we have 5 irreducible components of $\mathcal{W}_w$. The equivalence $w \asymp_G w_1$ implies

$$\sqrt[G]{w} = \sqrt[G]{w_1} = \{\omega \in F_2 \mid \omega \succcurlyeq_G w_1\}.$$

Proposition 2.4.

$$\begin{cases} \textbf{A}.\ \sqrt[G]{w}^3 = \langle\langle w_3 = [x,y]\rangle\rangle,\ \Delta_3 = F_2/\sqrt[G]{w}^3 \approx \mathbb{Z}^2 \textit{ is the } \mathcal{W}_w^{3\star}\textit{- group}, \\ \textbf{B}.\ \sqrt[G]{w}^2 = \langle\langle w_2' = xyxy^{-1}, w_2'' = y^4\rangle\rangle,\ \Delta_2 = F_2/\sqrt[G]{w}^2 \approx \mathbb{Z}\cdot\mathbb{Z}_4 \\ \quad \textit{is the } \mathcal{W}_w^{2\star}\textit{- group}, \\ \textbf{C}.\ \sqrt[G]{w}^{2+} = \langle\langle \{w_{2j} = [x, y^jxy^{-j}]\}_{j\in\mathbb{Z}}\rangle\rangle,\ \Delta_{2+} = F_2/\sqrt[G]{w}^{2+} \approx \mathbb{Z}^\infty\cdot\mathbb{Z} \\ \quad \textit{is the } \mathcal{W}_w^{2+\star}\textit{- group}, \\ \textbf{D}.\ \sqrt[G]{w}^{2-} = \langle\langle \{w_{2j} = [x, y^jxy^{-j}]\}_{j\in\mathbb{Z}}\rangle\rangle,\ \Delta_{2-} = F_2/\sqrt[G]{w}^{2-} \approx \mathbb{Z}^\infty\cdot\mathbb{Z} \\ \quad \textit{is the } \mathcal{W}_w^{2-\star}\textit{- group}, \\ \textbf{E}.\ \sqrt[G]{w}^{1} = \langle\langle w_1' = [x^2,y],\ w_1'' = x^4\rangle\rangle,\ \Delta_1 = F_2/\sqrt[G]{w}^1 \approx (\mathbb{Z}_4 * \mathbb{Z})/\mathcal{Z}, \\ \textit{where } \mathcal{Z} = \langle\langle[\sigma^2,\tau]\rangle\rangle,\ \langle\sigma,\tau\rangle \textit{ is the image of } (x,y) \textit{ in } \mathbb{Z}_4 * \mathbb{Z},\ \Delta_1 \textit{ is the } \mathcal{W}_w^{1\star}\textit{- group}. \end{cases} \tag{2.1}$$

Proof.

A. For any $\mathfrak{g} = (gt_1g^{-1}, gt_2g^{-1}), t_1, t_2 \in T, \operatorname{ord} t_1 = \operatorname{ord} t_2 = \infty,\ g \in G$ we have

$$\Delta_w^3 = \langle gt_1g^{-1}, gt_2g^{-1}\rangle \approx \mathbb{Z}^2.$$

Here $\dim W_w^3 = 4$ (see [GKP1]).

B. For any $\mathfrak{g} = (gt_1g^{-1}, gwt_2g^{-1}), t_1, t_2 \in T, \operatorname{ord} t_1 = \infty,\ g \in G$ we have

$$\Delta_w^2 = \langle gt_1g^{-1}, gwt_2g^{-1}\rangle \approx \mathbb{Z}\cdot\mathbb{Z}_4.$$

Here $\dim W_w^2 = 4$. (Indeed, consider the map $\chi : \mathcal{W}_w^2 \to T$ which is defined on dense subset $\{g(t_1, wt_2)g^{-1} \mid g \in G,\ (t_1,t_2) \in T \times T\}$ by the formula $\chi(g(t_1,wt_2)g^{-1}) = t_1$. For $t_1 \neq \pm 1$ we have $\dim \chi^{-1}(t_1) = 3$. Since $\dim \operatorname{Im} \chi = 1$ we have $\dim \mathcal{W}_w^2 = \dim \overline{\{g(t_1, wt_2)g^{-1} \mid g \in G,\ (t_1,t_2) \in T \times T\}} = 4$).

C. Let $\mathfrak{g} = (gug^{-1}, gtg^{-1})$ where

$$u = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix},\ r \neq 0,\ t = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$$ where $s \in \mathbb{C}$ and s is a transcendent number.

Then $\langle u,t\rangle \approx \mathbb{Z}^\infty \cdot \mathbb{Z}$ (see [KM]). Also, the relation on the generators (x,y) of $\mathbb{Z}^\infty \cdot \mathbb{Z}$ are generated by commutators $[y^ixy^{-i}, y^kxy^{-k}]$ for every $i,k\in\mathbb{Z}$. The commutator $[y^ixy^{-i}, y^kxy^{-k}]$, in its turn, is conjugate to $[x, y^{k-i}xy^{i-k}]$.

Here $\dim \mathcal{W}_w^{2+} = 4$ (the same arguments as in the case **B**).

D. Here we take $\mathfrak{g} = (g(-u)g^{-1}, gtg^{-1})$ and use the same arguments as in the case **C.**

E. Here any element $\mathfrak{g} = (g_1,g_2)\in C_4\times G$ satisfies the equations $g_1^4 = e = [g_1^2, g_2]$ and $\langle g_1,g_2\rangle \approx \big(\mathbb{Z}_4 * \mathbb{Z}\big)/\mathcal{Z}$ for every point $\mathfrak{g} = (g_1,g_2)\in\mathcal{X}$ where $\mathcal{X}$ is some c-open subset of $C_4\times G$ (It follows from the fact that the group $\mathrm{SL}_2(\mathbb{C})$ has no "identities with constants", see [G2]). Here $\dim \mathcal{W}_w^1 = \dim C_4 + \dim G = 5$.

□

Now consider the radical $\sqrt[G]{w}$.

Proposition 2.5. $\sqrt[G]{w} = \langle\langle \Big\{[x^2, y^jxy^{-j}]\Big\}_{j\in\mathbb{Z}}\rangle\rangle$.

Proof.

Lemma 2.6. *Let* $\Xi = \langle\langle \Big\{[x^2, y^jxy^{-j}]\Big\}_{j\in\mathbb{Z}}\rangle\rangle,\ \ \Xi_1 = \langle\langle [x^2,y], x^4\rangle\rangle,\ \ \Xi_2 = \langle\langle \Big\{[x, y^jxy^{-j}]\Big\}_{j\in\mathbb{Z}}\rangle\rangle.$ *Then* $\Xi \lhd \Xi_1,\ \ \Xi\lhd\Xi_2$.

Proof. Obviously,

$$[x, y^jxy^{-j}] = e \Rightarrow [x^2, y^jxy^{-j}] = e$$

and

$$[x^2, y] = e \Rightarrow [x^2, y^jxy^{-j}] = e \ \text{ for every }\ j\in\mathbb{Z}.$$

□

Lemma 2.7. *Let* $\Lambda = \langle \Big\{y^i\,x\,y^{-i}\Big\}_{i\in\mathbb{Z}}\rangle$. *Then*

i. $\Lambda \lhd F_2$*;*

ii. $\Lambda/\Xi_2 = \Lambda/[\Lambda,\Lambda]$ *is a free abelian group which is generated by elements* $\Big\{\bar{y}^j\bar{x}\bar{y}^{-j}\Big\}_{j\in\mathbb{Z}}$ *where* $\bar{x},\bar{y}$ *are the images of* x,y *in the quotient group* F_2/Ξ_2.

Proof.

i. We have the identity

$$x^{k_1}y^{m_1}x^{k_2}y^{m_2}\cdots x^{k_r}y^{m_r} = x^{k_1}\big(y^{m_1}x^{k_2}y^{-m_1}\big)\big(y^{m_1+m_2}x^{k_3}y^{-m_1-m_2}\big)\big(y^{m_1+m_2+m_3}\cdots \qquad (2.2)$$
$$\cdots\big(y^{m_1+\dots+m_{r-1}}x^{k_r}y^{-m_1-\dots-m_{r-1}}\big)y^{m_1+m_2+\dots+m_r}.$$

Hence $\Lambda = \{x^{k_1}y^{m_1}x^{k_2}y^{m_2}\cdots x^{k_r}y^{m_r} \ \mid\ m_1+m_2+\dots+m_r = 0\}\lhd F_2$.

ii. The group Λ is a free subgroup of F_2 of infinite rank with generators y^jxy^{-j}. The commutator group $[\Lambda,\Lambda]$ is generated by the commutators $[y^ixy^{-i}, y^kxy^{-k}]$ where $i,k\in\mathbb{Z}$. Since $\Lambda\lhd F_2$ and since the commutator $[\Lambda,\Lambda]$ is the subgroup invariant under any automorphism of Λ we have $[\Lambda,\Lambda]\lhd F_2$. On the other hand,

$$\Xi_2 = \langle\langle \Big\{[x, y^jxy^{-j}]\Big\}_{j\in\mathbb{Z}}\rangle\rangle = \langle\langle \Big\{[y^ixy^{-i}, y^kxy^{-k}]\Big\}_{i,k\in\mathbb{Z}}\rangle\rangle = [\Lambda,\Lambda].$$

Hence Λ/Ξ_2 is a free abelian group which is generated by elements $\left\{\bar{y}^j\bar{x}\bar{y}^{-j}\right\}_{j\in\mathbb{Z}}$ where $\bar{x},\bar{y}$ are the images of x,y in the quotient group F_2/Ξ_2. □

Lemma 2.8. *Let* $\Sigma = \langle\langle\left\{y^i\,x^2\,y^{-i}\right\}_{i\in\mathbb{Z}}\rangle\rangle$. *Then* $\Sigma/\Xi \le Z(\Lambda/\Xi)$.

Proof. We have $[x^2, y^i\,x\,y^{-i}] \in \Xi \Rightarrow [y^j x^2 y^{-j}, y^{i+j}\,x\,y^{-i-j}] \in \Xi$ for every $i, j \Rightarrow$

$$\Sigma/\Xi = \langle\langle\left\{\bar{y}^i\,\bar{x}^2\,\bar{y}^{-i}\right\}_{i\in\mathbb{Z}}\rangle\rangle/\Xi \le Z(\Lambda/\Xi).$$

□

Now we can prove Proposition 2.5.

Lemma 2.6 implies $\Xi \subset \Xi_1\cap\Xi_2$. Further, the groups $F_2/\Xi_1 \approx \big(\mathbb{Z}_4*\mathbb{Z}\big)/\mathcal{Z}$, $F_2/\Xi_2 \approx \mathbb{Z}^\infty\cdot\mathbb{Z}$ are $\mathcal{W}_w^{i\star}$-groups for components W_w^1 and $\mathcal{W}_w^{\pm 2}$ (see Proposition 2.4) and for the other components the $\mathcal{W}_w^{i\star}$-groups are quotient groups of F_2/Ξ_2. Thus, it is enough to prove (see, Theorem 2.1)

$$\Xi = \Xi_1 \cap \Xi_2. \tag{2.3}$$

Suppose $\omega \in \Xi_1\cap\Xi_2$. Since $\omega \in \Xi_2 = \langle\langle\left\{[x, y^j x y^{-j}]\right\}_{j\in\mathbb{Z}}\rangle\rangle$ we have

$$\omega = \prod_{i=1}^{r}\left[\left(y^{m_i}x^{p_i}y^{-m_i}\right),\left(y^{l_i}x^{q_i}y^{-l_i}\right)\right] \overset{\text{Lemma 2.8}}{\equiv} \prod_{i'=1}^{r'}\left[\left(y^{m_{i'}}x^{\pm1}y^{-m_{i'}}\right),\left(y^{l_{i'}}x^{\pm1}y^{-l_{i'}}\right)\right](\text{mod }\Xi).$$

Further,

$$\Xi = \langle\langle\left\{[x^2, y^j x y^{-j}]\right\}_{j\in\mathbb{Z}}\rangle\rangle \le \Xi_1 = \underbrace{\langle\langle\{[x^2,y],x^4\rangle\rangle}_{\ni\omega} \le \Xi^\star := \langle\langle x^2\rangle\rangle.$$

Hence

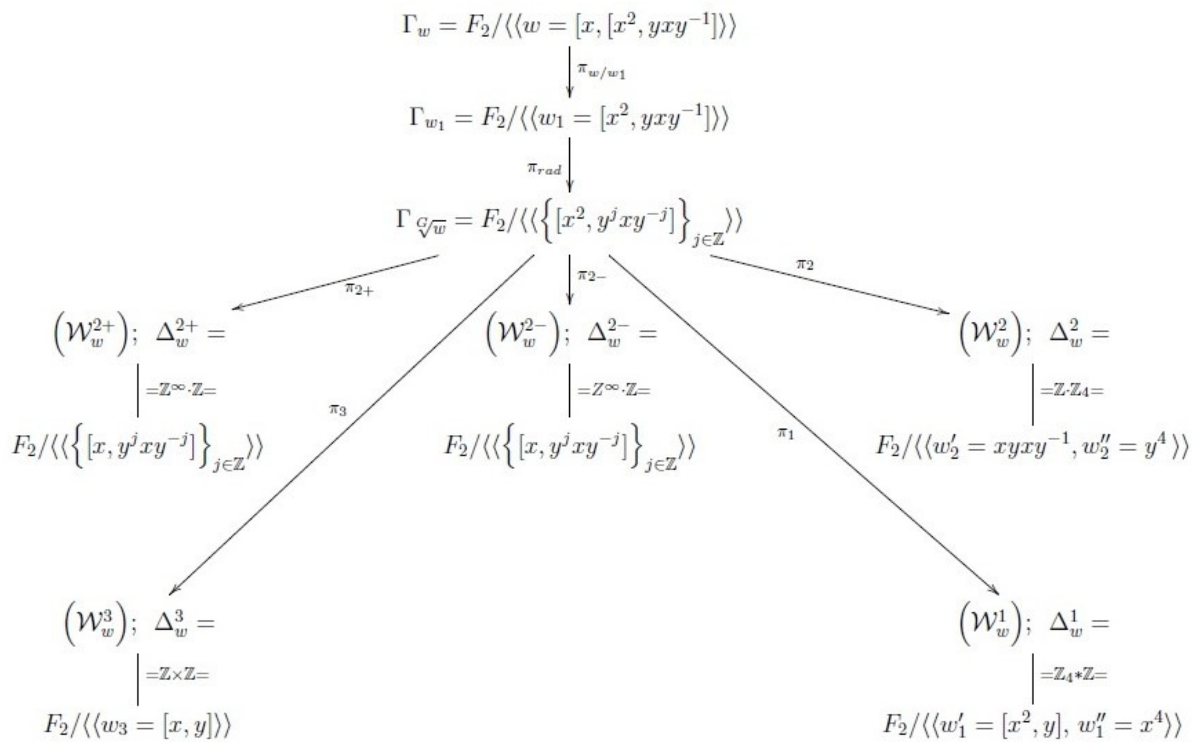
$$\prod_{i'=1}^{r'}\left[\left(y^{m_{i'}}x^{\pm1}y^{-m_{i'}}\right),\left(y^{l_{i'}}x^{\pm1}y^{-l_{i'}}\right)\right] = \prod_{j=1}^{s}\left(y^{k_j}x^{\pm1}y^{-k_j}\right) \equiv \omega \equiv e(\text{mod }\Xi^\star)$$

where $k_j \ne k_{j+1}$ for every $j = 1,\dots,s-1$. But $F_2/\Xi^\star$ is a free product $\approx \mathbb{Z}_2 * \mathbb{Z}$ and therefore $\omega \in \Xi$. □

Remark 2.9. *The equivalence* $w \asymp_G w_1$ *does not exist for simple algebraic groups of rank* > 2. *Indeed, if the rank of* G *is more than 2, then the Weyl group* W *of* G *contains an element* $w \in W$ *such that* $w^2 \notin Z(G)$ *and* $[t^2, w] \ne e$ *for* $t = g^{-1}wg \in T$ *for some* $g \in G$. *If we put* $x = t, y = g$, *we get* $[x^2, yxy^{-1}] \ne e$ *and* $[x,[x^2,yxy^{-1}]] = e$. *Hence for a group* G *when* $\operatorname{rank} G > 1$ *we have*

$$w \succeq_G w_1 \text{ and } w \not\asymp_G w_1.$$

The diagram of the descent from the group Γ_w *to* $\mathcal{W}_w^{i\star}$**- groups**



Here we present natural epimorphisms from the one-relator group Γ_w to the groups $\Delta_w^i \le \mathrm{SL}_2(\mathbb{C})$ that correspond to the group of general position of irreducible components $\mathcal{W}_w^i$ of the variety of representations $\Gamma_w \to \mathrm{SL}_2(\mathbb{C})$

Remark 2.10. *The set* $\left\{[x^2, y^j xy^{-j}]\right\}_{j=1}^{\infty}$ *is a minimal subset of generators (as a normal subgroup of F_2) of the radical $\sqrt[G]{w}$. Thus, the group* $F_2/\sqrt[G]{w} = F_2/\langle\langle\left\{[x^2, y^j xy^{-j}]\right\}_{j=1}^{\infty}\rangle\rangle$ *is not a finitely presented group.*

Remark 2.11. *Here neither the group Γ_w nor the group $\Gamma_{\sqrt[G]{w}}$ has no faithfull representation in* $\mathrm{SL}_2(\mathbb{C})$.

3. The projectivization

3.1. Extensions of word maps from the matrix groups to sets of matrices.

For a word

$$w = x_{i_1}^{a_{i_1}} x_{i_2}^{a_{i_2}} \cdots x_{i_k}^{a_{i_k}} \cdots x_{i_s}^{a_{i_s}} \in F_n \tag{3.1}$$

and any group G, a word map

$$\widetilde{w} : G^n \to G$$

is defined by the formula $\widetilde{w}(g_1, \dots, g_n) = w(g_1, \dots, g_n)$.

For any field K and any matrix $g \in \mathrm{M}_m(K)$ let us denote by g^* the transpose of the cofactor matrix for g. The matrix g^* satisfies $gg^* = g^*g = \big(\det g\big)E_m$. Then we may define the map

$$\widetilde{w}^* : \mathrm{M}_m(K)^n \to \mathrm{M}_m(K) \tag{3.2}$$

by the following rule:

for $(g_1, \ldots, g_n) \in \mathrm{M}_m(K)^n$ and for every power $x_{i_k}^{a_{i_k}}$ in (3.1) we write instead of this power the matrix $g_{i_k}^{a_{i_k}}$ if $a_{i_k} > 0$ and $(g_{i_k}^*)^{|a_{i_k}|}$ if $a_{i_k} < 0$.

Proposition 3.1. *Let w be a word (3.1) and let*

$$A_w^{j\,-} = \{a_{i_k} \ \mid \ a_{i_k} < 0, \ \ x_{i_k} = x_j\}, \ \ b_j^- = \sum_{a_{i_k} \in A_w^{j\,-}} \mid a_{i_k} \mid,$$

where $j = 1, \ldots, n$. Then for every $(g_1, \ldots, g_n) \in \mathrm{GL}_m(K)^n$ we have

$$\widetilde{w}^*(g_1, \ldots, g_n) = \Big(\prod_{j=1}^{n} \big(\det g_j\big)^{b_j^-}\Big)\widetilde{w}(g_1, \ldots, g_n).$$

Proof. Let $g \in \mathrm{GL}_m(K)$. Then $g^* = (\det g)g^{-1}$. From (3.1) we get

$$\widetilde{w}(g_1, \ldots, g_n) = g_{i_1}^{a_{i_1}} g_{i_2}^{a_{i_2}} \cdots g_{i_k}^{a_{i_k}} \cdots g_{i_s}^{a_{i_s}}. \tag{3.3}$$

The value of $\widetilde{w}^*(g_1, \ldots, g_n)$ is calculated by the same formula (3.3) by changing of powers $g_{i_k}^{a_{i_k}}$ in the cases when $a_{i_k} < 0$ for

$$(g_{i_k}^*)^{|a_{i_k}|} = \big(\det g_{i_k}\big)^{|a_{i_k}|}\big(g_{i_k}^{a_{i_k}}\big).$$

Hence the difference between $\widetilde{w}^*(g_1, \ldots, g_n)$ and $\widetilde{w}(g_1, \ldots, g_m)$ is the scalar

$$\Big(\prod_{j=1}^{n} \big(\det g_j\big)^{b_j^-}\Big).$$

□

Lemma 3.2. *Let $w \in [F_n, F_n]$ be the word of the form (3.1) and let*

$$A_w^{j\,-} = \{a_{i_k} \ \mid \ a_{i_k} < 0, \ \ x_{i_k} = x_j\}, \ \ A_w^{j\,+} = \{a_{i_k} \ \mid \ a_{i_k} > 0, \ \ x_{i_k} = x_j\},$$

$$b_j^- = \sum_{a_{i_k} \in A_w^{j\,-}} \mid a_{i_k} \mid, \ \ b_j^+ = \sum_{a_{i_k} \in A_w^{j\,+}} \mid a_{i_k} \mid$$

for every $j = 1, \ldots, n$. Then

i. $b_j^- = b_j^+$ for every $j = 1, \ldots, n$;

ii. for every $g_1, \ldots, g_n \in \mathrm{GL}_m(K)$ and every $\alpha_1, \ldots, \alpha_n \in K^*$

$$\widetilde{w}((\alpha_1 g_1), \ldots, (\alpha_n g_n)) = \widetilde{w}(g_1, \ldots, g_n).$$

Proof.

i. Since $w \in [F_n, F_n]$, we have

$$\widetilde{w}(e, \ldots, e, \underbrace{g}_{j^{th}-place}, e, \ldots, e) = e \Rightarrow b_j^+ - b_j^- = 0$$

for every j.

ii. From 3.1

$$\widetilde{w}((\alpha_1 g_1),\dots,(\alpha_n g_n)) = \left(\alpha_{i_1} g_{i_1}\right)^{a_{i_1}} \left(\alpha_{i_2} g_{i_2}\right)^{a_{i_2}} \cdots \left(\alpha_{i_k} g_{i_k}\right)^{a_{i_k}} \cdots \left(\alpha_{i_s} g_{i_s}\right)^{a_{i_s}} =$$

$$\underbrace{\left(\prod_{j=1}^{n} (\alpha_j)^{(b_j^+ - b_j^-)}\right)}_{=1 \text{ by the item i.}} g_{i_1}^{a_{i_1}} g_{i_2}^{a_{i_2}} \cdots g_{i_k}^{a_{i_k}} \cdots g_{i_s}^{a_{i_s}} = \widetilde{w}(g_1,\dots,g_n).$$

□

3.2. **Word maps over polynomial ring.**

Let

$$\mathcal{S} = K[\{x_{\mu\nu j}\}_{\{1\le\mu,\nu\le m,\,1\le j\le n\}}]$$

be the polynomial ring of $m^2 n$ variables $\{x_{\mu\nu j}\}_{\{1\le\mu,\nu\le m,\,1\le j\le n\}}$ over the field K. For $j = 1,\dots n$ we define the matrix

$$X_j = \begin{pmatrix} x_{11j} & x_{12j} & \cdots & x_{1mj} \\ x_{21j} & x_{22j} & \cdots & x_{2mj} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m1j} & x_{m2j} & \cdots & x_{mmj} \end{pmatrix} \in \mathrm{M}_m(\mathcal{S}).$$

Then the entries of the cofactor matrix X_j^* are homogeneous polynomials on the variables $\{x_{\mu\nu j}\}_{\{1\le\mu,\nu\le m,\,1\le j\le n\}}$ of degree $m-1$ and

$$X_j X_j^* = X_j^* X_j = \begin{pmatrix} \det X_j & 0 & \cdots & 0 \\ 0 & \det X_j & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \det X_j \end{pmatrix}$$

where $\det X_j$ is the homogenous polynomial of degree m. Moreover

$$\det X_j^* = \left(\det X_j\right)^{m-1}. \tag{3.4}$$

For a word $w \in F_n$ of the form (3.1) we have

$$\widetilde{w}^*(X_1,\dots,X_n) = \hat{X}_{i_1}^{|a_{i_1}|} \hat{X}_{i_2}^{|a_{i_2}|} \cdots \hat{X}_{i_k}^{|a_{i_k}|} \cdots \hat{X}_{i_s}^{|a_{i_s}|} \in \mathrm{M}_m(\mathcal{S}),$$

where

$$\hat{X}_{i_k} = \begin{cases} X_{i_k} & \text{if } a_{i_k} > 0, \\ X_{i_k}^* & \text{if } a_{i_k} < 0. \end{cases}$$

Proposition 3.3. *Let $w \in F_n$ be the map of the form (3.1) and let $\mathcal{X} = \widetilde{w}^*(X_1,\dots,X_n)$ be the corresponding word map matrix. Then every entry $\mathcal{X}_{pq}$ of the matrix $\mathcal{X}$ is a homogenous polynomial on $m^2 n$ variables $\{x_{\mu\nu j}\}_{\{1\le\mu,\nu\le m,\,1\le j\le n\}}$ of the degree*

$$\sum_{j=1}^{n} (b_j^+ + (m-1) b_j^-).$$

In particular, if $w \in [F_n, F_n]$ then

$$\deg \mathcal{X}_{pq} = m \sum_{j=1}^{n} b_j^{+} = m \frac{\big(\mid a_{i_1} \mid + \mid a_{i_2} \mid + \cdots + \mid a_{i_s} \mid \big)}{2}.$$

Proof. The entries of the matrix $\hat{X}^{|a_{i_k}|}$ are homogenous polynomials on $x_{\mu\nu j}$ of degree $\mid a_{i_k} \mid$ if $a_{i_k} > 0$ and of degree $(m-1) \mid a_{i_k} \mid$ if $a_{i_k} < 0$. Then the degree of $\mathcal{X}_{pq}$ is equal to

$$\sum_{j=1}^{n} \Big(\underbrace{\big(\sum_{i_k = j, a_{i_k} \in A_w^{j\,-}} \mid a_{i_k} \mid \big)}_{=b_j^{+}} + (m-1) \underbrace{\big(\sum_{i_k = j, a_{i_k} \in A_w^{j\,+}} \mid a_{i_k} \mid \big)}_{=b_j^{-}} \Big) = \sum_{j=1}^{n} (b_j^{+} + (m-1) b_j^{-}) =$$

$$\overset{if\, w \in [F_n, F_n]}{=} \sum_{j=1}^{n} (b_j^{+} + (m-1) b_j^{+}) = \sum_{j=1}^{n} m b_j^{+} = m \sum_{j=1}^{n} b_j^{+} = m \frac{\big(\mid a_{i_1} \mid + \mid a_{i_2} \mid + \cdots + \mid a_{i_s} \mid \big)}{2}.$$

□

Proposition 3.4. *Let $w \in F_n$ be the map of the form (3.1) and let $\mathcal{X} = \widetilde{w}^*(X_1, \ldots, X_n)$ be the corresponding word map matrix. Then*

$$\det \mathcal{X} = \prod_{j=1}^{n} \big(\det X_j \big)^{(b_j^{+} + (m-1) b_j^{-})}.$$

In particular, if $w \in [F_n, F_n]$ then

$$\det \mathcal{X} = \prod_{j=1}^{n} \big(\det X_j \big)^{m b_j^{+}}.$$

Proof. It follows from the definitions of $\mathcal{X} = w^*(X_1, \ldots, X_n)$, b_j^{+}, b_j^{-} and 3.4.

□

3.3. **Case $\mathbf{n = 2, m = 2}$.**

Now let $m = 2$, that is, we consider the group $F_2 = \langle x, y \rangle$ and

$$w(x, y) = x^{c_1} y^{d_1} x^{c_2} y^{d_2} \cdots x^{c_k} y^{d_k} \cdots x^{c_r} y^{d_r}, \tag{3.5}$$

where $\mid c_1 \mid, \mid d_r \mid \geq 0$ and $\mid d_1 \mid, \mid c_2 \mid, \ldots, \mid c_r \mid > 0$. Further, let

$C_w^{+} = \{c_j \ \mid \ c_j \geq 0\}$, $C_w^{-} = \{c_j \ \mid \ c_j < 0\}$, $D_w^{+} = \{d_j \ \mid \ d_j \geq 0\}$, $D_w^{-} = \{d_j \ \mid \ d_j < 0\}$.

Then put

$$c^{+} = \sum_{c_j \in C_w^{+}} \mid c_j \mid, \; c^{-} = \sum_{c_j \in C_w^{-}} \mid c_j \mid, \; d^{+} = \sum_{d_j \in D_w^{+}} \mid d_j \mid, \; d^{-} = \sum_{d_j \in D_w^{-}} \mid d_j \mid.$$

Here $\mathcal{S} = K[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]$ is the polynomial ring of 8 variables,

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \; Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \in \mathrm{M}_2(\mathcal{S}),$$

where

$$\hat{X}^{c_i} = \begin{cases} X^{c_i} & \text{if } c_i \in C_w^+, \\ \left(X^*\right)^{|c_i|} & \text{if } c_i \in C_w^-, \end{cases} \qquad \hat{Y}^{d_i} = \begin{cases} Y^{d_i} & \text{if } d_i \in D_w^+, \\ \left(Y^*\right)^{|d_i|} & \text{if } d_i \in D_w^-. \end{cases}$$

From Proposition 3.3 we get here

$$\deg \mathcal{X}_{pq} = \mid c_1 \mid + \cdots + \mid c_r \mid + \mid d_1 \mid + \cdots + \mid d_r \mid . \tag{3.6}$$

Now consider the maps

$$\widetilde{w}^* : \mathrm{M}_2(\mathbb{C})^2 \to \mathrm{M}_2(\mathbb{C}), \;\; \widetilde{w} : \mathrm{GL}_2(\mathbb{C})^2 \to \mathrm{GL}_2(\mathbb{C}).$$

By Proposition 3.1 for every pair $(g_1, g_2) \in \mathrm{GL}_2(\mathbb{C})^2$ we have

$$\widetilde{w}^*(g_1, g_2) = \left(\det g_1\right)^{c^-} \left(\det g_2\right)^{d^-} \widetilde{w}(g_1, g_2). \tag{3.7}$$

Theorem 3.5. *Let $w \in [F_2, F_2]$ and let $\alpha \in \mathbb{C}$. Then*
i. there exists a pair $(g_1, g_2) \in \mathrm{M}_2^2(\mathbb{C})$ such that

$$\mathcal{X}_{11}(g_1, g_2) = 0, \;\; \mathcal{X}_{22}(g_1, g_2) - \alpha\left(det g_1\right)^{c^-} \left(\det g_2\right)^{d^-} = 0,$$

where $\mathcal{X}_{pq}(g_1, g_2)$ is the (pq)-entry of the matrix $\widetilde{w}^*(g_1, g_2) \in \mathrm{M}_2(\mathbb{C})$;
ii. if $(g_1, g_2) \in \mathrm{GL}_2(\mathbb{C})^2$ where (g_1, g_2) is a pair which satisfies the condition i., then there exists a non-central element $g \in \mathrm{Im}\, \widetilde{w}$ such that $\mathrm{tr}\, g = \alpha$.

Proof. Let

$$\widetilde{w}^*(X, Y) = \mathcal{X} = \begin{pmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{21} & \mathcal{X}_{22} \end{pmatrix} =$$
$$= \hat{X}^{c_1}\hat{Y}^{d_1}\hat{X}^{c_2}\hat{Y}^{d_2} \cdots \hat{X}^{c_k}\hat{Y}^{d_k} \cdots \hat{X}^{c_r}\hat{Y}^{d_r} \in \mathrm{M}_2(\mathcal{S})$$

where

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \; Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

The entries $\mathcal{X}_{pq}$ are homogenous polynomials of degree $\sum_{i=1}^r \mid c_i \mid + \sum_{j=1}^r \mid d_j \mid$ (see (3.6)). Since $w \in [F_2, F_2]$, we have by Proposition 3.3

$$\deg \mathcal{X}_{pq} = 2c^- + 2d^-.$$

Thus we have the system of 2 homogenous equations

$$\begin{cases} \mathcal{X}_{11} = 0, \\ \mathcal{X}_{22} - \alpha\left(\det X\right)^{c^-} \left(\det Y\right)^{d^-} = 0 \end{cases} \tag{3.8}$$

on 8 variables x_{ij}, y_{kl}. The system of equations (3.8) has a non-zero solution $(g_1, g_2) \in \mathrm{M}_2(\mathbb{C})^2$ and, therefore, the item i. holds.

Now suppose that $(g_1, g_2) \in \mathrm{GL}_2(\mathbb{C})^2$. Then $g_1 = \delta g_1', g_2 = \gamma g_2'$ for some $g_1', g_2' \in \mathrm{SL}_2(\mathbb{C})$, $\gamma, \delta \in \mathbb{C}^*$ and $\delta^2 = \det g_1, \gamma^2 = g_2$. Now if we substitute in $\widetilde{w}^*(X, Y)$ the values of matrices g_1, g_1' instead of corresponding entries x_{ij} and the values of matrices g_2, g_2' instead of corresponding entries y_{kl} we get

$$\widetilde{w}^*(g_1, g_2) = \widetilde{w}^*(\gamma g_1', \delta g_2') = \gamma^{2c^-}\delta^{2d^-}\widetilde{w}^*(g_1', g_2') = \left(\det g_1\right)^{c^-} \left(\det g_2\right)^{d^-} \widetilde{w}^*(g_1', g_2').$$

The entries $\mathcal{X}_{pq}(g_1,g_2)$, $\mathcal{X}_{pq}(g_1',g_2')$ of the matrices $\mathcal{X}(g_1,g_2) := \widetilde{w}^*(g_1,g_2)$, $\mathcal{X}(g_1',g_2') := \widetilde{w}^*(g_1',g_2')$ satisfy the equations (see (3.8))

$$\begin{cases} \mathcal{X}_{11}(g_1,g_2) = \big(\det g_1\big)^{c^-}\big(\det g_2\big)^{d^-}\mathcal{X}_{11}(g_1',g_2') = 0, \\ \mathcal{X}_{22}(g_1,g_2) = \big(\det g_1\big)^{c^-}\big(\det g_2\big)^{d^-}\mathcal{X}_{22}(g_1',g_2') - \alpha\big(\det g_1\big)^{c^-}\big(\det g_2\big)^{d^-} = 0. \end{cases}$$

Hence

$$\begin{cases} \mathcal{X}_{11}(g_1',g_2') = 0, \\ \mathcal{X}_{22}(g_1',g_2') - \alpha = 0. \end{cases}$$

Since $g_1', g_2' \in \mathrm{SL}_2(\mathbb{C})$ then

$$g = \mathcal{X}(g_1',g_2') = \widetilde{w}^*(g_1',g_2') = \widetilde{w}(g_1',g_2') = \begin{pmatrix} 0 & * \\ * & \alpha \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \Rightarrow \operatorname{tr} g = \alpha.$$

□

Corollary 3.6. *Let $w \in [F_2,F_2]$. If there exists a pair $(g_1,g_2) \in \mathrm{GL}_2(\mathbb{C})^2$ that satisfies condition i. of Theorem 3.5 with $\alpha = 2$ or $\alpha = -2$ then the map*

$$\widetilde{w} : \mathrm{PGL}_2(\mathbb{C})^2 \to \mathrm{PGL}_2(\mathbb{C})$$

is surjective.

Proof. We may assume $(g_1,g_2) \in \mathrm{SL}_2(\mathbb{C})^2$ and $\operatorname{tr}\widetilde{w}(g_1,g_2) = 2$ or $\operatorname{tr}\widetilde{w}(g_1,g_2) = -2$ by ii. of Theorem 3.5. Moreover,

$$g = \widetilde{w}(g_1,g_2) = \begin{pmatrix} 0 & * \\ * & \pm 2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}).$$

Hence g is a non-trivial unipotent element u or $-u$. □

3.4. **The varieties $\mathcal{V}_{\widetilde{w}}^{\alpha}$, $\mathrm{P}(\mathcal{V}_{\widetilde{w}}^{\alpha})$.**

For a fixed $\alpha \in \mathbb{C}$ define

$$\mathcal{V}_{\widetilde{w}}^{\alpha} := \{(X,Y) \in \mathrm{M}_2(\mathbb{C})^2 \ \mid \ \mathcal{X}_{11}(X,Y) = 0, \ \mathcal{X}_{22}(X,Y) - \alpha\big(detX\big)^{c^-}\big(\det Y\big)^{d^-} = 0\}.$$

Then $\mathcal{V}_{\widetilde{w}}^{\alpha}$ are closed algebraic subsets of $\mathrm{M}_2(\mathbb{C}) \times \mathrm{M}_2(\mathbb{C}) \approx \mathrm{A}_{\mathbb{C}}^8$ and all their irreducible components have dimensions ≥ 6. Since the equations that define the variety $\mathcal{V}_{\widetilde{w}}^{\alpha}$ are homogenous with respect to both sets $\{x_{ij}\}, \{y_{kl}\}$ of variables, we define the corresponding closed projective sets

$$\mathrm{P}(\mathcal{V}_{\widetilde{w}}^{\alpha}) := \{(\bar{X},\bar{Y}) \in \mathrm{P}(\mathrm{M}_2(\mathbb{C})) \times \mathrm{P}(\mathrm{M}_2(\mathbb{C})) \ \mid \ (X,Y) \in \mathcal{V}_{\widetilde{w}}^{\alpha}\}$$

where $\bar{X}, \bar{Y}$ are the images of the matrices X, Y in $\mathrm{P}(\mathrm{M}_2(\mathbb{C}))$. All irreducible components of $\mathrm{P}(\mathcal{V}_{\widetilde{w}}^{\alpha})$ have the dimension ≥ 4.

Further, let

$$\mathcal{D}_{X,\widetilde{w}} := \{(X,Y) \in \mathrm{M}_2^2(\mathbb{C}) \ \mid \ \det X = 0, \ \mathcal{X}_{11}(X,Y) = 0, \ \mathcal{X}_{22}(X,Y) = 0\},$$

$$\mathcal{D}_{Y,\widetilde{w}} := \{(X,Y) \in \mathrm{M}_2^2(\mathbb{C}) \ \mid \ \det Y = 0, \ \mathcal{X}_{11}(X,Y) = 0, \ \mathcal{X}_{22}(X,Y) = 0\}$$

be the corresponding closed subsets of $A_{\mathbb{C}}^4 \times A_{\mathbb{C}}^4$ and let $\mathrm{P}(\mathcal{D}_{X,\widetilde{w}})$, $\mathrm{P}(\mathcal{D}_{Y,\widetilde{w}})$ be their projectiviation in $\mathrm{P}_{\mathbb{C}}^3 \times \mathrm{P}_{\mathbb{C}}^3$. Then we can reformulate Theorem 3.5.

Theorem 3.7. *Let $w \in [F_2, F_2]$ and let $\alpha \in \mathbb{C}$. Then*

i. $\dim \mathcal{V}^{\alpha}_{\widetilde{w}} \geq 6$ *(respectively,* $\dim \mathrm{P}(\mathcal{V}^{\alpha}_{\widetilde{w}}) \geq 4$*);*

ii. if $\mathcal{V}^{\alpha}_{\widetilde{w}} \not\subseteq \mathcal{D}_{X,\widetilde{w}} \cup \mathcal{D}_{Y,\widetilde{w}}$ *(respectively,* $\mathrm{P}(\mathcal{V}^{\alpha}_{\widetilde{w}}) \not\subseteq \mathrm{P}(\mathcal{D}_{X,\widetilde{w}}) \cup \mathrm{P}(\mathcal{D}_{Y,\widetilde{w}})$*), then there exists a non-central element* $g \in \mathrm{Im}\ \widetilde{w}$ *such that* $\mathrm{tr}\, g = \alpha$*.*

Proof.

i. Here the inequality $\dim \mathcal{V}^{\alpha}_{\widetilde{w}} \geq 6$ is some strengthening of the part *i.* of Theorem 3.5 which is equivalent to $\mathcal{V}^{\alpha}_{\widetilde{w}} \neq \emptyset$.

ii. The condition $(g_1, g_2) \in \mathrm{GL}_2(\mathbb{C})^2$ is equivalent to $\mathcal{V}^{\alpha}_{\widetilde{w}} \not\subseteq \mathcal{D}_{X,\widetilde{w}} \cup \mathcal{D}_{Y,\widetilde{w}}$. ☐

Note, that for every $\alpha \in \mathbb{C}$ we have

$$\mathcal{D}_{X,\widetilde{w}} \cup \mathcal{D}_{Y,\widetilde{w}} \subset \mathcal{V}^{\alpha}_{\widetilde{w}},\ \ \mathrm{P}(\mathcal{D}_{X,\widetilde{w}}) \cup \mathrm{P}(\mathcal{D}_{Y,\widetilde{w}}) \subset \mathrm{P}(\mathcal{V}^{\alpha}_{\widetilde{w}}). \tag{3.9}$$

Indeed, if the pair (X, Y) satisfies the equations $\det X = 0$ (or $\det Y = 0$), $\mathcal{X}_{11}(X, Y) = 0$, $\mathcal{X}_{22}(X, Y) = 0$, then this pair also satisfies the equation

$$\mathcal{X}_{22}(X, Y) - \alpha \underbrace{\left(detX\right)^{c^-} \left(\det Y\right)^{d^-}}_{\det X = 0 \text{ or } \det Y = 0} = 0.$$

There is a pair $(g_1, g_2) \in \mathrm{SL}_2(\mathbb{C})^2$ such that $\mathrm{tr}(\widetilde{w}(g_1, g_2)) = \alpha$ ([BZ]) for every $\alpha \in \mathbb{C}$. If $\alpha \neq \pm 2$ then the matrix $\widetilde{w}((g_1, g_2))$ is non-central and therefore it is conjugate to the matrix of the form $\begin{pmatrix} 0 & * \\ * & \alpha \end{pmatrix}$ ([EG1]). The image of the map $\widetilde{w}$ is invariant under conjugations. Hence we may assume that just for (g_1, g_2) the matrix $\widetilde{w}(g_1, g_2)$ has the appropriate form and therefore $(g_1, g_2) \in \mathcal{V}^{\alpha}_{\widetilde{w}} \setminus \left(\mathcal{D}_{X,\widetilde{w}} \cup D_{Y,\widetilde{w}}\right)$. Thus we have the family of closed subsets

$$\left\{\mathcal{V}^{\alpha}_{\widetilde{w}}\right\}_{\alpha \in \mathbb{C}},\ \ \left\{\mathrm{P}(\mathcal{V}^{\alpha}_{\widetilde{w}})\right\}_{\alpha \in \mathbb{C}} \tag{3.10}$$

such that the inclusions (3.9) hold for every α and

$$\mathcal{V}^{\alpha}_{\widetilde{w}} \neq \mathcal{D}_{X,\widetilde{w}} \cup \mathcal{D}_{Y,\widetilde{w}},\ \ \mathrm{P}(\mathcal{V}^{\alpha}_{\widetilde{w}}) \neq \mathrm{P}(\mathcal{D}_{X,\widetilde{w}}) \cup \mathrm{P}(\mathcal{D}_{Y,\widetilde{w}}) \text{ for every } \alpha \neq \pm 2. \tag{3.11}$$

In order to prove the surjectivity of $\widetilde{w} : \mathrm{PGL}_2(\mathbb{C})^2 \to \mathrm{PGL}_2(\mathbb{C})$ we have to prove that the inequality (3.11) holds also for $\alpha = 2$ or $\alpha = -2$ (see Corollary 3.6). For instance, it would hold if $\dim \left(\mathcal{D}_{X,\widetilde{w}} \cup \mathcal{D}_{Y,\widetilde{w}}\right) \leq 5$ (the set is defined by three equations in 8-dimensional space). Indeed, all components of $\mathcal{V}^{\alpha}_{\widetilde{w}}$ have dimension ≥ 6.

3.5. The varieties $\mathcal{V}^{\alpha,g}_{\widetilde{w}}$, $\mathcal{D}^{g}_{X,\widetilde{w}}$.

One of the problems to prove the inequality 3.11 for $\alpha = \pm 2$ is a rather big dimension of varieties. Here we propose a reduction of dimensions of the considered varieties using *the word maps with constants* ([G2], [GG], [GKP2]).

Let $g \in \mathrm{SL}_2(\mathbb{C}), g \neq \pm E_2$ and let

$$\mathcal{V}^{\alpha,g}_{\widetilde{w}} := \{(X, g) \in \mathcal{V}^{\alpha}_{\widetilde{w}}\},\ \ \mathcal{D}^{g}_{X,\widetilde{w}} = \{(X, g) \in \mathcal{D}_{X,\widetilde{w}}\},$$

$$\mathrm{P}(\mathcal{V}^{\alpha,g}_{\widetilde{w}}) := \{(X, g) \in \mathrm{P}(\mathcal{V}^{\alpha}_{\widetilde{w}})\},\ \ \mathrm{P}(\mathcal{D}^{g}_{X,\widetilde{w}}) := \{(X, g) \in \mathrm{P}(\mathcal{D}_{X,\widetilde{w}})\}.$$

Now we get the families of closed affine $\left\{\mathcal{V}^{\alpha,g}_{\widetilde{w}}\right\}_{\alpha\in\mathbb{C}}$ or projective $\left\{\mathrm{P}(\mathcal{V}^{\alpha,g}_{\widetilde{w}})\right\}_{\alpha\in\mathbb{C}}$ sets that satisfy the following conditions

$$\mathcal{D}^g_{X,\widetilde{w}} \subset \mathcal{V}^{\alpha,g}_{\widetilde{w}},\ \ \mathrm{P}(\mathcal{D}^g_{X,\widetilde{w}}) \subset \mathrm{P}(\mathcal{V}^{\alpha,g}_{\widetilde{w}})$$

for every $\alpha \in \mathbb{C}$. To prove the surjectivity of $\widetilde{w} : \mathrm{PGL}_2(\mathbb{C})^2 \to \mathrm{PGL}_2(\mathbb{C})$ we have to prove the inequality

$$\mathcal{V}^{\alpha,g}_{\widetilde{w}} \neq \mathcal{D}^g_{X,\widetilde{w}} \text{ or } \ \mathrm{P}(\mathcal{V}^{\alpha,g}_{\widetilde{w}}) \neq \mathrm{P}(\mathcal{D}^g_{X,\widetilde{w}}) \ \text{ for } \ \alpha = 2 \text{ or } \alpha = -2$$

for some appropriate g.

3.6. **Example:** $w = [x, y]$**.**
Here consider the simplest example when the word $w = [x, y]$ is just the commutator of x, y. It is a known fact that $\mathrm{Im}\ \widetilde{w} = \mathrm{SL}_2(\mathbb{C})$ (see [GKP3]). Let

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ X = \begin{pmatrix} x & y \\ z & t \end{pmatrix}.$$

Then

$$\widetilde{w}^*(X, g) = \begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} t & -y \\ -z & x \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} x & z \\ y & t \end{pmatrix} = \begin{pmatrix} x^2 + y^2 & ty + xz \\ ty + xz & z^2 + t^2 \end{pmatrix}.$$

Then

$$\mathcal{X}_{11} = x^2 + y^2,\ \ \mathcal{X}_{22} = z^2 + t^2,$$

$$\mathcal{D}^g_{X,\widetilde{w}} = \{(X, g)\ \mid\ \det X = 0,\ \ \mathcal{X}_{11} = 0,\ \ \mathcal{X}_{22} = 0\} =$$

$$\{(X, g)\ \mid\ x = iy, z = it\} \bigcup \{(X, g)\ \mid\ x = -iy, z = -it\}.$$

Thus, $\mathcal{D}^g_{X,\widetilde{w}}$ is a union of two planes in 4-dimensional space.
Further,

$$\mathcal{V}^{\alpha,g}_{\widetilde{w}} = \{(X, g)\ \mid\ x = \pm iy,\ z^2 + t^2 - \alpha(xt - yz) = 0\}.$$

We have

$$z^2 + t^2 - \alpha(\underbrace{x}_{=\pm iy}t - yz) = (z + it)(z - it) + \alpha y(z \pm it) \Rightarrow \mathcal{V}^{\alpha,g}_{\widetilde{w}} =$$

$$= \{(X, g)\ \mid\ x = iy,\ (z - it)(z + it + \alpha y) = 0\} \bigcup \{(X, g)\ \mid\ x = -iy,\ (z + it)(z - it + \alpha y) = 0\}.$$

Thus $\mathcal{V}^{\alpha,g}_{\widetilde{w}}$ is a union of four planes in 4-dimensional space and therefore $\mathcal{V}^{\alpha,g}_{\widetilde{w}} \neq \mathcal{D}^g_{X,\widetilde{w}}$ for every α. Hence, for every $\alpha \in \mathbb{C}$ there exists an element $g' \in \mathrm{GL}_2(\mathbb{C})$ such that $(g', g) \in \mathcal{V}^{\alpha}_{\widetilde{w}}$ and therefore in the image of the map $\widetilde{w} : \mathrm{SL}_2(\mathbb{C})^2 \to \mathrm{SL}_2(\mathbb{C})$ there exists a non-central element g'' such that $\mathrm{tr}(g'') = \alpha$. Then the map $\widetilde{w} : \mathrm{SL}_2(\mathbb{C})^2 \to \mathrm{SL}_2(\mathbb{C})$ is surjective.

References

[BZ] T. Bandman, Yu. G. Zarhin, *Surjectivity of certain word maps on $PSL(2,\mathbb{C})$ and $SL(2,\mathbb{C})$*, Eur. J. Math. **2** (2016), 614–643.

[Bo1] A. Borel, *On free subgroups of semisimple groups*, Enseign. Math. **29** (1983), 151–164.

[Bo2] A. Borel. Linear Algebraic groups. 2nd enl.ed., Graduate texts in mathematics **126**. Springer-Verlag New York Inc.1991.

[Bou] N. Bourbaki. Éléments de Mathématique. Groupes et Algèbres de Lie, Chap. IV, V, VI, 2ème édition. Masson, Paris 1981.

[EG1] E. Ellers, N. Gordeev, *Intersection of Conjugacy Classes of Chevalley Groups with Gauss Cells*. Journal Of Algebra, v. 220(1999), n. 2, 591–611.

[G] N. Gordeev *On Engel words on simple agebraic groups*. J. Algebra, v. 425 (2015), 215-244.

[G2] N. Gordeev *21. Freedom in conjugacy classes of simple algebraic groups and identities with constants*, Algebra and Analiz, vol.9 (1997),n.4, p.63-78 ; English translation: St. Petersburg Math.Jouranal, vol.9 (1998), 709–723.

[GG] N. Gordeev, F.Gnutov *Recursive sequences of surjective word maps on the group PGL_2*, Athive der Mathematik, v. 114 (2020), no. 6, 609–618.

[GKP1] N. Gordeev, B. Kunyavskii, E. Plotkin *, Word maps and word maps with constants of simple algebraic groups*, Dokl. Akad. Nauk 471(2) (2016) 136–138 (in Russian); Dokl. Math. 94 (2016) 632–634.

[GKP2] N. Gordeev, B. Kunyavskii, E. Plotkin, *Word maps, word maps with constants and representation varieties of one-relator groups—*, J. Algebra 500 (2018) 390–424.

[GKP3] N. Gordeev, B. Kunyavskii, E. Plotkin, *Geometry of word equations in simple algebraic groups over special fields*, Russian Mathematical Surveys, 2018, Volume 73, Issue 5, 753–796.

[GKP4] N. Gordeev, B. Kunyavskii, E. Plotkin *Word maps on perfect algebraic groups*, International Journal of Algebra and Computation Vol. 28, No. 8 (2018) 1487–1515.

[JS] U. Jezernick, J. Sanchez, *On surjectivity of word maps on PSL_2*, J.of Algebra, **587**, (2021), 613—627.

[KM] Fundamentals of the Theory of Groups, by M. I. Kargapolov and Ju I. Merzljakov. Translated by Robert G. Burns. Pp. 203. DM 35. 1979. ISBN 0 387 90396 8 (Springer)

[LM] A. Lubotzky, A. Magid, *Varieties of representations of finitely generated groups*, Memoirs of the American Mathematical Society 58 (1985), 1–116.

[PRR] V. Platonov, A. Rapinchuk, I. Rapinchuk, "Algebraic Groups and Number Theory", Cambridge University Press, 2023.

[Pl1] B. Plotkin, *Algebras with the same (algebraic) geometry*. In: Proc. Steklov Inst. Math. 242, (2003)176–207.

[Pl2] B. Plotkin, *Seven lectures on the universal algebraic geometry*. In: Groups, Algebras and Identities. Contemporary Mathematics, 726, ed. Plotkin E., Amer Mathematical Soc, (2019) 143–215.

[T] A. Thom, *Convergent sequences in discrete groups*, Canad. Math. Bull. 56 (2013) 424–433.

Nikolai Gordeev, Department of Mathematics, Russian State Pedagogical University, Moijka 48, St.Petersburg, 191-186, Russia, and Faculty of Mathematics and Mechanics of Sankt-Petersburg State University, Universitetsky prospekt, 28, Peterhof, St. Petersburg, 198504, Russia.

Email address: `nickgordeev@mail.ru`

Eugene Plotkin, Department of Mathematics, Bar Ilan University, Ramat Gan, Israel

Email address: `plotkin.evegeny@gmail.com`